Lecture I: Simplicial Complexes (Jan. 10, 2006)
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1. Introduction

The goal of these lectures is to introduce *combinatorial commutative algebra.* The foundations of this field were created by Mel Hochster and Richard Stanley. This area continues to be an active area of research.

Basic Idea: Combinatorial objects (simplicial complexes) are associated to algebraic objects (Stanley-Reisner ideals, facet ideals). The combinational objects are studied through their algebraic properties.

2. Definition of a Simplicial Complex.

Today's goal is to introduce the combinatorial objects we wish to study.

Recall: If S is a set, the power set of S, denoted $P(S)$, is the set of all subsets of S.

Example 2.1. If $S = \{a, b, c\}$, then $P(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ where $\emptyset$ = empty set.

Definition 2.2. Let $V = \{v_1, \cdots, v_n\}$ be a finite set. An (abstract) simplicial complex Δ on V is a subset of $P(V)$ such that

- $\{v_i\} \in \Delta$ for each i;
- if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.

Note: $\Delta \subseteq P(V)$

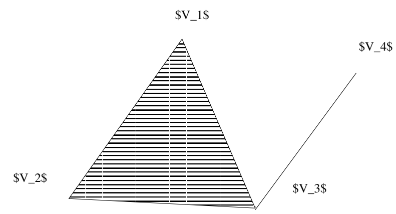
Example 2.3. If $V = \{v_1, v_2, v_3, v_4\}$ then

$$\Delta = \{\{v_1, v_2, v_3\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_3, v_4\}, \{v_4\}\}$$

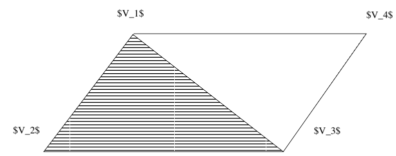
is a simplicial complex. On the other hand

$$\Delta' = \{\{v_1, v_2, v_3\}, \{v_2, v_4\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}\}$$

is not a simplicial complex since $\{v_1, v_2\} \subseteq \{v_1, v_2, v_3\}$, but $\{v_1, v_2\} \notin \Delta'$.
We can "draw" Δ:



Example 2.4. Consider the simplicial complex $\Delta =$



This is the simplicial complex

$$\Delta = \{\{v_1, v_2, v_3\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_1, v_4\}, \{v_3, v_4\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}\}$$

Remark 2.5. If Δ is a nonempty simplicial complex, then $\emptyset \in \Delta$ (when writing elements of Δ, we usually omit $\emptyset$).

Definition 2.6. The elements of Δ are called *faces*. The maximal faces under inclusion are called *facets*.

We use the notation $|S|$ to denote the number of elements in a set S.

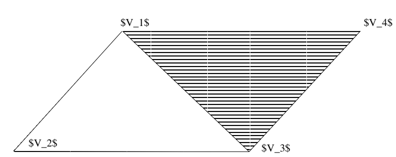
Definition 2.7. If $F \in \Delta$ is a face, then the dimension of F is

$$\dim F = |F| - 1.$$

The dimension of Δ is

$$\dim \Delta = \max\{\dim F \mid F \in \Delta\}.$$

Example 2.8. Suppose $\Delta =$



then $\dim \Delta = 2$ since $F = \{v_1, v_3, v_4\}$ has $\dim 2$, and all other faces have dimension ≤ 1

Definition 2.9. Let $F \in \Delta$ be a face.

- If $\dim F = 0$, i.e. $|F| = 1$, then F is a vertex.
- If $\dim F = 1$, i.e. $|F| = 2$, then F is an edge.
- If $F = \emptyset$ then $\dim F = -1$.

If $\{F_1, \cdots, F_s\}$ is a collection of subsets of V, then there is a unique smallest simplicial complex, denoted $< F_1, \cdots, F_s >$, which contains all F_i. We say $< F_1, \cdots, F_s >$ is generated by the F_i's. In set notation we have

$$< F_1, \cdots, F_s >= \{G \subseteq V \mid G \subseteq F_i \text{ for some } i\}.$$

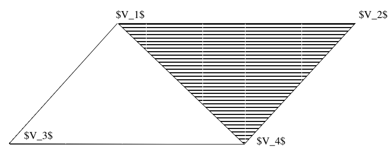
Note that if $F_1, \cdots, F_s$ are facets of a simplicial complex Δ, then

$$\Delta =< F_1, \cdots, F_s >$$

We call Δ a *simplex* if $\Delta =< F >$.

Definition 2.10. The simplicial complex Δ is *pure* if all its facets have the same dimension.

Example 2.11. The simplicial complex

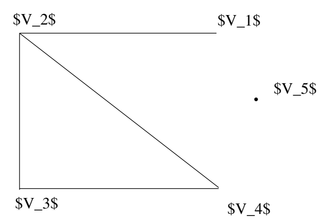


is not pure since $\{v_1, v_2, v_4\}$ and $\{v_1, v_3\}$ are two facets of different dimension.

3. Examples

3.1. **Graphs.** Let G be a finite simple graph. i.e., G has no loops or multiple edges. Then G is a 1-dimensional simplicial complex on vertex set $V = V_G = \{v_1, \cdots, v_n\}$, the vertex set of G.

