# Solutions Manual to Walter Rudin's *Principles of Mathematical Analysis*

Roger Cooke, University of Vermont

# Chapter 1

# The Real and Complex Number Systems

**Exercise 1.1** If $r$ is rational ($r \neq 0$) and $x$ is irrational, prove that $r+x$ and $rx$ are irrational.

*Solution.* If $r$ and $r+x$ were both rational, then $x = r + x - r$ would also be rational. Similarly if $rx$ were rational, then $x = \frac{rx}{r}$ would also be rational.

**Exercise 1.2** Prove that there is no rational number whose square is 12.

*First Solution.* Since $\sqrt{12} = 2\sqrt{3}$, we can invoke the previous problem and prove that $\sqrt{3}$ is irrational. If $m$ and $n$ are integers having no common factor and such that $m^2 = 3n^2$, then $m$ is divisible by 3 (since if $m^2$ is divisible by 3, so is $m$). Let $m = 3k$. Then $m^2 = 9k^2$, and we have $3k^2 = n^2$. It then follows that $n$ is also divisible by 3 contradicting the assumption that $m$ and $n$ have no common factor.

*Second Solution.* Suppose $m^2 = 12n^2$, where $m$ and $n$ have no common factor. It follows that $m$ must be even, and therefore $n$ must be odd. Let $m = 2r$. Then we have $r^2 = 3n^2$, so that $r$ is also odd. Let $r = 2s+1$ and $n = 2t+1$. Then

$$4s^2 + 4s + 1 = 3(4t^2 + 4t + 1) = 12t^2 + 12t + 3,$$

so that

$$4(s^2 + s - 3t^2 - 3t) = 2.$$

But this is absurd, since 2 cannot be a multiple of 4.

**Exercise 1.3** Prove Proposition 1.15, i.e., prove the following statements:

(*a*) *If* $x \neq 0$ *and* $xy = xz$, *then* $y = z$.
(*b*) *If* $x \neq 0$ *and* $xy = x$, *then* $y = 1$.
(*c*) *If* $x \neq 0$ *and* $xy = 1$, *then* $y = 1/x$.
(*d*) *If* $x \neq 0$, *then* $1/(1/x) = x$.

*Solution.* (*a*) Suppose $x \neq 0$ and $xy = xz$. By Axiom (M5) there exists an element $1/x$ such that $1/x = 1$. By (M3) and (M4) we have $(1/x)(xy) = ((1/x)x)y = 1y = y$, and similarly $(1/x)(xz) = z$. Hence $y = z$.

(*b*) Apply (*a*) with $z = 1$.

(*c*) Apply (*a*) with $z = 1/x$.

(*d*) Apply (*a*) with $x$ replaced by $1/x$, $y = 1/(1/x)$, and $z = x$.

**Exercise 1.4** Let $E$ be a nonempty subset of an ordered set; suppose $\alpha$ is a lower bound of $E$, and $\beta$ is an upper bound of $E$. Prove that $\alpha \leq \beta$.

*Solution.* Since $E$ is nonempty, there exists $x \in E$. Then by definition of lower and upper bounds we have $\alpha \leq x \leq \beta$, and hence by property *ii* in the definition of an ordering, we have $\alpha < \beta$ unless $\alpha = x = \beta$.

**Exercise 1.5** Let $A$ be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

Solution: We need to prove that $-\sup(-A)$ is the greatest lower bound of $A$. For brevity, let $\alpha = -\sup(-A)$. We need to show that $\alpha \leq x$ for all $x \in A$ and $\alpha \geq \beta$ if $\beta$ is any lower bound of $A$.

Suppose $x \in A$. Then, $-x \in -A$, and, hence $-x \leq \sup(-A)$. It follows that $x \geq -\sup(-A)$, i.e., $\alpha \leq x$. Thus $\alpha$ is a lower bound of $A$.

