Lecture 35

Adomian Method for Higher-Order Ordinary Differential Equations

It is possible to model many of the physical events that take place in nature using linear and nonlinear differential equations. This modelling enables us to understand and interpret the particular event in a much better manner. Thus, finding the analytical and approximate solutions of such models with initial and boundary conditions gain importance. Differential equations have had an important place in engineering since many years. Scientists and engineers generally examine systems that undergo changes.

Many methods have been developed to determine the analytical and approximate solutions of linear and nonlinear differential equations with initial and boundary value conditions and among these methods, the Adomian decomposition method (ADM), homotopy perturbation method, variational iteration method, and homotopy analysis method can be listed.

Recall that in previous lectures solving differential equations, solutions are usually obtained as exact solutions defined in closed form expressions, or as series solutions normally obtained from concrete problems.

To apply the Adomian decomposition method for solving nonlinear ordinary differential equations, we consider the equation

$$Ly + R(y) + F(y) = g(x), \tag{1}$$

where the differential operator L may be considered as the highest order derivative in the equation, R is the remainder of the differential operator, $F(y)$ expresses the nonlinear terms, and $g(x)$ is an inhomogeneous term. If L is a first order operator defined by

$$L = \frac{d}{dx},$$

then, we assume that L is invertible and the inverse operator L^{-1} is given by

$$L^{-1}(.) = \int_0^x (.)dx.$$

$$\Rightarrow L^{-1}(Ly) = y(x) - y(0).$$

However, if L is a second order differential operator given by

$$L(.) = \frac{d^2}{dx^2}(.),$$

and therefore the inverse operator L^{-1} is defined by

$$L^{-1}(.) = \int_0^x \int_0^x (.) dx dx$$

$$\Rightarrow L^{-1}L(y) = y(x) - y(0) - xy'(0)$$

In a parallel manner, if L is a third order differential operator, we can easily show that

$$L^{-1}L(y) = y(x) - y(0) - xy'(0) - \frac{1}{2!}x^2 y''(0)$$

For higher order operators we can easily define the related inverse operators in a similar way.

Applying L^{-1} to both sides of (1) gives

$$y(x) = \psi_0 + L^{-1}g(x) - L^{-1}R(y) - L^{-1}F(y), \qquad (2)$$

Where

$$\psi_0 = \begin{cases} y(0), & \text{for } L = \frac{d}{dx}, \\ y(0) + xy'(0), & \text{for } L = \frac{d^2}{dx^2}, \\ y(0) + xy'(0) + \frac{1}{2!}x^2 y''(0), & \text{for } L = \frac{d^3}{dx^3}, \\ y(0) + xy'(0) + \frac{1}{2!}x^2 y''(0) + \frac{1}{3!}x^3 y'''(0), & \text{for } L = \frac{d^4}{dx^4}, \\ y(0) + xy'(0) + \frac{1}{2!}x^2 y''(0) + \frac{1}{3!}x^3 y'''(0) + \frac{1}{4!}x^4 y^{(4)}(0), & \text{for } L = \frac{d^5}{dx^5}, \end{cases} \qquad (3)$$

and so on.

The Adomian decomposition method admits the decomposition of y into an infinite series of components

$$y(x) = \sum_{n=0}^{\infty} y_n, \tag{4}$$

and the nonlinear term $F(y)$ be equated to an infinite series of polynomials

$$F(y) = \sum_{n=0}^{\infty} A_n, \tag{5}$$

where A_n are the Adomian polynomials. Substituting (4) and (5) into (2) gives

$$\sum_{n=0}^{\infty} y_n = \psi_0 + L^{-1}g(x) - L^{-1}R\left(\sum_{n=0}^{\infty} y_n\right) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{6}$$

The various components y_n of the solution y can be easily determined by using the recursive relation

$$\begin{cases} y_0 = \psi_0 + L^{-1}g(x), \\ y_{n+1} = -L^{-1}R\left(y_n\right) - L^{-1}\left(A_n\right), \ \ n \geq 0. \end{cases} \tag{7}$$

Consequently, the first few components can be written as

$$\begin{cases} y_0 = \psi_0 + L^{-1}g(x), \\ y_1 = -L^{-1}R\left(y_0\right) - L^{-1}\left(A_0\right), \\ y_2 = -L^{-1}R\left(y_1\right) - L^{-1}\left(A_1\right), \\ y_3 = -L^{-1}R\left(y_2\right) - L^{-1}\left(A_2\right). \end{cases} \tag{8}$$

Having determined the components y_n, $n\geq 0$, the solution y in a series form follows immediately. As stated before, the series may be summed to provide the solution in a closed form. However, for concrete problems, the $n-$term partial sum

$$\phi_n = \sum_{k=0}^{n-1} y_k,$$

may be used to give the approximate solution.

