Lecture 31

Definition of a Partial Differential Equation (PDE)

A partial differential equation (PDE) is an equation that contains the dependent variable (the unknown function), and its partial derivatives. As in the ordinary differential equations (ODEs), the dependent variable $u = u(x)$ depends only on one independent variable x. However in the PDEs, the dependent variable, such as $u = u(x, t)$ or $u = u(x, y, t)$, must depend on more than one independent variable.

If $u = u(x, t)$, then the function u depends on the independent variable x, and on the time variable t. However, if $u = u(x, y, t)$, then the function u depends on the space variables x, y, and on the time variable t.

Examples

1. The heat equation
   $u_t = ku_{xx}$,
   $u_t = k(u_{xx} + u_{yy})$,
   in one dimensional space and two dimensional space respectively. The dependent variable $u = u(x, t)$ in first equation depends on the position x and on the time variable t. However, in second equation $u = u(x, y, t)$ depends on three independent variables, the space variables x, y and the time variable t.

2. The wave equations
   $u_{tt} = c^2(u_{xx} + u_{yy})$,
   $u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz})$,
   in two dimensional space and three dimensional space respectively.

3. The Laplace equation is
   $u_{xx} + u_{yy} = 0$,
   $u_{xx} + u_{yy} + u_{zz} = 0$.

Order of a PDE

The order of a PDE represents the order of the highest partial derivative that appears in the equation. For example, the following equations

$$u_t = u_{xx},$$
$$u_y - uu_{xxx} = 0,$$

are PDEs of second order, and third order respectively.

Example 1. Find the order of the following PDEs:

a. $u_t = u_{xx} + u_{yy}$
b. $u^4 u_{xx} + u_{xxy} = 0$

Solution

a. The highest partial derivative in this equation is u_{xx} or u_{yy}. The PDE is therefore of order two.
b. The highest partial derivative in this equation is u_{xxy}. The PDE is therefore of order three.

Linear and Nonlinear PDEs

Partial differential equations are classified as linear or nonlinear.

- A partial differential equation is called linear if
  1. the power of the dependent variable u and each partial derivative contained in the equation is one, and
  2. the coefficients of the dependent variable and the coefficients of each partial derivative are constants or independent variables.

- However, if any of these conditions is not satisfied, the equation is called nonlinear PDE.

Example 2. Classify the following PDEs as linear or nonlinear

a. $xu_{xx} + yu_{yy} = 0$
b. $u_x + \sqrt{u} = x$
c. $uu_t + xu_x = 0$

Solution

a. Here the power of each partial derivative u_{xx} and u_{yy} is one. Also, the coefficients of the partial derivatives are the independent variables x and y respectively. Hence, the PDE is linear.
b. The equation is nonlinear because of the term $\sqrt{u}$, as its power is ½.
c. In this case the power of each partial derivative is one, but u_t has the dependent variable u as its coefficient. Therefore, the PDE is nonlinear.

Homogeneous and Inhomogeneous PDEs

A partial differential equation of any order is called homogeneous if every term of the PDE contains the dependent variable u or one of its derivatives, otherwise, it is called an inhomogeneous PDE.

You can also say that a PDE is called homogeneous if the equation does not contain a term independent of the unknown function and its derivatives.

Example 3. Classify the following partial differential equations as homogeneous or inhomogeneous.

a. $u_t + 4u_x = 0$
b. $u_t = u_{xx} + x$
c. $u_x u_{xx} + \left(u_y\right)^2 = 0$

Solution.

a. The terms of the equation contain partial derivatives of u only, therefore it is a linear, homogeneous, 1st order PDE.
b. The equation is an inhomogeneous PDE, as one term contains the independent variable x.
c. This is nonlinear, 2nd order, homogeneous PDE.

Solution of a PDE

A solution of a PDE is a function such that it satisfies the equation under discussion and satisfies the given conditions as well. In other words, for u to satisfy the equation, the left hand side of the PDE and the right hand side should be the same upon substituting the resulting solution.

Example 4. Show that $u(x,y)=\sin x\sin y+x^2$ is a solution of the following PDE.

$u_{xx}=u_{yy}+2.$

Solution.