Now let $\beta$ be any lower bound of $A$. This means $\beta \leq x$ for all $x$ in $A$. Hence $-x \leq -\beta$ for all $x \in A$, which says $y \leq -\beta$ for all $y \in -A$. This means $-\beta$ is an upper bound of $-A$. Hence $-\beta \geq \sup(-A)$ by definition of sup, i.e., $\beta \leq -\sup(-A)$, and so $-\sup(-A)$ is the greatest lower bound of $A$.

**Exercise 1.6** Fix $b > 1$.

(*a*) If $m$, $n$, $p$, $q$ are integers, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$.

(*b*) Prove that $b^{r+s} = b^r b^s$ if $r$ and $s$ are rational.

($c$) If $x$ is real, define $B(x)$ to be the set of all numbers $b^t$, where $t$ is rational and $t \le x$. Prove that

$$b^r = \sup B(r)$$

when $r$ is rational. Hence it makes sense to define

$$b^x = \sup B(x)$$

for every real $x$.

($d$) Prove that $b^{x+y} = b^x b^y$ for all real $x$ and $y$.

*Solution.* ($a$) Let $k = mq = np$. Since there is only one positive real number $c$ such that $c^{nq} = b^k$ (Theorem 1.21), if we prove that both $(b^m)^{1/n}$ and $(b^p)^{1/q}$ have this property, it will follow that they are equal. The proof is then a routine computation: $\big((b^m)^{1/n}\big)^{nq} = (b^m)^q = b^{mq} = b^k$, and similarly for $(b^p)^{1/q}$.

($b$) Let $r = \frac{m}{n}$ and $s = \frac{v}{w}$. Then $r + s = \frac{mw+vn}{nw}$, and

$$b^{r+s} = (b^{mw+vn})^{1/nw} = \big((b^{mw}b^{vn})\big)^{1/nw},$$

by the laws of exponents for integer exponents. By the corollary to Theorem 1.21 we then have

$$b^{r+s} = (b^{mw})^{1/nw}(b^{nv})^{1/nw} = b^r b^s,$$

where the last equality follows from part ($a$).

($c$) It will simplify things later on if we amend the definition of $B(x)$ slightly, by defining it as $\{b^t : t \text{ rational}, t < x\}$. It is then slightly more difficult to prove that $b^r = \sup B(r)$ if $r$ is rational, but the technique of Problem 7 comes to our rescue. Here is how: It is obvious that $b^r$ is an upper bound of $B(r)$. We need to show that it is the least upper bound. The inequality $b^{1/n} < t$ if $n > (b-1)/(t-1)$ is proved just as in Problem 7 below. It follows that if $0 < x < b^r$, there exists an integer $n$ with $b^{1/n} < b^r/x$, i.e., $x < b^{r-1/n} \in B(r)$. Hence $x$ is not an upper bound of $B(r)$, and so $b^r$ is the least upper bound.

($d$) By definition $b^{x+y} = \sup B(x+y)$, where $B(x+y)$ is the set of all numbers $b^t$ with $t$ rational and $t < x+y$. Now any rational number $t$ that is less than $x+y$ can be written as $r+s$, where $r$ and $s$ are rational, $r < x$, and $s < y$. To do this, let $r$ be any rational number satisfying $t - y < r < x$, and let $s = t - r$. Conversely any pair of rational numbers $r$, $s$ with $r < x$, $s < y$ gives a rational sum $t = r + s < x + y$. Hence $B(x+y)$ can be described as the set of all numbers $b^r b^s$ with $r < x$, $s < y$, and $r$ and $s$ rational, i.e., $B(x+y)$ is the set of all products $uv$, where $u \in B(x)$ and $v \in B(y)$.