Example 1:

Use the Adomian decomposition method to find the solution of the following second order nonlinear ordinary differential equation

$$y'' + (y')^2 + y^2 = 1 - \sin x; \quad y(0) = 0, y'(0) = 1. \tag{9}$$

Solution

In an operator form, the given equation can be written as

$$Ly = 1 - \sin x - (y')^2 - y^2, \tag{10}$$

where L is a second order differential operator. It is clear that L^{-1} is invertible and given by

$$L^{-1}(.) = \int_0^x \int_0^x (.)dxdx.$$

Applying L^{-1} to both sides of (10) and using the initial condition we obtain

$$L^{-1}Ly = L^{-1}(1 - \sin x) - L^{-1}\left((y')^2 + y^2\right), \tag{11}$$

$$\Rightarrow y(x) = y(0) + xy'(0) + \frac{x^2}{2} + \sin x - x - L^{-1}\left((y')^2 + y^2\right),$$

$$= x + \frac{x^2}{2} + \sin x - x - L^{-1}\left((y')^2 + y^2\right),$$

$$y(x) = \frac{x^2}{2} + \sin x - L^{-1}\left((y')^2 + y^2\right). \tag{12}$$

The decomposition method suggests that the solution $y(x)$ be expressed by the decomposition series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{13}$$

and the nonlinear terms $(y')^2 + y^2$ be equated to

$$(y')^2 + y^2 = \sum_{n=0}^{\infty} A_n, \tag{14}$$

where $y_n(x)$, $n \geq 0$ are the components of $y(x)$ that will be determined recursively, and A_n, $n \geq 0$ are the Adomian polynomials that represent the nonlinear term $(y')^2 + y^2$.

Inserting equations (13) and (14) into (12), gives

$$\sum_{n=0}^{\infty} y_n(x) = \frac{x^2}{2} + \sin x - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right)$$

The zeroth component y_0 is usually defined by all terms that are not included under the operator L^{-1}. The remaining components can be determined recurrently such that each term is determined by using the previous component. Consequently, the components of $y(x)$ can be elegantly determined by using the recursive relation

$$\begin{aligned} y_0(x) &= \frac{x^2}{2} + \sin x, \\ y_{k+1}(x) &= -L^{-1}(A_k), \quad k \geq 0, \end{aligned} \tag{15}$$

Note that the Adomian polynomials A_n for the nonlinear term $(y')^2 + y^2$ were determined before by using Adomian algorithm and are calculated as

$$\begin{aligned} A_0 &= (y_0')^2 + y_0^2, \\ A_1 &= 2\left(y_0' y_1' + y_0 y_1\right), \\ A_2 &= 2\left(y_0' y_2' + y_0 y_2\right) + (y_1')^2 + y_1^2, \end{aligned} \tag{16}$$

and so on. Using these polynomials into (15), the first few components can be determined recursively by

$$\begin{aligned} y_0 &= \frac{x^2}{2} + \sin x, \\ y_1 &= -L^{-1}\left((y_0')^2 + y_0^2\right) = -\frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{12} - \frac{x^6}{120}, \\ y_2 &= -L^{-1}\left(2\left(y_0' y_1' + y_0 y_1\right)\right) = \frac{x^3}{3} + \frac{x^4}{3} + \frac{2x^5}{15} + \ldots, \end{aligned} \tag{17}$$

Consequently, the solution in a series form is given by

$$y(x) = \sin x + \frac{x^2}{2} - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{12} - \frac{x^6}{120} + \frac{x^3}{3} + \frac{x^4}{3} + \frac{2x^5}{15} + \ldots$$

After cancellation of terms, we have

$$y(x) = \sin x.$$

Example 2:

Find the solution of the following third order nonlinear ordinary differential equation by using the Adomian decomposition method.

