8.2.1 Calculation of Adomian Polynomials

It is well known now that Adomian decomposition method suggests that the unknown linear function u may be represented by the decomposition series

$$u = \sum_{n=0}^{\infty} u_n, \tag{8.10}$$

where the components $u_n, n \geqslant 0$ can be elegantly computed in a recursive way. However, the nonlinear term $F(u)$, such as u^2, u^3, u^4, $\sin u, e^u, uu_x, u_x^2$, etc. can be expressed by an infinite series of the so-called Adomian polynomials A_n given in the form

$$F(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \cdots, u_n), \tag{8.11}$$

where the so-called Adomian polynomials A_n can be evaluated for all forms of nonlinearity. Several schemes have been introduced in the literature by researchers to calculate Adomian polynomials. Adomian introduced a scheme for the calculation of Adomian polynomials that was formally justified. An alternative reliable method that is based on algebraic and trigonometric identities and on Taylor series has been developed and will be examined later. The alternative method employs only elementary operations and does not require specific formulas.

The Adomian polynomials A_n for the nonlinear term $F(u)$ can be evaluated by using the following expression

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[F\left(\sum_{i=0}^{n} \lambda^i u_i \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \cdots. \tag{8.12}$$

The general formula (8.12) can be simplified as follows. Assuming that the nonlinear function is $F(u)$, therefore by using (8.12), Adomian polynomials [3] are given by

$$\begin{aligned}
A_0 &= F(u_0), \\
A_1 &= u_1 F'(u_0), \\
A_2 &= u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0), \\
A_3 &= u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0), \\
A_4 &= u_4 F'(u_0) + (\frac{1}{2!} u_2^2 + u_1 u_3) F''(u_0) + \frac{1}{2!} u_1^2 u_2 F'''(u_0) + \frac{1}{4!} u_1^4 F^{(4)}(u_0).
\end{aligned} \tag{8.13}$$

Other polynomials can be generated in a similar manner.

Two important observations can be made here. First, A_0 depends only on u_0, A_1 depends only on u_0 and u_1, A_2 depends only on u_0, u_1 and u_2, and so on. Second, substituting (8.13) into (8.11) gives

$$\begin{aligned}
F(u) &= A_0 + A_1 + A_2 + A_3 + \cdots \\
&= F(u_0) + (u_1 + u_2 + u_3 + \cdots) F'(u_0)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2!}(u_1^2+2u_1u_2+2u_1u_3+u_2^2+\cdots)F''(u_0)+\cdots\\
&+\frac{1}{3!}(u_1^3+3u_1^2u_2+3u_1^2u_3+6u_1u_2u_3+\cdots)F'''(u_0)+\cdots\\
=\;& F(u_0)+(u-u_0)F'(u_0)+\frac{1}{2!}(u-u_0)^2F''(u_0)+\cdots.
\end{aligned}
$$

The last expansion confirms a fact that the series in A_n polynomials is a Taylor series about a function u_0 and not about a point as is usually used. The few Adomian polynomials given above in (8.13) clearly show that the sum of the subscripts of the components of u of each term of A_n is equal to n. As stated before, it is clear that A_0 depends only on u_0, A_1 depends only u_0 and u_1, A_2 depends only on u_0, u_1 and u_2. The same conclusion holds for other polynomials.

In the following, we will calculate Adomian polynomials for several forms of nonlinearity that may arise in nonlinear ordinary or partial differential equations.