Since any such product is less than $\sup B(x) \sup B(y)$, we see that the number $M = \sup B(x) \sup B(y)$ is an upper bound for $B(x+y)$. On the other hand, suppose $0 < c < \sup B(x) \sup B(y)$. Then $c/(\sup B(x)) < \sup B(y)$. Let $m = (1/2)\big(c/(\sup B(x)) + \sup B(y)\big)$. Then $c/\sup B(x) < m < \sup B(y)$, and there exist $u \in B(x)$, $v \in B(y)$ such that $c/m < u$ and $m < v$. Hence we have

$c = (c/m)m < uv \in B(x+y)$, and so $c$ is not an upper bound for $B(x+y)$. It follows that $\sup B(x) \sup B(y)$ is the least upper bound of $B(x+y)$, i.e.,

$$b^{x+y} = b^x b^y,$$

as required.

**Exercise 1.7** Fix $b > 1$, $y > 0$, and prove that there is a unique real $x$ such that $b^x = y$, by completing the following outline. (This $x$ is called the *logarithm of $y$ to the base $b$.*)

($a$) For any positive integer $n$, $b^n - 1 \geq n(b-1)$.

($b$) Hence $b - 1 \geq n(b^{1/n} - 1)$.

($c$) If $t > 1$ and $n > (b-1)/(t-1)$, then $b^{1/n} < t$.

($d$) If $w$ is such that $b^w < y$, then $b^{w+(1/n)} < y$ for sufficiently large $n$; to see this apply part ($c$) with $t = y \cdot b^{-w}$.

($e$) If $b^w > y$, then $b^{w-(1/n)} > y$ for sufficiently large $n$.

($f$) Let $A$ be the set of all $w$ such that $b^w < y$, and show that $x = \sup A$ satisfies $b^w = y$.

($g$) Prove that this $x$ is unique.

*Solution.* ($a$) The inequality $b^n - 1 \geq n(b-1)$ is equality if $n = 1$. Then, by induction $b^{n+1} - 1 = b^{n+1} - b + (b-1) = b(b^n - 1) + (b-1) \geq bn(b-1) + (b-1) = (bn+1)(b-1) \geq (n+1)(b-1)$.

($b$) Replace $b$ by $b^{1/n}$ in part ($a$).

($c$) The inequality $n > (b-1)/(t-1)$ can be rewritten as $n(t-1) > (b-1)$, and since $b - 1 \geq n(b^{1/n} - 1)$, we have $n(t-1) > n(b^{1/n} - 1)$, which implies $t > b^{1/n}$.

($d$) The application of part ($c$) with $t = y \cdot b^{-w} > 1$ is immediate.

($e$) The application of part ($c$) with $t = b^w \cdot (1/y)$ yields the result, as in part ($d$) above.

($f$) There are only three possibilities for the number $x = \sup A$: 1) $b^x < y$; 2) $b^x > y$; 3) $b^x = y$. The first assumption, by part ($d$), implies that $x + (1/n) \in A$ for large $n$, contradicting the assumption that $x$ is an upper bound for $A$. The second, by part ($e$), implies that $x - (1/n)$ is an upper bound for $A$ if $n$ is large, contradicting the assumption that $x$ is the smallest upper bound. Hence the only remaining possibility is that $b^x = y$.

($g$) Suppose $z \neq x$, say $z > x$. Then $b^z = b^{x+(z-x)} = b^x b^{z-x} > b^x = y$. Hence $x$ is unique. (It is easy to see that $b^w > 1$ if $w > 0$, since there is a positive rational number $r = \frac{m}{n}$ with $0 < r < w$, and $b^r = (b^m)^{1/n}$. Then $b^m > 1$ since $b > 1$, and $(b^m)^{1/n} > 1$ since $1^n = 1 < b^m$.)

**Exercise 1.8** Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint:* $-1$ is a square.

*Solution.* By Part ($a$) of Proposition 1.18, either $i$ or $-i$ must be positive. Hence $-1 = i^2 = (-i)^2$ must be positive. But then $1 = (-1)^2$, must also be positive, and this contradicts Part ($a$) of Proposition 1.18, since 1 and $-1$ cannot both be positive.

