Lecture V: Cohen-Macaulay Rings II (Feb. 7, 2006)
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Let $R = k[x_1, \ldots, x_n]$ and $I \subseteq R$ an ideal. Set $S = R/I$. Recall that $\dim S =$ length of longest chain of prime ideals in S. As well, a sequence $F_1, \ldots, F_m$ of elements in R is a regular sequence on $S = R/I$ if

(1) $\overline{F_1}$ is regular (a nonzero divisor) on R/I;
(2) $\overline{F_i}$ is regular on $R/(I, F_1, \ldots, F_{i-1})$ for $i = 2, \ldots, m$.

The depth of S is the length of the longest sequence $F_1, \ldots, F_m \in \mathfrak{m} = (x_1, \ldots, x_n) \subseteq R$ that is a regular sequence on S.

Definition 1. R/I is Cohen-Macaulay (CM) if depth $R/I = \dim R/I$.

If Δ is a simplicial complex on $V_\Delta = \{x_1, \ldots, x_n\}$, then the Stanley-Reisner ideal is the square-free monomial ideal

$$I_\Delta = < x_{i_1} \cdots x_{i_r} \mid \{x_{i_1}, \ldots, x_{i_r}\} \notin \Delta > .$$

We call Δ a Cohen-Macaulay simplicial complex if R/I_Δ is Cohen Macaulay.

The Cohen-Macaulay property is special; we should expect that the Cohen-Macaulay property of R/I_Δ puts bounds on the invariants of Δ. We will show how the hypothesis of CM forces a bound on the h-vector on Δ.

For each $d \in \mathbb{N}$ set

$$\begin{aligned} R_d &= \text{all homogeneous forms of degree } d \\ (I_\Delta)_d &= (I_\Delta) \cap R_d = \text{all homogeneous forms of degree } d \text{ in } I_\Delta. \end{aligned}$$

Both R_d and $(I_\Delta)_d$ are finite dimensional vector spaces over the field k. There is a formula for calculating the dimension of R_d.

Lemma 2. *If* $R = k[x_1, \ldots, x_n]$, *then* $\dim_k R_d = \binom{n+d-1}{d}$.

Definition 3. The Hilbert function of R/I_Δ is the function $H_{R/I_\Delta} : \mathbb{N} \to \mathbb{N}$ given by

$$H_{R/I_\Delta}(d) = \dim_k (R/I_\Delta)_d = \dim_k R_d - \dim_k (I_\Delta)_d = \binom{n+d-1}{d} - \dim_k (I_\Delta)_d.$$

Definition 4. The Hilbert series of R/I_Δ is the generating function for the sequence $\{H_{R/I_\Delta}(i)\}_{i \in \mathbb{N}}$, i.e.,

$$HS(R/I_\Delta, t) = \sum_{i=0}^{\infty} (H_{R/I_\Delta}(i)) t^i = \sum_{i=0}^{\infty} \dim_k (R/I_\Delta)_i t^i.$$

Example 5. The Hilbert series of $R = k[x_1, \ldots, x_n]$ is given by

$$HS(R,t) = 1 + \binom{n}{1}t + \binom{n+1}{2}t^2 + \binom{n+2}{3}t^3 + \cdots = \frac{1}{(1-t)^n}.$$

Theorem 6. *If $d = \dim R/I_\Delta$ (as a ring), then there exists some polynomial $h(t) \in \mathbb{Z}[t]$, with $\deg h(t) \leq d$, and $h(1) \neq 0$ such that*

$$HS(R/I_\Delta, t) = \frac{h(t)}{(1-t)^d}.$$

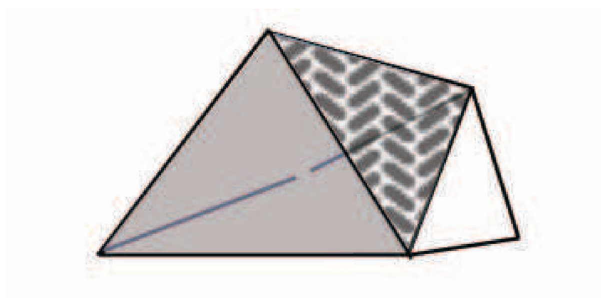
Since $h(t) = h_0 + h_1 t + h_2 t^2 + \cdots + h_d t^d$, we call

$$h(\Delta) = (h_0, h_1, \ldots, h_d)$$

the h-vector of Δ.