Calculation of Adomian Polynomials A_n

I. Nonlinear Polynomials

Case 1. $F(u)=u^2$

The polynomials can be obtained as follows:

$$
\begin{aligned}
A_0&=F(u_0)=u_0^2,\\
A_1&=u_1F'(u_0)=2u_0u_1,\\
A_2&=u_2F'(u_0)+\frac{1}{2!}u_1^2F''(u_0)=2u_0u_2+u_1^2,\\
A_3&=u_3F'(u_0)+u_1u_2F''(u_0)+\frac{1}{3!}u_1^3F'''(u_0)=2u_0u_3+2u_1u_2.
\end{aligned}
$$

Case 2. $F(u)=u^3$

The polynomials are given by

$$
\begin{aligned}
A_0&=F(u_0)=u_0^3,\\
A_1&=u_1F'(u_0)=3u_0^2u_1,\\
A_2&=u_2F'(u_0)+\frac{1}{2!}u_1^2F''(u_0)=3u_0^2u_2+3u_0u_1^2,\\
A_3&=u_3F'(u_0)+u_1u_2F''(u_0)+\frac{1}{3!}u_1^3F'''(u_0)=3u_0^2u_3+6u_0u_1u_2+u_1^3.
\end{aligned}
$$

Case 3. $F(u)=u^4$

Proceeding as before we find

$$A_0=u_0^4,$$

$$A_1 = 4u_0^3 u_1,$$
$$A_2 = 4u_0^3 u_2 + 6u_0^2 u_1^2,$$
$$A_3 = 4u_0^3 u_3 + 4u_1^3 u_0 + 12u_0^2 u_1 u_2.$$

In a parallel manner, Adomian polynomials can be calculated for nonlinear polynomials of higher degrees.

II. Nonlinear Derivatives

Case 1. $F(u) = (u_x)^2$

$$A_0 = u_{0_x}^2,$$
$$A_1 = 2u_{0_x} u_{1_x},$$
$$A_2 = 2u_{0_x} u_{2_x} + u_{1_x}^2,$$
$$A_3 = 2u_{0_x} u_{3_x} + 2u_{1_x} u_{2_x}.$$

Case 2. $F(u) = u_x^3$

The Adomian polynomials are given by

$$A_0 = u_{0_x}^3,$$
$$A_1 = 3u_{0_x}^2 u_{1_x},$$
$$A_2 = 3u_{0_x}^2 u_{2_x} + 3u_{0_x} u_{1_x}^2,$$
$$A_3 = 3u_{0_x}^2 u_{3_x} + 6u_{0_x} u_{1_x} u_{2_x} + u_{1_x}^3.$$

Case 3. $F(u) = uu_x = \frac{1}{2} L_x(u^2)$

The Adomian polynomials for this nonlinearity are given by

$$A_0 = F(u_0) = u_0 u_{0_x},$$
$$A_1 = \frac{1}{2} L_x(2u_0 u_1) = u_{0_x} u_1 + u_0 u_{1_x},$$
$$A_2 = \frac{1}{2} L_x(2u_0 u_2 + u_1^2) = u_{0_x} u_2 + u_{1_x} u_1 + u_{2_x} u_0,$$
$$A_3 = \frac{1}{2} L_x(2u_0 u_3 + 2u_1 u_2) = u_{0_x} u_3 + u_{1_x} u_2 + u_{2_x} u_1 + u_{3_x} u_0.$$