As

$$u_x = \cos x\sin y+2x$$
$$u_{xx}=-\sin x\sin y+2=L.H.S$$

Now

$$u_y=\sin x\cos y$$
$$u_{yy}=-\sin x\sin y$$
$$u_{yy}+2=-\sin x\sin y+2=R.H.S$$

Hence L.H.S=R.H.S

Example 5. Show that $u(x,\mathrm{t})=\sin xe^{-4t}$ is a solution of the following PDE

$u_t=4u_{xx}.$

Solution.

$$u_t=-4\sin xe^{-4t}=L.H.S$$
$$4u_{xx}=-4\sin xe^{-4t}=R.H.S$$

Boundary Conditions

For a given PDE that controls the mathematical behavior of physical phenomenon in a bounded domain *D*, the dependent variable *u* is usually prescribed at the boundary of the domain *D*. The boundary data is called boundary conditions. There are three types of boundary conditions (BCs) that can occur for heat flow problems. They are

- **Dirichlet Boundary Conditions**
  Consider heat flow problem in a rod ($0\le x\le L$). The specification of the temperatures $u(0,\ t)$ and $u(L,\ t)$ at the ends are classified as Dirichlet type BC. In this case, the function *u* is usually prescribed on the boundary of the bounded domain.

- **Neumann Boundary Conditions**
  The specification of the normal derivative (i.e., $\frac{\partial u}{\partial n}$, where *n* is the outward normal to the boundary) on the boundary is classified as Neumann type BCs. For a rod of length *L*,

Neumann boundary conditions are of the form $u_x(0,t)=\alpha, u_x(\mathrm{L},t)=\beta,$ where α and β are constants.

- **Mixed Boundary Conditions**
  If the condition on the boundary is a mixture of both Dirichlet and Neumann types i.e., a linear combination of the dependent variable u and the normal form $\frac{\partial u}{\partial n}$, then it is called Mixed BCs.
  It is not always necessary for the domain to be bounded, however one or more parts of the boundary may be at infinity.

Initial Conditions

PDEs mostly arise to govern physical phenomenon such as heat distribution, wave propagation and quantum mechanics. Most of the PDEs, such as the diffusion equation and the wave equation, depend on the time t. Accordingly, the initial values of the dependent variable u at the starting time $t = 0$ should be prescribed. For the heat case, the initial value u ($t = 0$), that defines the temperature at the starting time, should be prescribed. For the wave equation, the initial conditions u ($t = 0$) and u_t ($t = 0$) should also be prescribed.

Well-posed PDEs

A partial differential equation is said to be well-posed

- if a unique solution that satisfies the equation and the prescribed conditions exists, and
- Provided that the unique solution obtained is stable. The solution of a PDE is said to be stable if a small change in the conditions or the coefficients of the PDE results in a small change in the solution.

Exercises

1. Find the order of the following PDEs.
   a. $u_{xx}+2xu_{xy}+u_{yy}=e^y$
   b. $u_{xx}=u_{xxx}+u+1$
   c. $u_{xxy}+xu_{yy}+8u=7y$
   d. $u_t+u_{xxyy}=u$

2. Classify the following PDEs as linear or nonlinear.
   a. $yu_{xx} + 2xyu_{yy} + u = 1$
   b. $u_{xx} = u_{tt} - u^2$
   c. $u_t + uu_x + u_{xxx} = 0$
   d. $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$

3. Classify the following PDEs as homogeneous or inhomogeneous.
   a. $u_t = u_{xx} + x$
   b. $u_t + u_{xxy} = u$
   c. $u_x + u_y = u + 4$
   d. $u_{tt} = u_{xx} + u_{yy} + u_{zz}$

4. Verify that the following functions
   $u(x, y) = x^2 - y^2,$
   $u(x, y) = e^x \sin y,$
   $u(x, y) = 2xy$
   are the solutions of the equation $u_{xx} + u_{yy} = 0.$

5. Show that $u = f(xy)$, where f is an arbitrary differentiable function satisfies $xu_x - yu_y = 0$.
   Also verify that the functions $\sin(xy), \cos(xy), \log(xy), e^{xy}$ and $(xy)^3$ are solutions.

