Lecture 33

Applications of Adomian Decomposition Method

In this lecture the singular initial value problems, linear and nonlinear, homogeneous and nonhomogeneous, generalized Emden-Fowler equation and Bratu-type equations are investigated by using Adomian decomposition method. The solutions are constructed in the form of a convergent series.

ADOMIAN METHOD FOR SINGULAR INITIAL VALUE PROBLEMS IN SECOND-ORDER ODES

The studies of singular initial value problems in the second order ordinary differential equations (ODEs) have attracted the attention of many mathematicians and physicists. One of the equations describing this type is the Lane–Emden-type equations formulated as

$$\begin{aligned}&y'' + \frac{2}{x}y' + f(y) = 0, \ \ 0 < x \le 1,\\ &y(0) = A, \ y'(0) = B.\end{aligned} \qquad (1)$$

On the other hand, studies have been carried out on another class of singular initial value problems of the form

$$\begin{aligned}&y'' + \frac{2}{x}y' + f(x,y) = g(x), \qquad 0 < x \le 1,\\ &y(0) = A, \qquad y'(0) = B,\end{aligned} \qquad (2)$$

where A and B are constants, $f(x, y)$ is a continuous real valued function, and $g(x) \in C[0,1]$. Eq.(2) differs from the classical Lane–Emden-type equations (1) for the function $f(x, y)$ and for the inhomogeneous term $g(x)$.

Eq.(1) with specializing $f(y)$ was used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres, and theory of thermionic currents. Due to the significant applications of Lane–Emden-type equations in the scientific community, various forms of $f(y)$ have been investigated in many research works.

In recent years, a large amount of literature is developed concerning Adomian decomposition method, and the related modification to investigate various scientific models. The Adomian decomposition method provides the solution in a rapidly convergent series with components that are elegantly computed. A reliable part of this approach is how this method can be modified to address the concept of singular points. To properly address this question, we may require slight variation of the decomposition algorithm as described in previous lecture. An alternate framework can be designed to overcome the difficulty of the singular point at x=0.

The Adomian decomposition method usually defines the equation in an operator form by considering the highest-ordered derivative in the problem. To overcome the singularity behavior, we define the differential operator L in terms of the two derivatives contained in the problem. We rewrite (2) in the form

$$Ly = -f(x, y) + g(x), \tag{3}$$

where the differential operator L in terms of two derivatives, $y'' + \frac{2}{x}y'$, is defined by

$$L = x^{-2}\frac{d}{dx}\left(x^2\frac{d}{dx}\right). \tag{4}$$

The inverse operator L^{-1} is therefore considered a two-fold integral operator defined by

$$L^{-1}(.) = \int_0^x x^{-2}\int_0^x x^2(.)dxdx. \tag{5}$$

Operating with L^{-1} on (3), it follows

$$y(x) = A + Bx + L^{-1}g(x) - L^{-1}f(x, y). \tag{6}$$

As the Adomian decomposition method introduces the solution $y(x)$ by an infinite series of components

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{7}$$

and the nonlinear function $f(x, y)$ by an infinite series of polynomials

$$f(x, y) = \sum_{n=0}^{\infty} A_n, \tag{8}$$

where the components $y_n(x)$ of solution $y(x)$ will be determined recurrently, and A_n are Adomian polynomials constructed for non-linear function defined as $A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left((x, y)\lambda\right)\right]_{\lambda=0}$.

Substituting (7) and (8) into (6) gives

$$\sum_{n=0}^{\infty} y_n(x) = A + B(x) + L^{-1}g(x) - L^{-1}\sum_{n=0}^{\infty} A_n. \tag{9}$$

To determine the components $y_n(x)$, we use Adomian decomposition method that suggests the use of the recursive relation

$$\begin{aligned} &y_0(x) = A + B(x) + L^{-1}g(x), \\ &y_{k+1}(x) = -L^{-1}(A_k), \quad k \geq 0, \end{aligned} \tag{10}$$

which gives

$$\begin{aligned} &y_0(x) = A + Bx + L^{-1}g(x), \\ &y_1(x) = -L^{-1}(A_0), \\ &y_2(x) = -L^{-1}(A_1), \\ &y_3(x) = -L^{-1}(A_2), \\ &\vdots \end{aligned} \tag{11}$$

The series solution of $y(x)$ defined by (7) follows immediately.