III. Trigonometric Nonlinearity

Case 1. $F(u) = \sin u$

The Adomian polynomials for this form of nonlinearity are given by

$$A_0 = \sin u_0,$$
$$A_1 = u_1 \cos u_0,$$

$$A_2 = u_2 \cos u_0 - \frac{1}{2!} u_1^2 \sin u_0,$$

$$A_3 = u_3 \cos u_0 - u_1 u_2 \sin u_0 - \frac{1}{3!} u_1^3 \cos u_0.$$

Case 2. $F(u) = \cos u$

Proceeding as before gives

$$A_0 = \cos u_0,$$

$$A_1 = -u_1 \sin u_0,$$

$$A_2 = -u_2 \sin u_0 - \frac{1}{2!} u_1^2 \cos u_0,$$

$$A_3 = -u_3 \sin u_0 - u_1 u_2 \cos u_0 + \frac{1}{3!} u_1^3 \sin u_0.$$

IV. Hyperbolic Nonlinearity

Case 1. $F(u) = \sinh u$

The A_n polynomials for this form of nonlinearity are given by

$$A_0 = \sinh u_0,$$

$$A_1 = u_1 \cosh u_0,$$

$$A_2 = u_2 \cosh u_0 + \frac{1}{2!} u_1^2 \sinh u_0,$$

$$A_3 = u_3 \cosh u_0 + u_1 u_2 \sinh u_0 + \frac{1}{3!} u_1^3 \cosh u_0.$$

Case 2. $F(u) = \cosh u$

The Adomian polynomials are given by

$$A_0 = \cosh u_0,$$

$$A_1 = u_1 \sinh u_0,$$

$$A_2 = u_2 \sinh u_0 + \frac{1}{2!} u_1^2 \cosh u_0,$$

$$A_3 = u_3 \sinh u_0 + u_1 u_2 \cosh u_0 + \frac{1}{3!} u_1^3 \sinh u_0.$$

V. Exponential Nonlinearity

Case 1. $F(u) = e^u$

The Adomian polynomials for this form of nonlinearity are given by

$$A_0 = e^{u_0},$$

$$A_1 = u_1 e^{u_0},$$

$$A_2 = (u_2 + \frac{1}{2!}u_1^2)e^{u_0},$$

$$A_3 = (u_3 + u_1u_2 + \frac{1}{3!}u_1^3)e^{u_0}.$$

Case 2. $F(u) = e^{-u}$

Proceeding as before gives

$$A_0 = e^{-u_0},$$

$$A_1 = -u_1e^{-u_0},$$

$$A_2 = (-u_2 + \frac{1}{2!}u_1^2)e^{-u_0},$$

$$A_3 = (-u_3 + u_1u_2 - \frac{1}{3!}u_1^3)e^{-u_0}.$$

VI. Logarithmic Nonlinearity

Case 1. $F(u) = \ln u, u > 0$

The A_n polynomials for logarithmic nonlinearity are give by

$$A_0 = \ln u_0,$$

$$A_1 = \frac{u_1}{u_0},$$

$$A_2 = \frac{u_2}{u_0} - \frac{1}{2}\frac{u_1^2}{u_0^2},$$

$$A_3 = \frac{u_3}{u_0} - \frac{u_1u_2}{u_0^2} + \frac{1}{3}\frac{u_1^3}{u_0^3}.$$

Case 2. $F(u) = \ln(1+u), -1 < u \leqslant 1$

The A_n polynomials are give by

$$A_0 = \ln(1+u_0),$$

$$A_1 = \frac{u_1}{1+u_0},$$

$$A_2 = \frac{u_2}{1+u_0} - \frac{1}{2}\frac{u_1^2}{(1+u_0)^2},$$

$$A_3 = \frac{u_3}{1+u_0} - \frac{u_1u_2}{(1+u_0)^2} + \frac{1}{3}\frac{u_1^3}{(1+u_0)^3}.$$

8.2.2 Alternative Algorithm for Calculating Adomian Polynomials

It is worth noting that a considerable amount of research work has been invested to develop an alternative method to Adomian algorithm for calculating Adomian polynomials A_n. The aim was to develop a practical technique that will calculate Adomian polynomials in a practical way without any need to the formulae introduced before. However, the methods developed so far in this regard are identical to that used by Adomian.

We believe that a simple and reliable technique can be established to make the calculations less dependable on the formulae presented before.

In the following, we will introduce an alternative algorithm that can be used to calculate Adomian polynomials for nonlinear terms in an easy way. The newly developed method in [15–16] depends mainly on algebraic and trigonometric identities, and on Taylor expansions as well. Moreover, we should use the fact that the sum of subscripts of the components of u in each term of the polynomial A_n is equal to n.

The alternative algorithm suggests that we substitute u as a sum of components $u_n, n \geqslant 0$ as defined by the decomposition method. It is clear that A_0 is always determined independent of the other polynomials $A_n, n \geqslant 1$, where A_0 is defined by

$$A_0 = F(u_0). \tag{8.14}$$

The alternative method assumes that we first separate $A_0 = F(u_0)$ for every nonlinear term $F(u)$. With this separation done, the remaining components of $F(u)$ can be expanded by using algebraic operations, trigonometric identities, and Taylor series as well. We next collect all terms of the expansion obtained such that the sum of the subscripts of the components of u in each term is the same. Having collected these terms, the calculation of the Adomian polynomials is thus completed. Several examples have been tested, and the obtained results have shown that Adomian polynomials can be elegantly computed without any need to the formulas established by Adomian. The technique will be explained by discussing the following illustrative examples.