**Exercise 1.9** Suppose $z = a + bi$, $w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of order relation is called a *dictionary order*, or *lexicographic order*, for obvious reasons.) Does this ordered set have the least upper bound property?

*Solution.* We need to show that either $z < w$ or $z = w$, or $w < z$. Now since the *real* numbers are ordered, we have $a < c$ or $a = c$, or $c < a$. In the first case $z < w$; in the third case $w < z$. Now consider the second case. We must have $b < d$ or $b = d$ or $d < b$. In the first of these cases $z < w$, in the third case $w < z$, and in the second case $z = w$.

We also need to show that if $z < w$ and $w < u$, then $z < u$. Let $u = e + fi$. Since $z < w$, we have either $a < c$ or $a = c$ and $b < d$. Since $w < u$ we have either $c < f$ or $c = f$ and $d < g$. Hence there are four possible cases:

Case 1: $a < c$ and $c < f$. Then $a < f$ and so $z < u$, as required.

Case 2: $a < c$ and $c = f$ and $d < g$. Again $a < f$, and $z < u$.

Case 3: $a = c$ and $b < d$ and $c < f$. Once again $a < f$ and so $z < u$.

Case 4: $a = c$ and $b < d$ and $c = f$, and $d < g$. Then $a = f$ and $b < g$, and so $z < u$.

**Exercise 1.10** Suppose $z = a + bi$, $w = u + iv$, and

$$a = \left(\frac{|w| + u}{2}\right)^{1/2}, \quad b = \left(\frac{|w| - u}{2}\right)^{1/2}.$$

Prove that $z^2 = w$ if $v \geq 0$ and that $(\bar{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception) has two complex square roots.

*Solution.*

$$z^2 = (a + bi)^2 = (a^2 - b^2) + 2abi.$$

Now

$$a^2 - b^2 = \frac{|w| + u}{2} - \frac{|w| - u}{2} = u,$$

and, since $(xy)^{1/2} = x^{1/2}y^{1/2}$,

$$2ab = 2\left(\frac{|w| + u}{2}\frac{|w| - u}{2}\right)^{1/2} = 2\left(\frac{|w|^2 - u^2}{4}\right)^{1/2}.$$

Hence

$$2ab = 2\left(\left(\frac{v}{2}\right)^2\right)^{1/2}$$

Now $(x^2)^{1/2} = x$ if $x \geq 0$ and $(x^2)^{1/2} = -x$ if $x \leq 0$. We conclude that $2ab = v$ if $v \geq 0$ and $2ab = -v$ if $v \leq 0$. Hence $z^2 = w$ if $v \geq 0$. Replacing $b$ by $-b$, we find that $(\bar{z})^2 = w$ if $v \leq 0$.

Hence every non-zero complex number has (at least) two complex square roots.

**Exercise 1.11** If $z$ is a complex number, prove that there exists an $r \geq 0$ and a complex number $w$ with $|w| = 1$ such that $z = rw$. Are $w$ and $r$ always uniquely determined by $z$?

*Solution.* If $z = 0$, we take $r = 0$, $w = 1$. (In this case $w$ is not unique.) Otherwise we take $r = |z|$ and $w = z/|z|$, and these choices are unique, since if $z = rw$, we must have $r = r|w| = |rw| = |z|$, $z/r$.