The main advantage of using this choice for the operator L is that it tackles the differential equation directly without any need for a transformation formula.

Example 1:

Solve the following linear singular initial value problem by using Adomian decomposition method.

$$\begin{aligned} &y'' + \frac{2}{x}y' + y = 6 + 12x + x^2 + x^3, \\ &y(0) = 0, \qquad y'(0) = 0. \end{aligned} \tag{12}$$

Solution

Firstly, we have to re-write the given DE in an operator form as

$$Ly = 6 + 12x + x^2 + x^3 - y. \tag{13}$$

Applying L^{-1} to both sides of (13) and using the initial condition. we obtain

$$L^{-1}Ly = L^{-1}(6) + 12L^{-1}(x) + L^{-1}(x^2) + L^{-1}(x^3) - L^{-1}(y). \tag{14}$$

As,

$$L^{-1}L(y) = L^{-1}\left(y'' + \frac{2}{x}y'\right)$$
$$= \int_0^x x^{-2}\int_0^x x^2\left(y'' + \frac{2}{x}y'\right)dxdx$$
$$= \int_0^x x^{-2}\left(x^2y' - \int_0^x 2xy'dx + \int_0^x 2xy'dx\right)dx$$
$$= \int_0^x y'dx = y(x) - y(0) = y(x),$$

$$L^{-1}(6) = \int_0^x x^{-2}\int_0^x x^2\left(6\right)dxdx$$
$$= 6\int_0^x x^{-2}\left(\frac{x^3}{3}\right)dx = 2\int_0^x xdx = x^2,$$

$$12L^{-1}(x) = 12\int_0^x x^{-2}\int_0^x x^2\left(x\right)dxdx$$
$$= 12\int_0^x x^{-2}\left(\frac{x^4}{4}\right)dx = 3\int_0^x x^2dx = x^3,$$

$\approx L^{-1}(x^2) = \dfrac{x^4}{20}$ and $L^{-1}(x^3) = \dfrac{x^5}{30}$.

Putting all these values in Eq. (14),

$$y = x^2 + x^3 + \frac{1}{20}x^4 + \frac{1}{30}x^5 - L^{-1}y. \qquad (15)$$

Proceeding as before we obtain the recursive relationship

$$y_0(x) = x^2 + x^3 + \frac{1}{20}x^4 + \frac{1}{30}x^5$$
$$y_{k+1}(x) = -L^{-1}(y_k), \qquad k \geq 0. \qquad (16)$$

Consequently, the first few components are as

$$
\begin{aligned}
y_0 &= x^2 + x^3 + \frac{1}{20}x^4 + \frac{1}{30}x^5, \\
y_1 &= -L^{-1}(y_0) = -\frac{1}{20}x^4 - \frac{1}{30}x^5 - \frac{1}{840}x^6 - \frac{1}{1680}x^7, \\
y_2 &= -L^{-1}(y_1) = -\frac{1}{840}x^6 + \frac{1}{1680}x^7 + \frac{1}{60480}x^8 + \frac{1}{151200}x^9, \\
y_3 &= -L^{-1}(y_2) = -\frac{1}{60480}x^8 - \frac{1}{151200}x^9 - \ldots, \qquad (17)
\end{aligned}
$$

Other components can be evaluated in a similar manner. Substituting these values in Eq. (7) and after cancellation, we have

$$y(\mathrm{x}) = x^2 + x^3, \qquad (18)$$

which is the exact solution.