Classifications of Second-order PDEs

Classification of PDEs is an important concept because the general theory and methods of solution usually apply only to a given class of equations. Let us first discuss the classification of PDEs involving two independent variables.

1. Classification with two independent variables

Consider the following general second order linear partial differential equation in two independent variables

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F_u = G, \qquad (1)$$

where *A, B, C, D, E, F*, and *G* are constants or functions of the independent variables *x* and *y*. The classification of partial differential equations is suggested by the classification of the quadratic equation of conic sections in analytic geometry. The equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

represents hyperbola, parabola, or ellipse accordingly as the discriminant ($B^2 - 4AC$) is positive, zero, or negative.

Similarly, the nature of the second order partial differential equation (1) is determined by the principal part containing the highest partial derivatives, that is,

$$Lu \equiv Au_{xx} + Bu_{xy} + Cu_{yy}$$

Now depending on the sign of the discriminant, PDE (1) is usually classified into three basic classes of equations,

1. **Parabolic equation** is an equation which satisfies the property
   $B^2 - 4AC = 0,$
2. **Hyperbolic equation** is an equation which satisfies the property
   $B^2 - 4AC > 0,$
3. **Elliptic equation** is an equation which satisfies the property
   $B^2 - 4AC < 0.$

Note. The classification of eq. (1) as parabolic, hyperbolic or elliptic depends only on the coefficients of the second derivatives. It has nothing to do with the first derivative terms, the term in *u*, or the non-homogeneous term.

Example 1. Classify the following equations as parabolic, hyperbolic or elliptic.

a. $u_{xx} + u_{yy} = 0$ (Laplace equation)

b. $u_t = u_{xx}$ (Heat equation)

c. $u_{tt} - u_{xx} = 0$ (Wave equation)

d. $u_{xx} + xu_{yy} = 0;\ x \neq 0$ (Tricomi equation).

Solution.

a. Here A = 1, B = 0, C = 1 and $B^2 - 4AC = -4 < 0.$ Therefore, it is an elliptic equation.

b. Here A = -1, B = 0, C = 0 and $B^2 - 4AC = 0.$ Thus, it is parabolic type equation.

c. Here A = -1, B = 0, C = 1 and $B^2 - 4AC = 4 > 0.$ Hence, it is of hyperbolic type.

d. Here A = 1, B = 0, C = x and $B^2 - 4AC = -4x.$ Therefore, the equation is parabolic if $x = 0$, hyperbolic if $x < 0$, and elliptic if $x > 0$. This example shows that equations with variable coefficients can change form in the different regions of the domain.

2. Classification with more than two independent variables

Consider the second-order PDE in general form

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu + d = 0 \tag{2}$$

where the coefficients a_{ij}, b_i, c and d are functions of $x = (x_1, x_2, \ldots, x_n)$ alone and $u = u(x_1, x_2, \ldots, x_n)$.

Its principal part is

$$L \equiv \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} \tag{3}$$

It is enough to assume that $A = [a_{ij}]$ is symmetric if not, let $\overline{a}_{ij} = \frac{1}{2}\left(a_{ij} + a_{ji}\right)$ and rewrite

$$L \equiv \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{a}_{ij}\frac{\partial^2}{\partial x_i \partial x_j} \tag{4}$$

Note that $\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial x_j \partial x_i}$. As in two-space dimension, let us attach a quadratic form P with (4) (i.e., replacing $\frac{\partial u}{\partial x_i}$ by x_i).

$$P(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j \tag{5}$$

Since A is a real valued symmetric ($a_{ij} = a_{ji}$) matrix, it is diagonalizable with real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. In other words, there exists a corresponding set of orthonormal set of n eigenvectors, say $\sigma_1, \sigma_2, \ldots, \sigma_n$ with $R = [\sigma_1, \sigma_2, \ldots, \sigma_n]$ as column vectors such that

$$R^T AR = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & \bigcirc & \\ & & \cdot & & \\ & & & \cdot & \\ & \bigcirc & & & \cdot & \\ & & & & & \lambda_n \end{bmatrix} = D \tag{6}$$

We now classify (2) depending on sign of eigenvalues of A:

- If one or more of the $\lambda_i = 0$ then PDE (2) is of parabolic type.
- If one of the $\lambda_i > 0$ or $\lambda_i < 0$, and all the remaining have opposite sign then PDE (2) is of hyperbolic type.
- If $\lambda_i > 0\ \forall i$ or $\lambda_i < 0\ \forall i$ then PDE (2) is said to be of elliptic type.