$$y^{(3)}(x)+\left(y''(x)\right)^2+\left(y'(x)\right)^2=2+\cos x;$$
$$y(0)=0, y'(0)=2, y''(0)=0. \tag{18}$$

Solution

In an operator form, the given equation becomes

$$Ly=2+\cos x-\left(y'\right)^2-\left(y''\right)^2, \tag{19}$$

where L is a third order differential operator. It is clear that L^{-1} is invertible and given by

$$L^{-1}(.)=\int_0^x\int_0^x\int_0^x(.)dxdxdx$$
$$L^{-1}L(y)=y(x)-y(0)-xy'(0)-\frac{1}{2!}x^2y''(0).$$

So by applying L^{-1} on both sides of (19), we have

$$y(x)-y(0)-xy'(0)-\frac{1}{2!}x^2y''(0)=L^{-1}\left(2+\cos x\right)-L^{-1}\left(\left(y'\right)^2+\left(y''\right)^2\right),$$
$$y(x)-2x=\frac{x^3}{3}+x+\sin x-L^{-1}\left(\left(y'\right)^2+\left(y''\right)^2\right),$$
$$y(x)=\frac{x^3}{3}+3x+\sin x-L^{-1}\left(\left(y'\right)^2+\left(y''\right)^2\right), \tag{20}$$

We next represent the linear term $y(x)$ by the decomposition series of components $y_n(x)$, $n\geq 0$, equate the nonlinear term y'^2 by the Adomian polynomials A_n, $n\geq 0$, and equate the nonlinear term y''^2 by the series of Adomian polynomials B_n, $n\geq 0$, to find

$$\sum_{n=0}^{\infty}y_n(x)=\frac{x^3}{3}+3x+\sin x-L^{-1}\left(\sum_{n=0}^{\infty}A_n+\sum_{n=0}^{\infty}B_n\right),$$

Identifying the zeroth component y_0, and following the decomposition method we set the recursive relation

$$\begin{aligned} y_0(x) &= \frac{x^3}{3} + 3x + \sin x, \\ y_{k+1}(x) &= -L^{-1}\left(A_k + B_k\right), \quad k \ge 0, \end{aligned} \tag{21}$$

Consequently, for finding the first few components of the solution proceeds as

$$y_1(x) = -L^{-1}\left(A_0 + B_0\right)$$

As

$$A_0 = y_0'^2 = \left(x^2 + 3 - \cos x\right)^2$$

$$A_0 = x^4 + 6x^2 + 9 - 6\cos x - 2x^2 \cos x + \cos^2 x$$

and

$$B_0 = y_0''^2 = \left(2x + \sin x\right)^2 = 4x^2 + 4x\sin x + \sin^2 x$$

$$y_1 = -L^{-1}\left(A_0 + B_0\right) = -L^{-1}\left(x^4 + 10x^2 + 10 - 6\cos x - 2x^2 \cos x + 4x\sin x\right)$$

$$y_1 = -\frac{2}{3}x^3 - \frac{2}{15}x^5 - \frac{1}{210}x^7$$

Similarly you can find A_1 and B_1 by using Adomian algorithm, as

$$A_1 = 2y_0' y_1'$$

$$A_1 = -\frac{1}{15}x^8 - \frac{23}{15}x^6 - 8x^4 - 12x^2 + \frac{1}{15}x^6 \cos x + \frac{4}{3}x^4 \cos x + 4x^2 \cos x$$

and

$$B_1 = 2y_0'' y_1'' = -\frac{4}{5}x^6 - \frac{32}{3}x^4 - 16x^2 - 8x\sin x - \frac{2}{5}x^5 \sin x - \frac{16}{3}x^3 \sin x$$

$$y_2 = -L^{-1}\left(A_1 + B_1\right)$$

$$y_2 = \frac{2}{5}x^5 + \frac{26}{315}x^7 + \frac{17}{3780}x^9 + \frac{1}{14850}x^{11}$$

Consequently, the solution in a series form is given by

$$\begin{aligned} y(x) &= \frac{x^3}{3} + 3x + \sin x - \frac{2}{3}x^3 - \frac{2}{15}x^5 - \frac{1}{210}x^7 + \frac{2}{5}x^5 + \frac{26}{315}x^7 + \frac{17}{3780}x^9 + \frac{1}{14850}x^{11} + \dots \\ &= \sin x + 3x - \frac{1}{3}x^3 + \frac{4}{15}x^5 + \dots \end{aligned}$$

Example 3:

Consider the linear boundary value problem

$$y^{(3)} = y - 3e^{x},$$
$$y'(0) = 0,\ y(1) = 0,\ y(0) = 1. \qquad (22)$$

Find its approximate solution by using Adomian decomposition method.