For example, the graph



is the 1-dimensional simplicial complex

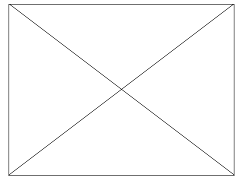
$$\Delta = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}\}.$$

Note that any 1-dimensional simplicial complex can be represented as a graph.

3.2. **Clique Complexes.** We now present another way to associate to a graph a simplicial complex.

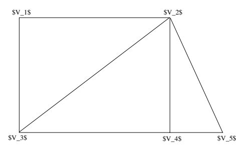
Definition 3.1. A clique of size n, denoted K_n, is the graph on n vectors with an edge between every pair of vertex.

Example 3.2. The graph for the clique K_4 is given below:

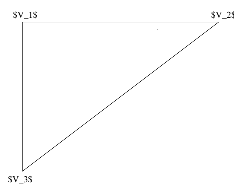


Definition 3.3. If $F \subseteq V_G$ the induced graph of G on F, denoted G_F, is the subgraph consisting of vertex of F and all edges of G between vertex of F.

Example 3.4. If G is the graph



then the induced graph $G_{\{v_1, v_2, v_3\}}$ is



Definition 3.5. If G is a finite simple graph, the clique complex of G is the simplicial complex

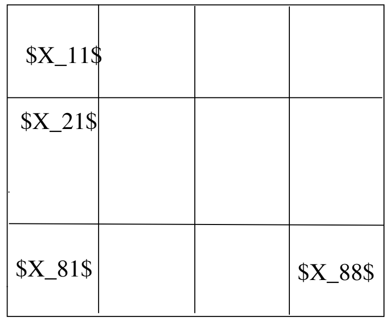
$$\Delta(G) = \{F \subseteq V_G \mid G_F \text{ is a clique}\}$$

Note that $\Delta(G)$ is a simplicial complex because:

- $x_i \in \Delta(G)$ since for all i, $G_{\{x_i\}} = K_1$.
- if $F \in \Delta(G)$ and $H \subseteq F$, then G_H is an induced subgraph of the clique G_F. So G_H is also a clique, so $H \in \Delta(G)$.

3.3. **Chess Problem.** A famous chess question asks how many queens can be placed on a chessboard so that no two can kill each other. We can build a simplicial complex to answer this question.

We begin by labeling the entries of the chessboard with x_{ij} for $1 \leq i \leq 8$ and $1 \leq j \leq 8$.



Definition 3.6. Let $V = \{x_{ij} \mid 1 \leq i \leq 8, 1 \leq j \leq 8\}$. We call $F \subseteq V$ a *safe queen configuration*, if one can place a queen at each $x_{ij} \in F$, so that no two queens can kill each other.

Example 3.7. The set $F = \{x_{11}, x_{23}\}$ is a safe-queen-configuration.

Now set $\Delta = \{F \subseteq V \mid F \text{ is a safe queen configuration}\}$.

Claim 3.8. *Δ as defined above is a simplicial complex.*

Proof. We verify that Δ satisfies both conditions of the definition.

- for each $x_{ij} \in V, \{x_{ij}\}$ is clearly a safe queen configuration.
- if $F \in \Delta$ is a safe-queen-configuration, then any subset of F will also be a safe-queen-configuration, and thus, this subset will also be in Δ.

□

Theorem 3.9. *The maximal number of queens that can be placed on the chessboard with no two killing each other is* $\dim \Delta + 1$.

Proof. Recall that $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$. Let $F \in \Delta$ be the facet with $\dim \Delta = \dim F$. Then

$$\begin{aligned} \dim \Delta + 1 &= \dim F + 1 = |F| + 1 - 1 = |F| \\ &= \text{maximal number of queens in a safe queen configuration.} \end{aligned}$$

□

Problems from Lecture 1

1. In class we looked at the chess problem of putting queens on an 8×8 chessboard so that no two can kill each other. We saw that we could associate to this problem a simplicial complex Δ such that $\dim \Delta + 1$ is the maximal number of queens that can be placed on chessboard. Repeat this problem for a 4×4 chessboard. Write out all the elements of the corresponding simplicial complex.

2. Let Δ and Γ be simplicial complexes on disjoint vertex sets V and W. The **join** $\Delta * \Gamma$ is the set on the vertex set $V \cup W$ with elements $F \cup G$ where $F \in \Delta$ and $G \in \Gamma$. Show that $\Delta * \Gamma$ is a simplicial complex.

3. Let Δ be a $(d-1)$-dimensional simplicial complex. For $0 \leq r \leq d-1$, define the r-**skeleton** of Δ to be the set

$$\Delta_r = \{F \in \Delta \mid \dim F \leq r\}.$$

   (a) Show Δ_r is also a simplicial complex for each r.
   (b) If $\Delta = \langle \{x_1, x_2, x_3, x_4\}, \{x_3, x_4, x_5\} \rangle$, find Δ_r for each $0 \leq r \leq \dim \Delta$.