Example 2. Classify the following equation as parabolic, hyperbolic or elliptic.

$$u_{x_1x_1} + 2(1+cx_2)u_{x_2x_3} = 0.$$

Solution. The equation can be rewritten as

$$u_{x_1x_1} + (1+cx_2)u_{x_2x_3} + (1+cx_2)u_{x_3x_2} = 0$$

For i, j=1, 2, 3 the eq(3) becomes

$$\begin{aligned} L &\equiv \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} \\ &= a_{11}\frac{\partial^2 u}{\partial x_1\partial x_1} + a_{12}\frac{\partial^2 u}{\partial x_1\partial x_2} + a_{13}\frac{\partial^2 u}{\partial x_1\partial x_3} + a_{21}\frac{\partial^2 u}{\partial x_2\partial x_1} + a_{22}\frac{\partial^2 u}{\partial x_2\partial x_2} + a_{23}\frac{\partial^2 u}{\partial x_2\partial x_3} \\ &\quad + a_{31}\frac{\partial^2 u}{\partial x_3\partial x_1} + a_{32}\frac{\partial^2 u}{\partial x_3\partial x_2} + a_{33}\frac{\partial^2 u}{\partial x_3\partial x_3} \end{aligned} \tag{7}$$

On comparison of given PDE with eq. (7),

$$\begin{aligned} &a_{11} = 1, a_{12} = 0, a_{13} = 0, \\ &a_{21} = 0, a_{22} = 0, a_{23} = (1 + cx_2), \\ &a_{31} = 0, a_{32} = (1 + cx_2), a_{33} = 0. \end{aligned}$$

Hence $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & (1+cx_2) \\ 0 & (1+cx_2) & 0 \end{bmatrix}$

Now the Eigen values of matrix A are

$$\det(A - \lambda I) = 0$$
$$\Rightarrow \lambda = 1, (1 + cx_2), -(1 + cx_2).$$

The corresponding Eigen vectors and normalized vectors are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ respectively.

$$\Rightarrow R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$R^T A R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1+cx_2) & 0 \\ 0 & 0 & -(1+cx_2) \end{bmatrix}$$

which is a diagonal matrix. Now we classify the given PDE depending on the sign of Eigen values of matrix A.

For c=0, the equation is hyperbolic type.

For $c \neq 0$, equation is

- parabolic if $x_2 = -\frac{1}{c}$,
- hyperbolic if $x_2 > -\frac{1}{c}$ *and* $x_2 < -\frac{1}{c}$.

Exercises

1. Classify the following equations into hyperbolic, elliptic or parabolic type.
   a. $5u_{xx} - 3u_{yy} + (\cos x)u_x + e^y u_y + u = 0.$
   b. $\sin^2 xu_{xx} + \sin^2 xu_{xy} + \cos^2 xu_{yy} = x.$
   c. $e^x u_{xx} + e^y u_{yy} - u = 0.$
   d. $8u_{xx} + u_{yy} - u_x + [\log(2 + x^2)]u = 0.$
   e. $xu_{xx} + u_{yy} = 0.$

2. Classify the following equations into elliptic, parabolic, or hyperbolic type.
   a. $e^z u_{xy} - u_{xx} = \log[x^2 + y^2 + z^2 + 1].$
   b. $u_{xx} + 2u_{yz} + (\cos x)u_z - e^{y^2} u = \cosh z.$
   c. $u_{xx} + 2u_{xy} + u_{yy} + 2u_{zz} - (1 + xy)u = 0.$

3. Determine the regions where $u_{xx} - 2x^2 u_{xz} + u_{yy} + u_{zz} = 0$ is of hyperbolic, elliptic and parabolic type.