Solution

In operator form the given differential equation becomes

$$Ly = y - 3e^{x},$$

where L is a third order differential operator. It is clear that L^{-1} is invertible and given by

$$L^{-1}(.) = \int_0^x\int_0^x\int_0^x (.)dxdxdx$$

So by applying L^{-1} on both sides of above equation, we have

$$y = 1 + \frac{1}{2}Ax^2 - L^{-1}(3e^{x}) + L^{-1}(y), \qquad (23)$$

where $A = y''(0)$. Using $y(x) = \sum_{n=0}^{\infty} y_n(x),$ in (23)

$$\sum_{n=0}^{\infty} y_n(x) = 1 + \frac{1}{2}Ax^2 - L^{-1}(3e^{x}) + L^{-1}\left(\sum_{n=0}^{\infty} y_n(x)\right),$$

$$\Rightarrow y_0(x) = 1 + \frac{1}{2}Ax^2 - L^{-1}(3e^{x}),$$
$$y_{n+1}(x) = L^{-1}\left(y_n\right);\ n \ge 0$$

The constant A in the reduction formula will be determined using boundary conditions (22) after finding the decomposition series. From the reduction relation, we can obtain the solution terms of the decomposition series as

$$y_0(x) = 4 + 3x - 3e^x + \frac{1}{2}(A+3)x^2,$$
$$y_1(x) = L^{-1}\left(y_0\right) = 3 + 3x - 3e^x + \frac{3}{2}x^2 + \frac{2}{3}x^3 + \frac{x^4}{8} + \frac{(A+3)}{120}x^5,$$
$$y_2(x) = L^{-1}\left(y_1\right) = 1 + 3x - 3e^x + \frac{3}{2}x^2 + \frac{x^3}{2} + \frac{x^4}{8} + \frac{x^5}{40} + \frac{x^6}{180}$$
$$+ \frac{x^7}{1680} + \frac{(A+3)}{40320}x^8,$$
$$\vdots$$

Consequently, the approximate solution obtained by using Adomian decomposition method using the first three terms of the given problem in a series form is given by

$$y(x) = 8 + 9x - 9e^x + \frac{1}{2}(A+9)x^2 + \frac{7}{6}x^3 + \frac{x^4}{4} + \frac{(A+6)}{120}x^5 + \frac{x^6}{180} + \frac{x^7}{1680} + \frac{(A+3)}{40320}x^8.$$

Example 4:

Consider the fourth order linear nonhomogeneous differential equation

$$y^{(4)} - 2y'' + y = -8e^x,$$

with the boundary conditions

$$y(0) = y''(0) = 0,\ y'(1) = y''(1) = -e. \tag{24}$$

Solution

In operator form the given differential equation becomes

$$Ly = 2y'' - y - 8e^x.$$

Here,

$$L = \frac{d^4}{dx^4}, \qquad L^{-1}(.) = \int_0^x\int_0^x\int_0^x\int_0^x (.)dxdxdxdx,$$

are the derivative and integral operators.

Apply the inverse operator and initial conditions are taken, we find

$$y(x) = Ax + \frac{1}{3!}Bx^3 - L^{-1}\left(8e^x\right) + 2L^{-1}\left(y''\right) - L^{-1}\left(y\right), \tag{25}$$

where $A = y'(0)$ *and* $B = y'''(0)$. Using $y(x) = \sum_{n=0}^{\infty} y_n(x),$ in (25)

$$\sum_{n=0}^{\infty} y_n(x) = Ax + \frac{1}{3!}Bx^3 - L^{-1}\left(8e^x\right) + 2L^{-1}\left(\sum_{n=0}^{\infty} y_n''(x)\right) - L^{-1}\left(\sum_{n=0}^{\infty} y_n(x)\right), \tag{26}$$

whereas the reduction formula given below can be written using (26),

$$\begin{aligned} y_0(x) &= Ax + \frac{1}{3!}Bx^3 - L^{-1}\left(8e^x\right), \\ y_{n+1}(x) &= 2L^{-1}\left(y_n''\right) - L^{-1}\left(y_n\right); n \geq 0 \end{aligned}$$

