Lecture 34

Convergence of Adomian Decomposition Method

In this lecture, the rate of convergence of Adomian decomposition method will be discussed. Cherruault, proposed a new definition of the technique to prove the convergence of this method, under suitable and reasonable hypothesis. We used Cherruault's definition and consider the order of convergence of the method.

THE ADOMIAN DECOMPOSITION METHOD FOR FUNCTIONAL EQUATIONS

Consider the functional equation

$$y - Ny = f, \tag{1}$$

where N is a non-linear operator from a Hilbert space H into H, f is a given function in H and we are looking for $y \in H$ satisfying (1).

As the initial Adomian technique considers of representing y as a series

$$y = \sum_{i=0}^{\infty} y_i, \tag{2}$$

and the non-linear operator as the sum of the series

$$Ny = \sum_{n=0}^{\infty} A_n,$$

The method consist of the scheme:

$$\begin{cases} y_0 = f, \\ y_{k+1} = A_n(y_0, y_1, \ldots, y_n), \end{cases} \tag{3}$$

where A_n's are polynomials in $y_0, y_1, \ldots, y_n$ called Adomian polynomials, obtained by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty} \lambda^i y_i\right)\right]_{\lambda=0} ; n = 0,1,2,\ldots$$

The Adomian technique is equivalent to determining the sequence

$$S_n = y_1 + y_2 + \ldots + y_n,$$

by using the iterative scheme

$$\begin{cases} S_0 = 0, \\ S_{n+1} = N(y_0 + S_n). \end{cases} \qquad (4)$$

Associated with the functional equation

$$S = N(y_0 + S). \qquad (5)$$

For the study of the numerical resolution of (4) Cherruault used fixed point theorem.

Theorem.

Let N be an operator from a Hilbert space H in to H and y be the exact solution of (1). $\sum_{i=0}^{\infty} y_i$, which is obtained by (3), converges to y when $\exists\ 0 \le \alpha < 1, \|y_{k+1}\| \le \alpha \|y_k\|, \ \forall k \in N \cup \{0\}$.

Proof.

We have

$$\begin{aligned} S_0 &= 0, \\ S_1 &= y_1, \\ S_2 &= y_1 + y_2, \\ &\vdots \\ S_n &= y_1 + y_2 + \ldots + y_n, \end{aligned}$$

and we show that, $\{S_n\}_{n=0}^{+\infty}$ is a Cauchy sequence in the Hilbert space H. For this reason, consider,

$$\|S_{n+1} - S_n\| = \|y_{n+1}\| \le \alpha \|y_n\| \le \alpha^2 \|y_{n-1}\| \le \ldots \le \alpha^{n+1} \|y_0\|.$$

But for every $n, m \in N, n \ge m,$ we have

$$\begin{aligned} \|S_n - S_m\| &= \|(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + \ldots + (S_{m+1} - S_m)\| \\ &\le \|(S_n - S_{n-1})\| + \|(S_{n-1} - S_{n-2})\| + \ldots + \|(S_{m+1} - S_m)\| \le \alpha^n \|y_o\| + \alpha^{n-1} \|y_o\| + \ldots + \alpha^{m+1} \|y_o\| \\ &\le (\alpha^{m+1} + \alpha^{m+2} + \ldots) \|y_o\| = \frac{\alpha^{m+1}}{1-\alpha} \|y_o\|. \end{aligned}$$

Hence, $\lim_{n,m\to+\infty}\|S_n - S_m\| = 0$, i.e., $\{S_n\}_{n=0}^{+\infty}$ is a Cauchy sequence in the Hilbert space H and it implies that $\exists\ S, S \in H,\quad \lim_{n\to+\infty} S_n = S,$

i.e., $S = \sum_{n=0}^{+\infty} y_n$. But, to solve Eq. (1) is equivalent to solving Eq. (5) and it implies that if N be a continuous operator then

$$N(y_0 + S) = N\left(\lim_{n\to+\infty}(y_0 + S_n)\right) = \lim_{n\to+\infty} N(y_0 + S_n) = \lim_{n\to+\infty} S_{n+1} = S,$$

i.e., S is a solution of Eq. (1), too.