Adomian Polynomials by Using the Alternative Method

I. Nonlinear Polynomials

Case 1. $F(u) = u^2$

We first set

$$u = \sum_{n=0}^{\infty} u_n. \tag{8.15}$$

Substituting (8.15) into $F(u) = u^2$ gives

$$F(u) = (u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + \cdots)^2. \tag{8.16}$$

Expanding the expression at the right hand side gives

$$F(u) = u_0^2 + 2u_0u_1 + 2u_0u_2 + u_1^2 + 2u_0u_3 + 2u_1u_2 + \cdots. \tag{8.17}$$

The expansion in (8.17) can be rearranged by grouping all terms with the sum of the subscripts is the same. This means that we can rewrite (8.17) as

$$\begin{aligned} F(u) = & \underbrace{u_0^2}_{A_0} + \underbrace{2u_0u_1}_{A_1} + \underbrace{2u_0u_2 + u_1^2}_{A_2} + \underbrace{2u_0u_3 + 2u_1u_2}_{A_3} \\ & + \underbrace{2u_0u_4 + 2u_1u_3 + u_2^2}_{A_4} + \underbrace{2u_0u_5 + 2u_1u_4 + 2u_2u_3}_{A_5} + \cdots. \end{aligned} \tag{8.18}$$

This completes the determination of Adomian polynomials given by

$$\begin{aligned} A_0 &= u_0^2, \\ A_1 &= 2u_0u_1, \\ A_2 &= 2u_0u_2 + u_1^2, \\ A_3 &= 2u_0u_3 + 2u_1u_2, \\ A_4 &= 2u_0u_4 + 2u_1u_3 + u_2^2, \\ A_5 &= 2u_0u_5 + 2u_1u_4 + 2u_2u_3. \end{aligned}$$

Case 2. $F(u) = u^3$

Proceeding as before, we set

$$u = \sum_{n=0}^{\infty} u_n. \tag{8.19}$$

Substituting (8.19) into $F(u) = u^3$ gives

$$F(u) = (u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + \cdots)^3. \tag{8.20}$$

Expanding the right hand side yields

$$\begin{aligned} F(u) = & u_0^3 + 3u_0^2u_1 + 3u_0^2u_2 + 3u_0u_1^2 + 3u_0^2u_3 + 6u_0u_1u_2 + u_1^3 \\ & + 3u_0^2u_4 + 3u_1^2u_2 + 3u_2^2u_0 + 6u_0u_1u_3 \cdots. \end{aligned} \tag{8.21}$$

The expansion in (8.21) can be rearranged by grouping all terms with the sum of the subscripts is the same. This means that we can rewrite (8.21) as

$$\begin{aligned} F(u) = & \underbrace{u_0^3}_{A_0} + \underbrace{3u_0^2u_1}_{A_1} + \underbrace{3u_0^2u_2 + 3u_0u_1^2}_{A_2} + \underbrace{3u_0^2u_3 + 6u_0u_1u_2 + u_1^3}_{A_3} \\ & + \underbrace{3u_0^2u_4 + 3u_1^2u_2 + 3u_2^2u_0 + 6u_0u_1u_3}_{A_4} + \cdots. \end{aligned} \tag{8.22}$$

Consequently, Adomian polynomials can be written by

$$\begin{aligned} A_0 &= u_0^3, \\ A_1 &= 3u_0^2u_1, \\ A_2 &= 3u_0^2u_2 + 3u_0u_1^2, \end{aligned}$$

$$A_3 = 3u_0^2u_3 + 6u_0u_1u_2 + u_1^3,$$
$$A_4 = 3u_0^2u_4 + 3u_1^2u_2 + 3u_2^2u_0 + 6u_0u_1u_3.$$