The A and B constants in the reduction formula will be determined using boundary conditions (24) after finding the decomposition series. From the reduction relation, we can obtain the solution terms of the decomposition series as

$$\begin{aligned} y_0(x) &= 8 - 8e^x + (8+A)x + 4x^2 + \frac{1}{6}(8+B)x^3, \\ y_1(x) &= 2L^{-1}\left(y_0''\right) - L^{-1}\left(y_0\right) \\ &= 8 - 8e^x + 8x + 4x^2 + \frac{4x^3}{3} + \frac{x^4}{3} - \frac{1}{120}(-8+A-2B)x^5 \\ &\quad - \frac{x^6}{90} - \frac{(8+B)}{5040}x^7, \\ y_2(x) &= 2L^{-1}\left(y_1''\right) - L^{-1}\left(y_1\right) \\ &= 8 - 8e^x + 8x + 4x^2 + \frac{4x^3}{3} + \frac{x^4}{3} - \frac{x^5}{15} + \frac{x^6}{90} - \frac{1}{2520}(A-2B-4)x^7 \\ &\quad - \frac{x^8}{1680} + \frac{(\mathrm{A}-4B-24)}{362880}x^9 + \frac{x^{10}}{453600} - \frac{(8+B)}{39916800}x^{11}, \\ &\vdots \end{aligned}$$

Consequently, the approximate solution obtained by using Adomian decomposition method using the first three terms of the given problem in a series form is given by

$$y(x)=24-24e^{x}+(24+A)x+12x^{2}+\frac{1}{6}(24+B)x^{3}+\frac{2x^{4}}{3}+\frac{1}{120}(16-A+2B)x^{5}$$
$$-\frac{(2\mathrm{A}-3B)}{5040}x^{7}-\frac{x^{8}}{1680}+\frac{(\mathrm{A}-4B-24)}{362880}x^{9}+\frac{x^{10}}{453600}-\frac{(8+B)}{39916800}x^{11}.$$

Example 5:

Consider the linear boundary value problem

$$y^{(5)}(x)=y-15e^{x}-10xe^{x}, \quad 0<x<1$$

subject to the boundary conditions

$$y(0)=y''(0)=0, \quad y'(0)=1, \quad y(1)=0, \quad y'(1)=-e. \tag{27}$$

Find out the recursive relation for it, by using Adomian decomposition method from which the various components y_n of the solution y can be determined.

Solution

In operator form the given differential equation becomes

$$Ly=y-15e^{x}-10xe^{x}.$$

Here,

$$L=\frac{d^{5}}{dx^{5}}, \qquad L^{-1}(.)=\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x(.)\,dxdxdxdxdx,$$

are the derivative and integral operators. Apply the inverse operator and initial conditions are taken, we find

$$L^{-1}Ly(x)=-L^{-1}\left(15e^{x}\right)-L^{-1}\left(10xe^{x}\right)+L^{-1}\left(y\right)$$
$$\Rightarrow y(x)=-35-24x-\frac{15}{2}x^{2}+\left(\frac{A}{6}-\frac{5}{6}\right)x^{3}+\left(\frac{B}{24}+\frac{5}{24}\right)x^{4}$$
$$+(35-10x)e^{x}+L^{-1}\left(y\right), \tag{28}$$

where the constants $A=y'''(0)$ *and* $B=y^{(4)}(0)$. Using $y(x)=\sum_{n=0}^{\infty}y_n(x),$ in (28)

$$\sum_{n=0}^{\infty} y_n(x) = -35 - 24x - \frac{15}{2}x^2 + \left(\frac{A}{6} - \frac{5}{6}\right)x^3 + \left(\frac{B}{24} + \frac{5}{24}\right)x^4 + (35 - 10x)e^x + L^{-1}\left(\sum_{n=0}^{\infty} y_n(x)\right), \qquad (29)$$

whereas the recursive relation given below can be written using (29),

$$y_0(x) = -35 - 24x - \frac{15}{2}x^2 + \left(\frac{A}{6} - \frac{5}{6}\right)x^3 + \left(\frac{B}{24} + \frac{5}{24}\right)x^4 + (35 - 10x)e^x,$$
$$y_{n+1}(x) = L^{-1}\left(y_n\right); n \geq 0$$

The A and B constants in the reduction formula will be determined using boundary conditions (27) after finding the decomposition series.

