Lecture II: Kruskal-Katona Theorem (Jan. 17, 2006)

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1. F-VECTOR.

Recall that if Δ is a simplicial complex on V, and if $F \in \Delta$, then F is a face. Also, $\dim F = |F| - 1$, and $\dim \Delta = \max_{F \in \Delta}\{\dim F\}$.

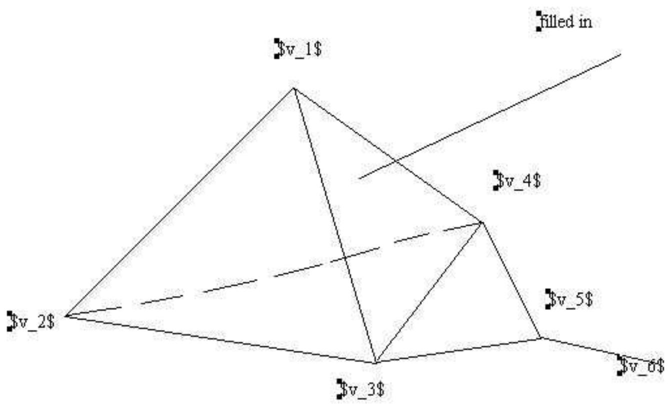
Definition 1.1. If Δ is a simplicial complex, then set

$$f_i = f_i(\Delta) = \text{number of faces of dimension } i.$$

Remark 1.2. $f_{-1} = 1$ since $\emptyset \in \Delta$ and $\dim \emptyset = -1$. Also $f_0 = |V|$, the number of vertices.

Definition 1.3. If $\dim \Delta = d$, then the f-vector of Δ is the $d-$tuple $f(\Delta) = (f_0, f_1, \cdots, f_{d-1})$.

Example 1.4. Consider the simplicial complex Δ:



The f-vector is then $f(\Delta) = (6, 9, 4, 1)$ since Δ has 6 vertices, 9 edges, 4 triangles (from the tetrahedron) and 1 tetrahedron.

Question: What can be the f-vector of a simplicial complex? That is, if $(f_0, f_1, \ldots, f_{d-1})$ is a sequence of numbers, is there a simplicial complex Δ such that $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$?

2. MACAULAY REPRESENTATIONS.

Recall that the binomial coefficient $\binom{a}{b} = \frac{a!}{b!(a-b)!}$. We make the convention that $\binom{a}{b} = 0$ if $a < b$.

Lemma 2.1. *Let k be a positive integer. Then each $a \in \mathbb{N}$ can be written uniquely in the form*

$$a = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s} \text{ with } a_k > a_{k-1} > \cdots > a_s \geq 1.$$

Definition 2.2. The sum above is called the k^{th} Macaulay representation of a.

Example 2.3. (How to find a representation) Consider Pascal's "Rectangle" (Pascal's triangle but written as a rectangle):

	0	1	2	3	4	......	i
0	1	1	1	1	1		
1	1	2	3	4	5		
2	1	3	6	10	15		
3	1	4	10	20	35		
4	1	5	15	35	70		
⋮							
j							$\binom{i+j}{i}$

We use this rectangle to find the 4^{th}-Macaulay representation of $a = 23$:
In column 4, largest number ≤ 23 is $15 = \binom{2+4}{4} = \binom{6}{4}$.
In column 3, largest number $\leq 23 - 15$ is $4 = \binom{3+1}{3} = \binom{4}{3}$.
In column 2, largest number $\leq 23 - 15 - 4$ is $3 = \binom{2+1}{2} = \binom{3}{2}$.
In column 1, largest number $\leq 23 - 15 - 4 - 3$ is $1 = \binom{1}{1}$.
So the 4^{th}-Macaulay representation of $a = 23$ is:

$$23 = \binom{6}{4} + \binom{4}{3} + \binom{3}{2} + \binom{1}{1}.$$

Definition 2.4. Let $a = \binom{a_k}{k} + \cdots + \binom{a_s}{s}$ be the k^{th}-Macaulay representation of a. Define a function $^{(k)} : \mathbb{N} \to \mathbb{N}$ by

$$a \mapsto a^{(k)} = \binom{a_k}{k+1} + \binom{a_{k-1}}{k} + \cdots + \binom{a_s}{s+1}.$$

Example 2.5. $23^{(4)} = \binom{6}{5} + \binom{4}{4} + \binom{3}{3} + \binom{1}{2} = 6 + 1 + 1 + 0 = 8$.