Definition. For every $i \in N \cup \{0\}$ we define

$$\alpha_i = \begin{cases} \dfrac{\|y_{i+1}\|}{\|y_i\|}, & \|y_i\| \neq 0, \\ 0, & \|y_i\| = 0. \end{cases} \tag{6}$$

Corollary.

In above theorem, $\sum_{i=0}^{\infty} y_i$ converges to exact solution y, when $0 \le \alpha_i < 1,\ i = 1, 2, 3, \ldots$.

The standard Adomian decomposition method usually defines the equation in an operator form by considering the highest-ordered derivative in the problem

$$L = \frac{d^n}{dx^n},$$

So

$$L^{-1}(.) = \int_0^x \int_0^x \ldots \int_0^x (.)dxdx\ldots dx.$$

Example 1.

Consider the initial value problem

$$\begin{aligned} &y' + (1+x^2)y^2 = x^4 + 2x^3 + 2x^2 + 2x + 2, \\ &y(0) = 1. \end{aligned} \tag{7}$$

Solution:

In an operator form, (7) becomes

$$Ly = \left(x^4 + 2x^3 + 2x^2 + 2x + 2\right) - (1+x^2)y^2, \tag{8}$$

where

$$L = \frac{d}{dx},$$

and

$$L^{-1}(.) = \int_0^x (.)dx.$$

By applying L^{-1} on the both sides of (8), we obtain

$$y = L^{-1}\left(x^4 + 2x^3 + 2x^2 + 2x + 2\right) - L^{-1}\left((1+x^2)y^2\right) + y(0). \tag{9}$$

So, we have

$$\begin{cases} y_0 = L^{-1}\left(x^4 + 2x^3 + 2x^2 + 2x + 2\right) + y(0), \\ y_{n+1} = -L^{-1}\left((1+x^2)A_n\right), \quad n \geq 0 \end{cases} \tag{10}$$

where A_n*'s* are Adomian polynomials for the nonlinear term y^2, as

$$\begin{aligned} A_0 &= y_0^2, \\ A_1 &= 2y_0y_1, \\ A_2 &= 2y_0y_2 + y_1^2, \\ A_3 &= 2y_0y_3 + 2y_1y_2, \\ &\vdots \end{aligned}$$

which are obtained by using a reference list of the Adomian polynomials given in lecture 32.

Hence

$$
\begin{aligned}
y_0 &= \frac{x^5}{5} + \frac{x^4}{2} + \frac{2x^3}{3} + x^2 + 2x + 1, \\
y_1 &= -L^{-1}\left((1+x^2)y_0^2\right) = -L^{-1}\left((1+x^2)y_0^2\right) \\
&= -\frac{x^{13}}{325} - \frac{x^{12}}{60} - \frac{167x^{11}}{3300} - \frac{19x^{10}}{150} - \frac{497x^9}{1620} - \frac{3x^8}{5} - \frac{311x^7}{315} - \frac{68x^6}{45} - \frac{32x^5}{15} - \frac{7x^4}{3} - \frac{7x^3}{3} - 2x^2 - x, \\
&\vdots
\end{aligned}
$$

By computing α_i's for this problem, we have

$$
\begin{aligned}
\alpha_0 &= \frac{\|y_1\|}{\|y_0\|} = 1.5750915 > 1, \\
\alpha_1 &= \frac{\|y_2\|}{\|y_1\|} = 2.9244392 > 1, \\
&\vdots
\end{aligned}
$$

Here, α_i's are not less than one and thus standard Adomian decomposition method is not convergent. So, the standard Adomian decomposition method may be divergent, even, to solve a simple nonsingular initial value problem.