We saw there exists a relation between before $h(\Delta)$ and $f(\Delta)$, the f-vector of Δ. Here is a quick way to pass from $f(\Delta)$ to $h(\Delta)$ (as illustrated by example):

Consider the simplicial complex $\Delta =$



with the f-vector $f(\Delta) = (5, 8, 2)$. We can then construct a triangle similar to a Pascal Triangle. The difference is that we put the entries of $f(\Delta)$ down one side of the triangle, and we subtract entries instead of add them as in the usual Pascal triangle. For our example we have

		1	5
	1	4	8
1	3	4	2
1	2	1	−2

Consider a regular element F on R/I. We compare the Hilbert function and series of R/I and $R/(I, F)$.

Theorem 7. *Suppose F is a homogeneous element of degree l of R and F is regular on R/I, where I is a homogeneous ideal. Then*

(1) $H_{R/(I,F)}(i) = H_{R/I}(i) - H_{R/I}(i-l)$ *for all* $i \geq 0$, *where* $H_{R/I}(j) = 0$ *if* $j < 0$.
(2) *If* $\dim R/I = d$ *and* $HS(R/I,t) = \frac{h(t)}{(1-t)^d}$, *then*

$$HS(R/(I,F),t) = \frac{h(t)(1-t^l)}{(1-t)^d}.$$

Proof. (i) We need to show that for all $i \in \mathbb{N}$

$$\dim_k (R/(I,F))_i = \dim_k (R/I)_i - \dim_k (R/I)_{i-l}.$$

Define a map $\varphi : (R/I)_{i-l} \to (R/I)_i$ by

$$\overline{G} \mapsto \overline{GF}.$$

The map φ is injective since $\ker \varphi = \overline{0}$. To see this, note that if $\varphi(\overline{G}) = \overline{GF} = \overline{0}$, then since F is a regular element. $\overline{G} = \overline{0}$. Thus $(R/I)_{i-l} \cong \operatorname{Im} \varphi$ as vector spaces.

We also have a surjective linear transformation $\Phi : (R/I)_i \to (R/(I,F))_i$ given by

$$\overline{G} \mapsto \overline{G}.$$

Since Φ is surjective

$$\dim_k (R/(I,F))_i = \dim_k (R/I)_i - \dim_k \ker \Phi.$$

Claim: $\ker \Phi = \operatorname{Im} \varphi$

The element $\overline{H} \in \operatorname{Im} \varphi$ implies $\overline{H} = \overline{H_1 F}$ for some $\overline{H_1} \in (R/I)_{i-l}$. Then $\Phi(\overline{H}) = \Phi(\overline{H_1 F}) \in R/(I,F)$. But $H_1 F \in (I,F)$, so $\overline{H_1 F} = \overline{0}$ in $R/(I,F)$. Thus $\operatorname{Im} \varphi \subseteq \ker \Phi$.

An element $\overline{H} \in \ker \Phi$ implies $H \in (I,F)$. So $H = H_1 + H_2 F$ with $H_1 \in I$ and $H_2 \in R_{i-l}$. So $\overline{H} = \overline{H_2 F} \in R/I$. But then $\overline{H} = \varphi(\overline{H_2})$. So $\overline{H} \in \operatorname{Im} \varphi$. Hence $\operatorname{Im} \varphi \supseteq \ker \Phi$, thus finishing the proof of the claim.

From the claim, we have $\dim_k \ker \Phi = \dim_k \operatorname{Im} \varphi = \dim_k (R/I)_{i-l}$. So

$$\dim_k (R/(I,F))_i = \dim_k (R/I)_i - \dim_k (R/I)_{i-l}.$$

(ii) We use (i) to prove (ii). We have

$$\begin{aligned} HS(R/(I,F),t) &= \sum_{i=0}^{\infty} H_{R/(I,F)}(i)t^i \\ &= \sum_{i=0}^{\infty}[H_{R/I}(i) - H_{R/I}(i-l)]t^i \\ &= \sum_{i=0}^{\infty} H_{R/I}(i)t^i - \sum_{i=0}^{\infty} H_{R/I}(i-l)t^i \\ &= \sum_{i=0}^{\infty} H_{R/I}(i)t^i - t^l \sum_{i=0}^{\infty} H_{R/I}(i)t^i \\ &= \frac{h(t)}{(1-t)^d} - t^l \frac{h(t)}{(1-t)^d} = \frac{h(t)(1-t^l)}{(1-t)^d}. \end{aligned}$$

□

Theorem 8. *If R/I is Cohen-Macaulay with $\dim R/I = d$, then there exists a regular sequence $F_1, \cdots, F_d$ on R/I such that F_i is homogeneous of degree 1 for each i.*