Lemma 2.6. *If $a \leq b$, then $a^{(k)} \leq b^{(k)}$.*

3. Kruskal-Katona Theorem.

Kruskal and Katona independently gave a characterization of the f-vectors of simplicial complexes.

Theorem 3.1 (Kruskal-Katona Theorem). *The d-tuple $(f_0, f_1, \cdots, f_{d-1})$ is the f-vector of a $(d-1)$-dimensional simplicial complex if and only if $0 \leq f_{i+1} \leq f_i^{(i+1)}$ for $0 \leq i \leq d-2$.*

Example 3.2. We saw $(6, 9, 4, 1)$ was a f-vector. So, $f_1 \leq f_0^{(0)}$ should be true. Indeed, since $6 = \binom{6}{1}, 6^{(1)} = \binom{6}{2} = 15$, thus $9 \leq 6^{(1)} = 15$. Also, $4 \leq 9^{(2)}$, since $9 = \binom{4}{2} + \binom{3}{1}$ which implies $9^{(2)} = \binom{4}{3} + \binom{3}{2} = 7$.

We will sketch out the main ideas of the proof. Our presentation is based upon P. Frankl's proof (his result is slightly more general).

($\Leftarrow$) We will show, via an example, how given a valid f-vector one constructs Δ such that $f(\Delta) = f$.

Definition 3.3. Suppose $(a_1, \ldots, a_r), (b_1, \ldots, b_r) \in \mathbb{N}^r$. Then $(a_1, \ldots, a_r) >_{rlex} (b_1, \ldots, b_r)$ if right most nonzero entry of $(a_1 - b_1, \ldots, a_r - b_r)$ is negative. This is called the reverse lexicographical order, and is a total ordering on $\mathbb{N}^r$.

Example 3.4. $(3, 2, 1) >_{rlex} (4, 9, 2)$ since $(3 - 4, 2 - 9, 1 - 2) = (-1, -7, -1)$.

The vector $f = (6, 8, 3)$ is a valid f-vector (check the details). Let $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ since $f_0 = 6$, implies we need 6 vertices.

Now write out all two element subsets of V in reverse lexicographical order. i.e.

$$v_{i_1} v_{j_1} > v_{i_2} v_{j_2} \Leftrightarrow (i_1, j_1) >_{rlex} (i_2, j_2).$$

These elements are:

$$v_1v_2, v_1v_3, v_1v_4, v_2v_4, v_3v_4, v_1v_5, v_2v_5, v_3v_5, v_4v_5, v_1v_6, v_2v_6, v_3v_6, v_4v_6, v_5v_6.$$

Let Δ_1 =first $f_1 = 8$ elements of the above set, that is

$$\Delta_1 = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_1, v_4\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_1, v_5\}, \{v_2, v_5\}\}.$$

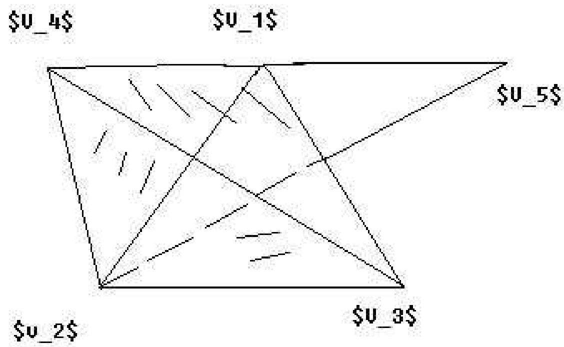
Now write out all three element subsets of V in reverse lexicographical order. These elements are

$$v_1v_2v_3, v_1v_2v_4, v_1v_3v_4, v_2v_3v_4, v_1v_2v_5, v_1v_3v_5, v_2v_3v_5, v_1v_4v_5, v_2v_4v_5, v_3v_4v_5, v_1v_2v_6, v_1v_3v_6,$$
$$v_2v_3v_6, v_1v_4v_6, v_2v_4v_6, v_3v_4v_6, v_1v_5v_6, v_2v_5v_6, v_3v_5v_6, v_4v_5v_6.$$