Example 6:

Consider the nonlinear boundary value problem

$$y^{(6)}(x) = e^x y^2(x), \quad 0 < x < 1$$

subject to the boundary conditions

$$\begin{aligned} &y(0) = 1, \quad y'(0) = -1, \quad y''(0) = 1, \\ &y(1) = e^{-1}, \quad y'(1) = -e^{-1}, \quad y''(1) = e^{-1}. \end{aligned} \qquad (30)$$

Find out the recursive relation for it, by using Adomian decomposition method from which the various components y_n of the solution y can be determined.

Solution

In operator form the given differential equation becomes

$$Ly = e^x y^2(x).$$

Here,

$$L = \frac{d^6}{dx^6}, \qquad L^{-1}(.) = \int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x (.)\, dxdxdxdxdxdx,$$

are the derivative and integral operators.

Apply the inverse operator and initial conditions are taken, we find

$$L^{-1}Ly(x) = L^{-1}\left(e^x y^2(x)\right)$$
$$\Rightarrow y(x) = 1 - x + \frac{1}{2}x^2 + \frac{A}{6}x^3 + \frac{B}{24}x^4 + \frac{C}{120}x^5 + L^{-1}\left(e^x y^2(x)\right), \qquad (31)$$

where $A = y'''(0), B = y^{(4)}(0)$ *and* $C = y^{(5)}(0)$ are the constants that will be determined later. Using $y(x) = \sum_{n=0}^{\infty} y_n(x)$ and $y^2(x) = \sum_{n=0}^{\infty} A_n$ in (31)

$$\sum_{n=0}^{\infty} y_n(x) = 1 - x + \frac{1}{2}x^2 + \frac{A}{6}x^3 + \frac{B}{24}x^4 + \frac{C}{120}x^5 + L^{-1}\left(e^x\left(\sum_{n=0}^{\infty} A_n\right)\right), \qquad (32)$$

where A_n are Adomian polynomials. Hence the recursive relation is as

$$y_0(x) = 1 - x + \frac{1}{2}x^2 + \frac{A}{6}x^3 + \frac{B}{24}x^4 + \frac{C}{120}x^5,$$
$$y_{n+1}(x) = L^{-1}\left(e^x\left(\sum_{n=0}^{\infty} A_n\right)\right); n \geq 0$$

Exercises

Use the Adomian decomposition method to find the series solution of the following nonlinear ordinary differential equations:

1. $y''(x) - y^3(x) = 0; \quad y(0) = 1,\ y'(0) = 0.$

2. $y''(x) - ye^y = 0; \quad y(0) = 1,\ y'(0) = 0.$

3. $$y^{(3)}(x) + \left(y''(x)\right)^2 + \left(y'(x)\right)^2 = 1 - \sin x;$$
   $$y(0) = y'(0) = 0,\ y''(0) = 1.$$

4. $$y^{(3)}(x) - \left(y''(x)\right)^2 + \left(y'(x)\right)^2 = 1 + \cosh x;$$
   $$y(0) = 0,\ y'(0) = 1,\ y''(0) = 0.$$

5. $$y^{(4)}(x) - 18y''(x) + 81y(x) = 0;$$
   $$y(0) = 0,\ y'(0) = -1,\ y''(0) = y'''(0) = 0.$$

Find out the recursive relation for the following ordinary differential equations, by using Adomian decomposition method from which the various components y_n of the solution y can be determined.

6. $$y^{(5)}(x) = e^x \left(y\right)^4;\ 0 < x < 1$$
   $$y(0) = 1,\ y'(0) = -\frac{1}{3},\ y''(0) = \frac{1}{9},\ y(1) = e^{-1/3},\ y'(1) = \left(-\frac{1}{3}\right)e^{-1/3}.$$

7. $y^{(6)}(x) = e^{-x} y^2(x), \quad 0 < x < 1$

   $$y(0) = y''(0) = y^{(4)}(0) = 1,$$
   $$y(1) = y''(1) = y^{(4)}(1) = e.$$