II. Nonlinear Derivatives

Case 1. $F(u) = u_x^2$

We first set

$$u_x = \sum_{n=0}^{\infty} u_{n_x}. \tag{8.23}$$

Substituting (8.23) into $F(u) = u_x^2$ gives

$$F(u) = (u_{0_x} + u_{1_x} + u_{2_x} + u_{3_x} + u_{4_x} + \cdots)^2. \tag{8.24}$$

Squaring the right side gives

$$F(u) = u_{0_x}^2 + 2u_{0_x}u_{1_x} + 2u_{0_x}u_{2_x} + u_{1_x}^2 + 2u_{0_x}u_{3_x} + 2u_{1_x}u_{2_x} + \cdots. \tag{8.25}$$

Grouping the terms as discussed above we find

$$\begin{aligned} F(u) = {} & \underbrace{u_{0_x}^2}_{A_0} + \underbrace{2u_{0_x}u_{1_x}}_{A_1} + \underbrace{2u_{0_x}u_{2_x} + u_{1_x}^2}_{A_2} \\ & + \underbrace{2u_{0_x}u_{3_x} + 2u_{1_x}u_{2_x}}_{A_3} + \underbrace{u_{2_x}^2 + 2u_{0_x}u_{4_x} + 2u_{1_x}u_{3_x}}_{A_4} + \cdots. \end{aligned} \tag{8.26}$$

Adomian polynomials are given by

$$A_0 = u_{0_x}^2,$$
$$A_1 = 2u_{0_x}u_{1_x},$$
$$A_2 = 2u_{0_x}u_{2_x} + u_{1_x}^2,$$
$$A_3 = 2u_{0_x}u_{3_x} + 2u_{1_x}u_{2_x},$$
$$A_4 = 2u_{0_x}u_{4_x} + 2u_{1_x}u_{3_x} + u_{2_x}^2.$$

Case 2. $F(u) = uu_x$

We first set

$$\begin{aligned} u &= \sum_{n=0}^{\infty} u_n, \\ u_x &= \sum_{n=0}^{\infty} u_{n_x}. \end{aligned} \tag{8.27}$$

Substituting (8.27) into $F(u) = uu_x$ yields

$$\begin{aligned} F(u) = {} & (u_0 + u_1 + u_2 + u_3 + u_4 + \cdots)\times \\ & (u_{0_x} + u_{1_x} + u_{2_x} + u_{3_x} + u_{4_x} + \cdots). \end{aligned} \tag{8.28}$$

Multiplying the two factors gives

$$\begin{aligned} F(u) = {} & u_0 u_{0_x} + u_{0_x} u_1 + u_0 u_{1_x} + u_{0_x} u_2 + u_{1_x} u_1 + u_{2_x} u_0 + u_{0_x} u_3 \\ & + u_{1_x} u_2 + u_{2_x} u_1 + u_{3_x} u_0 + u_{0_x} u_4 + u_0 u_{4_x} + u_{1_x} u_3 \\ & + u_1 u_{3_x} + u_2 u_{2_x} + \cdots . \end{aligned} \tag{8.29}$$

Proceeding with grouping the terms we obtain

$$\begin{aligned} F(u) = {} & \underbrace{u_{0_x} u_0}_{A_0} + \underbrace{u_{0_x} u_1 + u_{1_x} u_0}_{A_1} + \underbrace{u_{0_x} u_2 + u_{1_x} u_1 + u_{2_x} u_0}_{A_2} \\ & + \underbrace{u_{0_x} u_3 + u_{1_x} u_2 + u_{2_x} u_1 + u_{3_x} u_0}_{A_3} \\ & + \underbrace{u_{0_x} u_4 + u_{1_x} u_3 + u_{2_x} u_2 + u_{3_x} u_1 + u_{4_x} u_0}_{A_4} + \cdots . \end{aligned} \tag{8.30}$$