Example 2:

Solve the following nonlinear singular initial value problem by using Adomian decomposition method.

$$
\begin{aligned}
& y'' + \frac{2}{x}y' - 6y = 4y\ln y, \\
& y(0) = 1, \quad y'(0) = 0. \qquad (19)
\end{aligned}
$$

Solution

Re-write the given DE in an operator form as

$$Ly = 4y\ln y + 6y. \qquad (20)$$

Applying L^{-1} to both sides of (20) and using the initial condition we obtain

$$
\begin{aligned}
L^{-1}Ly &= 4L^{-1}\left(y\ln y\right) + 6L^{-1}\left(y\right), \\
y(x) &= 1 + 4L^{-1}\left(y\ln y\right) + 6L^{-1}\left(y\right). \qquad (21)
\end{aligned}
$$

Proceeding as previous example, we obtain the recursive relationship

$$
\begin{aligned}
& y_0(x) = 1, \\
& y_{k+1}(x) = 6L^{-1}(y_k) + 4L^{-1}(A_k), \qquad k \geq 0. \qquad (22)
\end{aligned}
$$

The Adomian polynomials for the nonlinear term $F(y) = y\ln y$ are computed as follows:

$$\begin{aligned}
A_0 &= y_0 \ln(y_0),\\
A_1 &= y_1F'(y_0) = y_1(1+\ln y_0),\\
A_2 &= y_2F'(y_0) + \frac{y_1^2}{2}F''(y_0) = y_2(1+\ln y_0) + \frac{y_1^2}{2y_0},\\
A_3 &= y_3F'(y_0) + y_1y_2F''(y_0) + \frac{y_1^3}{6}F'''(y_0) = y_3(1+\ln y_0) + \frac{y_1y_2}{y_0} - \frac{y_1^3}{6y_0^2},
\end{aligned} \tag{23}$$

which are obtained by using reference list of the Adomian polynomials given in lecture 32.

By putting (23) into (22), we get the following components

$$\begin{aligned}
y_0 &= 1,\\
y_1 &= 6L^{-1}(y_0) + 4L^{-1}(A_0) = x^2,\\
y_2 &= 6L^{-1}(y_1) + 4L^{-1}(A_1) = \frac{1}{2!}x^4,\\
y_3 &= 6L^{-1}(y_2) + 4L^{-1}(A_2) = \frac{1}{3!}x^6,\\
y_4 &= 6L^{-1}(y_3) + 4L^{-1}(A_3) = \frac{1}{4!}x^8,\\
y_5 &= 6L^{-1}(y_4) + 4L^{-1}(A_4) = \frac{1}{5!}x^{10},
\end{aligned} \tag{24}$$

and so on. In view of above equation, the solution in a series form is given by

$$y(x) = 1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \frac{1}{4!}x^8 + \frac{1}{5!}x^{10} + ..., \tag{25}$$

and in the closed form

$$y(x) = e^{x^2}. \tag{26}$$

GENERALIZATION:

Replace the standard coefficient of y' in (2) by n/x, for real n; $n\geq 0$. In other words, a general equation

$$y''+\frac{n}{x}y'+f(x,y)=g(x), \qquad n\geq 0, \qquad (27)$$

with initial conditions

$$y(0)=A, \qquad y'(0)=B, \qquad (28)$$

can be formulated.

Here, the differential operator is defined as

$$L_n = x^{-n}\frac{d}{dx}\left(x^n\frac{d}{dx}\right), \qquad (29)$$

for which the inverse operator L^{-1} is expressed by

$$L_n^{-1}(.)=\int_0^x x^{-n}\int_0^x x^n(.)dxdx. \qquad (30)$$

Applying L_n^{-1} to both sides of (27) yields

$$y(x)=A+Bx+L_n^{-1}g(x)-L_n^{-1}f(x,y). \qquad (31)$$

Proceeding as before we obtain

$$\begin{aligned} y_0(x) &= A+Bx+L_n^{-1}g(x),\\ y_{k+1}(x) &= -L_n^{-1}A_k, \qquad k\geq 0, \end{aligned} \qquad (32)$$

where A_k are Adomian polynomials that represent the nonlinear term $f(x, y)$. In view of (32), the components of the function $y(x)$ can be elegantly determined. The slight change we imposed on defining the operator L_n in (29), in terms of the first two derivatives, was successful to overcome the singularity issue for $n\geq 0$. To illustrate the generalization discussed above, we will discuss an example.