Now we focus on singular ODEs. For this reason, consider the Lane–Emden equation formulated as

$$
\begin{aligned}
& y'' + \frac{2}{x}y' + F(x,y) = g(x), \qquad 0 < x \le 1, \qquad (11) \\
& y(0) = A, \qquad y'(0) = B,
\end{aligned}
$$

where A and B are constants, $F(x, y)$ is a continuous real valued function, and $g(x) \in C[0,1]$. Usually, the standard Adomian decomposition method may be divergent to solve singular Lane–Emden equations. To overcome the singularity behavior, Wazwaz defined the differential operator L in terms of two derivatives contained in the problem. He rewrote (11) in the form

$$Ly = -F(x,y) + g(x),$$

where the differential operator L is defined by

$$L = x^{-2}\frac{d}{dx}\left(x^2\frac{d}{dx}\right).$$

There is an example of the form (11) that both standard and modified Adomian decomposition methods are convergent.

Example 2.

Consider the linear singular initial value problem

$$y''+\frac{2}{x}y'+y=x^5+30x^3,$$
$$y(0)=0, \qquad y'(0)=0. \tag{12}$$

Solution:

Standard Adomian method:

In the operator form, (12) becomes

$$Ly=x^5+30x^3-y-\frac{2}{x}y', \tag{13}$$

where

$$L=\frac{d^2}{dx^2},$$

so

$$L^{-1}(.)=\int_0^x\int_0^x(.)dxdx.$$

By applying L^{-1} on the both sides of (13), we obtain

$$y=L^{-1}\left(x^5+30x^3\right)+y(0)+xy'(0)-L^{-1}\left(\frac{2}{x}y'+\mathrm{y}\right).$$

We obtain the recursive relationship

$$\begin{cases} y_0=L^{-1}\left(x^5+30x^3\right)+y(0)+xy'(0), \\ y_{n+1}=-L^{-1}\left(\dfrac{2}{x}y'+\mathrm{y}\right), \quad n\geq 0 \end{cases}$$

Consequently, the first few components are as

$$\begin{aligned}
y_0 &= \frac{3x^5}{2} + \frac{x^7}{42}, \\
y_1 &= -\frac{3x^5}{4} - \frac{11x^7}{252} - \frac{x^9}{3024}, \\
y_2 &= \frac{3x^5}{8} + \frac{7x^7}{216} + \frac{25x^9}{36288} + \frac{x^{11}}{332640}, \\
&\vdots
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
\alpha_0 &= 0.5178432480 < 1, \\
\alpha_1 &= 0.5118817363 < 1, \\
\alpha_2 &= 0.5079364636 < 1, \\
&\vdots
\end{aligned}$$

Hence, the standard Adomian decomposition method is convergent.

Modified Adomian method:

In an operator form Eq. (12) becomes

$$Ly = x^5 + 30x^3 - y, \tag{15}$$

where

$$L = x^{-2} \frac{d}{dx}\left(x^2 \frac{d}{dx} \right).$$

The inverse operator L^{-1} is therefore expressed as

$$L^{-1}(.) = \int_0^x x^{-2} \int_0^x x^2 (.) dx dx.$$

Operating with L^{-1} on (15), it follows

$$y(x) = y(0) + L^{-1}\left(x^5 + 30x^3 \right) - L^{-1}(y),$$

Proceeding as in previous lecture, we obtain

$$y_0(x) = y(0) + L^{-1}\left(x^5 + 30x^3\right),$$
$$y_{n+1}(x) = -L^{-1}(y_n), \qquad n \geq 0.$$

This gives the first few components

$$\begin{aligned}
y_0 &= x^5 + \frac{x^7}{56}, \\
y_1 &= -\frac{x^7}{56} - \frac{x^9}{5040}, \\
y_2 &= \frac{x^9}{5040} + \frac{x^{11}}{665280}, \\
&\vdots
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
\alpha_0 &= 0.01521198194 < 1, \\
\alpha_1 &= 0.009843639085 < 1, \\
&\vdots
\end{aligned} \tag{17}$$

The obtained results in (14) and (16) show that the rate of convergence of modified Adomian method is higher than standard Adomian method for this problem.

Exercises

1. Check the convergence of linear nonsingular initial value problem,

$$y'' + y' = 2x + 2,$$
$$y(0) = 0,\ y'(0) = 0.$$

2. Check the convergence of linear singular initial value problem

$$y'' + \frac{2}{x}y' + y = 6 + 12x + x^2 + x^3,$$
$$y(0) = 0, \qquad y'(0) = 0.$$