It then follows that Adomian polynomials are given by

$$\begin{aligned} A_0 &= u_{0_x} u_0, \\ A_1 &= u_{0_x} u_1 + u_{1_x} u_0, \\ A_2 &= u_{0_x} u_2 + u_{1_x} u_1 + u_{2_x} u_0, \\ A_3 &= u_{0_x} u_3 + u_{1_x} u_2 + u_{2_x} u_1 + u_{3_x} u_0, \\ A_4 &= u_{0_x} u_4 + u_{1_x} u_3 + u_{2_x} u_2 + u_{3_x} u_1 + u_{4_x} u_0. \end{aligned}$$

III. Trigonometric Nonlinearity

Case 1. $F(u) = \sin u$

Note that algebraic operations cannot be applied here. Therefore, our main aim is to separate $A_0 = F(u_0)$ from other terms. To achieve this goal, we first substitute

$$u = \sum_{n=0}^{\infty} u_n, \tag{8.31}$$

into $F(u) = \sin u$ to obtain

$$F(u) = \sin[u_0 + (u_1 + u_2 + u_3 + u_4 + \cdots)]. \tag{8.32}$$

To calculate A_0, recall the trigonometric identity

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi. \tag{8.33}$$

Accordingly, Equation (8.32) becomes

$$\begin{aligned} F(u) = {} & \sin u_0 \cos(u_1 + u_2 + u_3 + u_4 + \cdots) \\ & + \cos u_0 \sin(u_1 + u_2 + u_3 + u_4 + \cdots). \end{aligned} \tag{8.34}$$

Separating $F(u_0) = \sin u_0$ from other factors and using Taylor expansions for $\cos(u_1 + u_2 \cdots)$ and $\sin(u_1 + u_2 + \cdots)$ give

$$\begin{aligned} F(u) = \sin u_0 \left(1 - \frac{1}{2!}(u_1 + u_2 + \cdots)^2 + \frac{1}{4!}(u_1 + u_2 + \cdots)^4 - \cdots\right) \\ + \cos u_0 \left((u_1 + u_2 + \cdots) - \frac{1}{3!}(u_1 + u_2 + \cdots)^3 + \cdots\right), \end{aligned} \tag{8.35}$$

so that

$$\begin{aligned} F(u) = \sin u_0 \left(1 - \frac{1}{2!}(u_1^2 + 2u_1u_2 + \cdots)\right) \\ + \cos u_0 \left((u_1 + u_2 + \cdots) - \frac{1}{3!}u_1^3 + \cdots\right). \end{aligned} \tag{8.36}$$

Note that we expanded the algebraic terms; then few terms of each expansion are listed. The last expansion can be rearranged by grouping all terms with the same sum of subscripts. This means that Eq. (8.36) can be rewritten in the form

$$\begin{aligned} F(u) = \underbrace{\sin u_0}_{A_0} + \underbrace{u_1 \cos u_0}_{A_1} + \underbrace{u_2 \cos u_0 - \frac{1}{2!}u_1^2 \sin u_0}_{A_2} \\ + \underbrace{u_3 \cos u_0 - u_1u_2 \sin u_0 - \frac{1}{3!}u_1^3 \cos u_0}_{A_3} + \cdots \end{aligned} \tag{8.37}$$

Case 2. $F(u) = \cos u$

Proceeding as before we obtain

$$\begin{aligned} F(u) = \underbrace{\cos u_0}_{A_0} - \underbrace{u_1 \sin u_0}_{A_1} + \underbrace{(-u_2 \sin u_0 - \frac{1}{2!}u_1^2 \cos u_0)}_{A_2} \\ + \underbrace{(-u_3 \sin u_0 - u_1u_2 \cos u_0 + \frac{1}{3!}u_1^3 \sin u_0)}_{A_3} + \cdots \end{aligned} \tag{8.38}$$