Theorem 9. *Suppose Δ is a Cohen-Macaulay simplicial complex of dimension $d-1$. If $h(\Delta) = (h_0, \ldots, h_d)$ is the h-vector of R/I_Δ then*

$$0 \le h_i \le \binom{i+n-d-1}{i} \text{ for } 0 \le i \le d$$

Proof. Since $\dim \Delta = d-1$, $\dim R/I_\Delta = d$. Since R/I_Δ is Cohen-Macaulay there exists a regular sequence $F_1, \ldots, F_d \in (x_1, \cdots, x_n)$ with $\deg F_1 = \cdots = \deg F_d = 1$. By repeated applying Theorem 7

$$HS(R/(I, F_1, \ldots, F_d), t) = \frac{h(t)(1-t)\cdots(1-t)}{(1-t)^d} = \frac{h(t)(1-t)^d}{(1-t)^d} = h(t).$$

So $H_{R/(I,F_1,\ldots,F_d)}(i) = h_i$ for $0 \le i \le d$ and 0 otherwise.

Note that

$$\frac{R}{(I, F_1, \ldots, F_d)} \cong \frac{R/(F_1, \ldots, F_d)}{(I, F_1, \ldots, F_d)/(F_1, \ldots, F_d).}$$

So

$$0 \le h_i = \dim_k \left[\frac{R}{(I, F_1, \ldots, F_d)}\right]_i \le \dim_k \left[\frac{R}{(F_1, \ldots, F_d)}\right]_i$$

But $F_1, \ldots, F_d$ is also a regular sequence on R. Since $HS(R,t) = \frac{1}{(1-t)^n}$, by Theorem 7 we get

$$HS(R/(F_1, \ldots, F_d), t) = \frac{(1-t)^d}{(1-t)^n} = \frac{1}{(1-t)^{n-d}}.$$

So $\dim_k(R/(F_1,\ldots,F_d))_i$ equals the coefficient of t^i in the expansion of $\frac{1}{(1-t)^{n-d}}$. Hence

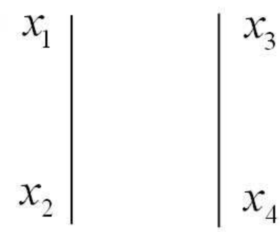
$$h_i \leq \dim_k(R/(F_1,\ldots,F_d))_i = \binom{n-d+i-1}{i} \text{ for } i = 0,\ldots,d.$$

□

Example 10. If

$$I = (x_1,x_2)\cap(x_3,x_4) = (x_1x_3, x_2x_4)$$

then $I = I_\Delta$ is the Stanley-Reisner ideal of $\Delta =$



Then R/I_Δ is not Cohen-Macaulay because $f(\Delta) = (4,2)$ implies

$$\begin{array}{ccccccc} & & 1 & & 4 & & \\ & 1 & & 3 & & 2 & \\ \hline 1 & & 2 & & -1 & & \end{array}$$

the h-vector has a negative entry! If Δ was Cohen-Macaulay, $h(\Delta)$ would only have positive entries.

Problems from Lecture 5

1. Let F be a homogeneous polynomial of degree d in the polynomial ring $R = k[x_1,\ldots,x_n]$.

   (i) If $I = (F)$, show that the Hilbert function of R/I is given by
$$H_{R/I}(i) = \binom{n+i-1}{i} - \binom{n-d+i-1}{i}.$$
   (ii) Find the Hilbert series of R/I.

2. Let $R = k[x_1, \ldots, x_n]$. If $d \in \mathbb{N}$, we let $R(-d)$ denote the R-module obtained by shifting the grading of R. More precisely, $R(-d)_i = R_{i-d}$. So, for example, if $d = 5$, then x_1^2 has degree 7 in $R(-5)$ since $x_1^2 \in R(-5)_7 = R_{7-5}$.

   Show that the Hilbert series of $R(-d)$ is given by

$$HS(R(-d), t) = \frac{t^d}{(1-t)^n}.$$

3. Let $R = k[x_1, x_2, x_3]$ be a polynomial ring, and let $I = (x_1^2, x_2^2 x_3, x_2^3)$. Prove that $H_{R/I}(i) = 4$ for $i \geq 4$.

4. Suppose that F, G is a regular sequence on $R = k[x_1, x_2]$ and $\deg F = a \leq \deg G = b$. What is the Hilbert series of R/I when $I = (F, G)$?