Example 3:

Solve the following nonlinear singular initial value problem by using Adomian decomposition method.

$$y''+\frac{6}{x}y'+14y=-4y\ln y,$$
$$y(0)=1, \quad y'(0)=0. \qquad (33)$$

Solution

In an operator form, given differential equation becomes

$$L_n(y)=-14y-4y\ln y. \qquad (34)$$

Recall that the operator L_n is defined by

$$L_n=x^{-6}\frac{d}{dx}\left(x^6\frac{d}{dx}\right), \qquad (35)$$

for which the inverse operator L_n^{-1} is expressed by

$$L_n^{-1}(.)=\int_0^x x^{-6}\int_0^x x^6(.)dxdx. \qquad (36)$$

Operating L_n^{-1} on both sides of (34), we have

$$y=1-14L_n^{-1}(y)-4L_n^{-1}(y\ln y). \qquad (37)$$

Proceeding as before we obtain the recursive relationship

$$y_0(x)=1,$$
$$y_{k+1}(x)=-14L_n^{-1}(y_k)-4L_n^{-1}(A_k), \qquad k\geq 0. \qquad (38)$$

The Adomian polynomials for the nonlinear term $F(y)=y\ln y$ are computed before in (23). Substituting (23) into (38) gives the components

$$
\begin{aligned}
y_0 &= 1, \\
y_1 &= -14L_n^{-1}(y_0) - 4L_n^{-1}(A_0) = -x^2, \\
y_2 &= -14L_n^{-1}(y_1) - 4L_n^{-1}(A_1) = \frac{1}{2!}x^4, \\
y_3 &= -14L_n^{-1}(y_2) - 4L_n^{-1}(A_2) = -\frac{1}{3!}x^6, \\
y_4 &= -14L_n^{-1}(y_3) - 4L_n^{-1}(A_3) = \frac{1}{4!}x^8, \\
y_5 &= -14L_n^{-1}(y_4) - 4L_n^{-1}(A_4) = -\frac{1}{5!}x^{10}, \qquad (39)
\end{aligned}
$$

so that other components can be evaluated in a similar manner. In view of (39), the solution in a series form is given by

$$y(x) = 1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \frac{1}{4!}x^8 - \frac{1}{5!}x^{10} + ..., \qquad (40)$$

and in closed form

$$y(x) = e^{-x^2}. \qquad (41)$$

ADOMIAN DECOMPOSITION METHOD FOR EMDEN–FOWLER EQUATION

Many problems in the literature of mathematical physics can be distinctively formulated as equations of Emden–Fowler type defined in the form

$$y'' + \frac{2}{x}y' + af(x)g(y) = 0, \qquad y(0) = y_0, \qquad y'(0) = 0, \qquad (42)$$

where $f(x)$ and $g(y)$ are some given functions of x and y respectively. For $f(x) = 1$ and $g(y) = y^n$, Eq. (42) becomes the standard Lane–Emden equation.

The standard coefficient of y' in Emden–Fowler equation is $2/x$. However, if we replace $2/x$ by r/x, for real r, $r \geq 0$, then we write down Emden–Fowler equation in general as

$$y'' + \frac{r}{x}y' + af(x)g(y) = 0, \qquad r \geq 0 \qquad (43)$$

with boundary conditions given by

$$y(0) = \alpha, \qquad y'(0) = 0. \qquad (44)$$

Introducing the differential operator

$$L = x^{-r} \frac{d}{dx}\left(x^r \frac{d}{dx} \right), \tag{45}$$

for which the inverse operator L^{-1} is expressed by

$$L^{-1}(.) = \int_0^x x^{-r} \int_0^x x^r (.)dxdx. \tag{46}$$

In an operator form, Eq. (43) may be rewritten as

$$Ly = -af(x)g(y). \tag{47}$$

Operating with L^{-1} on (47), we have

$$y = \alpha - aL^{-1}\left(f(x)g(y) \right). \tag{48}$$

The slight change we imposed in defining the operator L in (45), in terms of the first two derivatives, was successful to overcome the singularity issue for $r \neq 0$.