IV. Hyperbolic Nonlinearity

Case 1. $F(u) = \sinh u$

To calculate the A_n polynomials for $F(u) = \sinh u$, we first substitute

$$u = \sum_{n=0}^{\infty} u_n, \tag{8.39}$$

into $F(u) = \sinh u$ to obtain

$$F(u) = \sinh(u_0 + (u_1 + u_2 + u_3 + u_4 + \cdots)). \tag{8.40}$$

To calculate A_0, recall the hyperbolic identity

$$\sinh(\theta + \phi) = \sinh\theta\cosh\phi + \cosh\theta\sinh\phi. \tag{8.41}$$

Accordingly, Eq. (8.40) becomes

$$\begin{aligned} F(u) = {} & \sinh u_0 \cosh(u_1 + u_2 + u_3 + u_4 + \cdots) \\ & + \cosh u_0 \sinh(u_1 + u_2 + u_3 + u_4 + \cdots). \end{aligned} \tag{8.42}$$

Separating $F(u_0) = \sinh u_0$ from other factors and using Taylor expansions for $\cosh(u_1 + u_2 + \cdots)$ and $\sinh(u_1 + u_2 + \cdots)$ give

$$\begin{aligned} F(u) = {} & \sinh u_0 \\ & \times\left(1 + \frac{1}{2!}(u_1 + u_2 + \cdots)^2 + \frac{1}{4!}(u_1 + u_2 + \cdots)^4 + \cdots\right) \\ & + \cosh u_0\left((u_1 + u_2 + \cdots) + \frac{1}{3!}(u_1 + u_2 + \cdots)^3 + \cdots\right) \\ = {} & \sinh u_0\left(1 + \frac{1}{2!}(u_1^2 + 2u_1u_2 + \cdots)\right) \\ & + \cosh u_0\left((u_1 + u_2 + \cdots) + \frac{1}{3!}u_1^3 + \cdots\right). \end{aligned}$$

By grouping all terms with the same sum of subscripts we find

$$\begin{aligned} F(u) = {} & \underbrace{\sinh u_0}_{A_0} + \underbrace{u_1 \cosh u_0}_{A_1} + \underbrace{u_2 \cosh u_0 + \frac{1}{2!}u_1^2 \sinh u_0}_{A_2} \\ & + \underbrace{u_3 \cosh u_0 + u_1 u_2 \sinh u_0 + \frac{1}{3!}u_1^3 \cosh u_0}_{A_3} + \cdots. \end{aligned} \tag{8.43}$$

Case 2. $F(u) = \cosh u$

Proceeding as in $\sinh x$ we find

$$\begin{aligned} F(u) = {} & \underbrace{\cosh u_0}_{A_0} + \underbrace{u_1 \sinh u_0}_{A_1} + \underbrace{u_2 \sinh u_0 + \frac{1}{2!}u_1^2 \cosh u_0}_{A_2} \\ & + \underbrace{u_3 \sinh u_0 + u_1 u_2 \cosh u_0 + \frac{1}{3!}u_1^3 \sinh u_0}_{A_3} + \cdots. \end{aligned} \tag{8.44}$$

V. Exponential Nonlinearity

Case 1. $F(u) = e^u$

Substituting

$$u = \sum_{n=0}^{\infty} u_n, \tag{8.45}$$

into $F(u) = e^u$ gives

$$F(u) = e^{(u_0+u_1+u_2+u_3+\cdots)}, \tag{8.46}$$

or equivalently

$$F(u) = e^{u_0} e^{(u_1+u_2+u_3+\cdots)}. \tag{8.47}$$

Keeping the term e^{u_0} and using the Taylor expansion for the other factor we obtain

$$F(u) = e^{u_0} \times \left(1 + (u_1+u_2+u_3+\cdots) + \frac{1}{2!}(u_1+u_2+u_3+\cdots)^2 + \cdots\right). \tag{8.48}$$