Let Δ_2 =first $f_2 = 3$ elements of the above set, that is

$$\Delta_2 = \{\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}\}.$$

Set $\Delta = V \cup \Delta_1 \cup \Delta_2$. It can now be checked that Δ is indeed a simplicial complex, and that Δ has f-vector $f(\Delta) = (6, 8, 3)$. A picture of this simplicial complex is given below:



($\Rightarrow$) Let Δ be a $(d-1)$ dimensional simplicial complex and let

$$f_{i+1} = \binom{a_{i+2}}{i+2} + \cdots + \binom{a_s}{s}.$$

be the $(i+2)^{th}$-Macaulay expansion of f_{i+1}. Our goal is to prove for $i = 0, \cdots, d-2$

$$\binom{a_{i+2}}{i+1} + \cdots + \binom{a_s}{s-1} \leq f_i. \quad (3.1)$$

If we can prove (3.1), then when then apply $^{(i+1)}$ to both sides of the above equation we get

$$\left[\binom{a_{i+2}}{i+1} + \cdots + \binom{a_s}{s-1}\right]^{(i+1)} = f_{i+1} \leq f_i^{(i+1)}.$$

The first step at the proof is to "shift" the simplicial complex Δ, i.e. we replace Δ with Δ' such that

$$f_i(\Delta') \leq f_i(\Delta)$$

where Δ' is in some sense more simple. The shift we need is given in the exercises.

Let $\Delta_1 = \{F \backslash \{v_1\} \mid v_1 \in F \in \Delta, \text{ and } \dim F = i+1\}$. Then

$$f_i(\Delta) \leq f_i(\Delta_1) + f_{i-1}(\Delta_1).$$

To see this, if $G \in \Delta_1$ and $\dim G = i$, then $G \subseteq F \setminus \{v_1\} \subseteq F \in \Delta$. So $G \in \Delta$ and $v_1 \notin G$. If $G \in \Delta_1$ and $\dim G = i-1$, then $G \subseteq F \setminus \{v_1\}$ for some $F \in \Delta$. But then $G \cup \{v_1\} \subseteq F \in \Delta$. So $G \cup \{v_1\} \in \Delta$, and $G \cup \{v_1\} \notin \Delta_1$, and thus it is not counted by $f_i(\Delta_1)$.

One proceeds by doing induction on f_{i+1} and i. First, one shows that $f_i(\Delta) \geq \binom{a_{i+2}-1}{i+1} + \cdots + \binom{a_s-1}{s-1}$. Then, by induction

$$
\begin{aligned}
f_i(\Delta) &\geq f_i(\Delta_1) + f_{i-1}(\Delta_1) \\
&\geq \left[\binom{a_{i+2}-1}{i+1} + \cdots + \binom{a_s-1}{s-1}\right] + \left[\binom{a_{i+2}-1}{i} + \cdots + \binom{a_s-1}{s-2}\right] \\
&\geq \binom{a_{i+2}-1}{i+1} + \cdots + \binom{a_s}{s-1}.
\end{aligned}
$$

Problems from Lecture 2

1. Let Δ be a simplex, i.e., $\Delta = \langle F \rangle$ has a single facet. If $\dim \Delta = d - 1$, determine the f-vector of Δ.

2. Let Δ be a simplicial complex on $\{x_1, \ldots, x_n\}$. For each $1 \leq j \leq n$, and each $F \in \Delta$, define

$$
S_j(F) = \begin{cases} F \backslash \{x_j\} \cup \{x_1\} & \text{if } x_j \in F, x_1 \notin F, \text{ and } F\backslash\{x_j\} \cup \{x_1\} \notin \Delta \\ F & \text{otherwise} \end{cases}
$$

Set $S_j(\Delta) = \{S_j(F) \mid F \in \Delta\}$.

(a) Show $S_j(\Delta)$ is a simplicial complex for each j.
(b) Show $f_i(S_j(\Delta)) \leq f_i(\Delta)$

(If you read the paper by P. Frankl on the Kruskal-Katona Theorem, this is the method by which he simplifies the simplicial complex.)