As discussed above, Adomian decomposition method introduces the decomposition series

$y(x) = \sum_{n=0}^{\infty} y_n(x)$ and the infinite series of polynomials

$$f(y) = \sum_{n=0}^{\infty} A_n(y_0, y_1, \ldots, y_n), \tag{49}$$

where the components $y_n(x)$ of the solution $y(x)$ will be determined recurrently, and A_n are Adomian polynomials. Substituting the value of $y(x)$ and (49) into (48), it gives

$$\sum_{n=0}^{\infty} y_n(x) = \alpha - aL^{-1}\left(f(x)\sum_{n=0}^{\infty} A_n(y_0, y_1, \ldots, y_n) \right). \tag{50}$$

Identifying $y_0(x)=\alpha$, the recursive relation

$$\begin{aligned} &y_0(x) = \alpha, \\ &y_{k+1}(x) = -aL^{-1}\left(f(x)A_k \right), \qquad k \geq 0 \end{aligned} \tag{51}$$

or equivalently

$$\begin{aligned} &y_0(x) = \alpha, \\ &y_{k+1}(x) = -a\int_0^x x^{-r} \int_0^x x^r \left(f(x)A_k \right) dxdx, \qquad k \geq 0 \end{aligned} \tag{52}$$

will lead to the complete determination of the components $y_n(x)$ of $y(x)$. The series solution of $y(x)$ follows immediately.

Example 4:

Solve the following equation by using Adomian decomposition method.

$$y'' + \frac{8}{x}y' + 18ay = -4y\ln y,$$

where the boundary conditions are given by

$$y(0) = 1, \qquad y'(0) = 0.$$

Solution

Using the recursive relation (52) yields

$$\begin{aligned} &y_0(x) = 1 \\ &y_{k+1}(x) = -18aL^{-1}(y_k) - 4L^{-1}(A_k), \qquad k \geq 0 \end{aligned} \tag{53}$$

The first few Adomian polynomials for $g(\mathrm{y}) = y\ln y$ are given by

$$\begin{aligned} A_0 &= y_0 \ln y_0, \\ A_1 &= y_1(1+\ln y_0), \\ A_2 &= y_2(1+\ln y_0) + \frac{y_1^2}{2y_0}. \end{aligned} \tag{54}$$

Using (53) yields

$$\begin{aligned} y_0 &= 1, \\ y_1 &= -18aL^{-1}(y_0) - 4L^{-1}(A_0) = -ax^2, \\ y_2 &= -18aL^{-1}(y_1) - 4L^{-1}(A_1) = \frac{a^2}{2}x^4, \\ y_3 &= -18aL^{-1}(y_2) - 4L^{-1}(A_2) = \frac{a^3}{6}x^6. \end{aligned} \tag{55}$$

Consequently, the series solution is

$$y(x) = 1 - ax^2 + \frac{a^2}{2!}x^4 - \frac{a^3}{3!}x^3 + \ldots \tag{56}$$

and in a closed form it becomes $y(x) = e^{-ax^2}$.

ADOMIAN DECOMPOSITION METHOD FOR BRATU-TYPE EQUATIONS

The standard Bratu's boundary value problem in one-dimensional planar coordinates is of the form

$$u'' + \lambda e^u = 0, \qquad 0 < x < 1,$$
$$u(0) = u(1) = 0.$$

The Bratu model appears in a number of applications such as the fuel ignition of the thermal combustion theory. It stimulates a thermal reaction process in a rigid material where the process depends on the balance between chemically generated heat and heat transfered by conduction.

Example 5:

Solve the following Bratu-type model equation by using Adomian decomposition method.

$$u'' - \pi^2 e^u = 0, \qquad 0 < x < 1, \tag{57}$$
$$u(0) = u(1) = 0.$$

Solution

The given problem can be written in an operator form as

$$Lu = \pi^2 e^u = 0, \qquad 0 < x < 1, \tag{58}$$
$$u(0) = u(1) = 0,$$

where L is the differential operator given by

$$L = \frac{\partial^2}{\partial x^2}.$$

The inverse L^{-1} is assumed to be a two-fold integral operator given by

$$L^{-1}(.) = \int_0^x \int_0^x (.)dxdx.$$

Applying the inverse operator L^{-1} on both sides of (58) and using the initial condition u (0) = 0, we find