By grouping all terms with identical sum of subscripts we find

$$\begin{aligned} F(u) = {} & \underbrace{e^{u_0}}_{A_0} + \underbrace{u_1 e^{u_0}}_{A_1} + \underbrace{(u_2 + \frac{1}{2!}u_1^2)e^{u_0}}_{A_2} + \underbrace{(u_3 + u_1 u_2 + \frac{1}{3!}u_1^3)e^{u_0}}_{A_3} \\ & + \underbrace{(u_4 + u_1 u_3 + \frac{1}{2!}u_2^2 + \frac{1}{2!}u_1^2 u_2 + \frac{1}{4!}u_1^4)e^{u_0}}_{A_4} + \cdots. \end{aligned} \tag{8.49}$$

Case 2. $F(u) = e^{-u}$

Proceeding as before we find

$$\begin{aligned} F(u) = {} & \underbrace{e^{-u_0}}_{A_0} + \underbrace{(-u_1)e^{-u_0}}_{A_1} + \underbrace{(-u_2 + \frac{1}{2!}u_1^2)e^{-u_0}}_{A_2} \\ & + \underbrace{(-u_3 + u_1 u_2 - \frac{1}{3!}u_1^3)e^{-u_0}}_{A_3} \\ & + \underbrace{(-u_4 + u_1 u_3 + \frac{1}{2!}u_2^2 - \frac{1}{2!}u_1^2 u_2 + \frac{1}{4!}u_1^4)e^{-u_0}}_{A_4} + \cdots. \end{aligned} \tag{8.50}$$

VI. Logarithmic Nonlinearity

Case 1. $F(u) = \ln u, u > 0$

Substituting

$$u = \sum_{n=0}^{\infty} u_n, \tag{8.51}$$

into $F(u) = \ln u$ gives

$$F(u) = \ln(u_0 + u_1 + u_2 + u_3 + \cdots). \tag{8.52}$$

Equation (8.52) can be written as

$$F(u) = \ln\left(u_0\left(1 + \frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \cdots\right)\right). \tag{8.53}$$

Using the fact that $\ln(\alpha\beta) = \ln\alpha + \ln\beta$, Equation (8.53) becomes

$$F(u) = \ln u_0 + \ln\left(1 + \frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \cdots\right). \tag{8.54}$$

Separating $F(u_0) = \ln u_0$ and using the Taylor expansion for the remaining term we obtain

$$\begin{aligned} F(u) = \ln u_0 + \Bigg\{ &\left(\frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \cdots\right) - \frac{1}{2}\left(\frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \cdots\right)^2 \\ &+ \frac{1}{3}\left(\frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \cdots\right)^3 - \frac{1}{4}\left(\frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \cdots\right)^4 + \cdots \Bigg\}. \end{aligned} \tag{8.55}$$

Proceeding as before, Equation (8.55) can be written as

$$F(u) = \underbrace{\ln u_0}_{A_0} + \underbrace{\frac{u_1}{u_0}}_{A_1} + \underbrace{\frac{u_2}{u_0} - \frac{1}{2}\frac{u_1^2}{u_0^2}}_{A_2} + \underbrace{\frac{u_3}{u_0} - \frac{u_1 u_2}{u_0^2} + \frac{1}{3}\frac{u_1^3}{u_0^3}}_{A_3} + \cdots. \tag{8.56}$$

Case 2. $F(u) = \ln(1+u), -1 < u \leqslant 1$

In a like manner we obtain

$$\begin{aligned} F(u) = &\underbrace{\ln(1+u_0)}_{A_0} + \underbrace{\frac{u_1}{1+u_0}}_{A_1} + \underbrace{\frac{u_2}{1+u_0} - \frac{1}{2}\frac{u_1^2}{(1+u_0)^2}}_{A_2} \\ &+ \underbrace{\frac{u_3}{1+u_0} - \frac{u_1 u_2}{(1+u_0)^2} + \frac{1}{3}\frac{u_1^3}{(1+u_0)^3}}_{A_3} + \cdots. \end{aligned} \tag{8.57}$$

As stated before, there are other methods that can be used to evaluate Adomian polynomials. However, these methods suffer from the huge size of calculations. For this reason, the most commonly used methods are presented in this text.