$$u(x) = ax + L^{-1}(\pi^2 e^u), \tag{59}$$

where a=u'(0). Substituting (7) and (8) into the functional equation (59) gives

$$\sum_{n=0}^{\infty} u_n(x) = ax + L^{-1}\left(\pi^2 \sum_{n=0}^{\infty} A_n\right), \tag{60}$$

where A_n are the so-called Adomian polynomials. Identifying the zeroth component $u_0(x)$ by ax, the remaining components $u_n(x)$, $n \geq 1$ can be determined by using the recurrence relation

$$\begin{aligned} &u_0(x) = ax, \\ &u_{k+1}(x) = \pi^2 L^{-1}(A_k), \qquad k \geq 0 \end{aligned} \tag{61}$$

where A_k are Adomian polynomials that represent the nonlinear term e^u and given by

$$\begin{aligned} A_0 &= e^{u_o}, \\ A_1 &= u_1 e^{u_o}, \\ A_2 &= \left(u_2 + \frac{1}{2}u_1^2\right)e^{u_o}, \\ A_3 &= \left(u_3 + u_1u_2 + \frac{1}{6}u_1^3\right)e^{u_o}, \\ A_4 &= \left(u_4 + u_1u_3 + \frac{1}{2}u_2^2 + \frac{1}{2}u_1^2u_2 + \frac{1}{24}u_1^4\right)e^{u_o}. \\ &\cdots \end{aligned} \tag{62}$$

Other polynomials can be generated in a similar way to enhance the accuracy of approximation. Combining (61) and (62) yields

$$\begin{aligned} u_0(x) &= ax, \\ u_1(x) &= -\frac{\pi^2}{a^2}(-e^{ax} + ax + 1), \\ u_2(x) &= -\frac{\pi^4}{4a^4}(-e^{2ax} + 4axe^{ax} - 4e^{ax} + 2ax + 5), \\ u_3(x) &= \frac{\pi^6}{12a^6}(e^{3ax} + 6e^{2ax}(1-ax) + 3e^{ax}(2a^2x^2 - 6ax + 5) - 6ax - 22), \\ &\ldots \end{aligned} \tag{63}$$

In view of (63), the solution $u(x)$ is readily obtained in a series form by

$$\begin{aligned} u(x) = ax &- \frac{\pi^2}{a^2}(-e^{ax} + ax + 1) - \frac{\pi^4}{4a^4}(-e^{2ax} + 4axe^{ax} - 4e^{ax} + 2ax + 5) \\ &+ \frac{\pi^6}{12a^6}(e^{3ax} + 6e^{2ax}(1-ax) + 3e^{ax}(2a^2x^2 - 6ax + 5) - 6ax - 22) \\ &+ \ldots, \end{aligned}$$

or equivalently

$$
\begin{aligned}
u(x) &= ax + \frac{\pi^2}{2!}x^2 + \frac{\pi^2 a}{3!}x^3 + \left(\frac{\pi^2 a^2 + \pi^4}{4!}\right)x^4 + \left(\frac{\pi^2 a^3 + 4\pi^4 a}{5!}\right)x^5 \\
&\quad + \left(\frac{11\pi^4 a^2 + \pi^2 a^4 + 4\pi^6}{6!}\right)x^6 + \left(\frac{26\pi^4 a^3 + \pi^2 a^5 + 34\pi^6 a}{6!}\right)x^7 \\
&\quad + \ldots, \\
\Rightarrow u(x) &= -\ln\left(1 + \cos\left(\left(\frac{1}{2} + x\right)\pi\right)\right).
\end{aligned}
$$

Exercises

Solve the following problems by using Adomian decomposition method.

1. $$y'' + \frac{2}{x}y' = 2(2x^2 + 3)y, \qquad y(0) = 1, \quad y'(0) = 0.$$

2. $$y'' + \frac{5}{x}y' + 8a(e^y + 2e^{y/2}) = 0, \qquad y(0) = 0, \quad y'(0) = 0.$$

3. $$u'' + \pi^2 e^{-u} = 0, \qquad 0 < x < 1, \qquad u(0) = u(1) = 0.$$