**Exercise 1.12** If $z_1, \ldots, z_n$ are complex, prove that

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

*Solution.* The case $n = 2$ is Part (e) of Theorem 1.33. We can then apply this result and induction on $n$ to get

$$\begin{aligned} |z_1 + z_2 + \cdots z_n| &= |(z_1 + z_2 + \cdots + z_{n-1}) + z_n| \\ &\leq |z_1 + z_2 + \cdots + z_{n-1}| + |z_n| \\ &\leq |z_1| + |z_2| + \cdots + |z_{n-1}| + |z_n|. \end{aligned}$$

**Exercise 1.13** If $x$, $y$ are complex, prove that

$$\big||x| - |y|\big| \leq |x - y|.$$

*Solution.* Since $x = x - y + y$, the triangle inequality gives

$$|x| \leq |x - y| + |y|,$$

so that $|x| - |y| \leq |x - y|$. Similarly $|y| - |x| \leq |x - y|$. Since $|x| - |y|$ is a real number we have either $\big||x| - |y|\big| = |x| - |y|$ or $\big||x| - |y|\big| = |y| - |x|$. In either case, we have shown that $\big||x| - |y|\big| \leq |x - y|$.

**Exercise 1.14** If $z$ is a complex number such that $|z| = 1$, that is, such that $z\bar{z} = 1$, compute

$$|1+z|^2 + |1-z|^2.$$

*Solution.* $|1+z|^2 = (1+z)(1+\bar{z}) = 1 + \bar{z} + z + z\bar{z} = 2 + z + \bar{z}$. Similarly $|1-z|^2 = (1-z)(1-\bar{z}) = 1 - z - \bar{z} + z\bar{z} = 2 - z - \bar{z}$. Hence

$$|1+z|^2 + |1-z|^2 = 4.$$

**Exercise 1.15** Under what conditions does equality hold in the Schwarz inequality?

*Solution.* The proof of Theorem 1.35 shows that equality can hold if $B = 0$ or if $Ba_j - Cb_j = 0$ for all $j$, i.e., the numbers $a_j$ are proportional to the numbers $b_j$. (In terms of linear algebra this means the vectors $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ in complex $n$-dimensional space are linearly dependent. Conversely, if these vectors are linearly independent, then strict inequality holds.)

**Exercise 1.16** Suppose $k \geq 3$, $\mathbf{x}, \mathbf{y} \in R^k$, $|\mathbf{x} - \mathbf{y}| = d > 0$, and $r > 0$. Prove:

(*a*) If $2r > d$, there are infinitely many $\mathbf{z} \in R^k$ such that

$$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r.$$

(*b*) If $2r = d$, there is exactly one such $\mathbf{z}$.

(*c*) If $2r < d$, there is no such $\mathbf{z}$.

How must these statements be modified if $k$ is 2 or 1?

*Solution.* (*a*) Let $\mathbf{w}$ be any vector satisfying the following two equations:

$$\begin{aligned} \mathbf{w} \cdot (\mathbf{x} - \mathbf{y}) &= 0, \\ |\mathbf{w}|^2 &= r^2 - \frac{d^2}{4}. \end{aligned}$$

From linear algebra it is known that all but one of the components of a solution $\mathbf{w}$ of the first equation can be arbitrary. The remaining component is then uniquely determined. Also, if $\mathbf{w}$ is any non-zero solution of the first equation, there is a unique positive number $t$ such that $t\mathbf{w}$ satisfies both equations. (For example, if $x_1 \neq y_1$, the first equation is satisfied whenever

$$z_1 = \frac{z_2(x_2 - y_2) + \cdots + z_k(x_k - y_k)}{y_1 - x_1}.$$

If $(z_1, z_2, \ldots, z_k)$ satisfies this equation, so does $(tz_1, tz_2, \ldots, tz_k)$ for any real number $t$.) Since at least two of these components can vary independently, we can find a solution with these components having any prescribed ratio. This ratio does not change when we multiply by the positive number $t$ to obtain a solution of both equations. Since there are infinitely many ratios, there are infinitely many distinct solutions. For each such solution $\mathbf{w}$ the vector $\mathbf{z} = \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y} + \mathbf{w}$ is a solution of the required equation. For

$$\begin{aligned}
|\mathbf{z}-\mathbf{x}|^2 &= \left|\frac{\mathbf{y}-\mathbf{x}}{2} + \mathbf{w}\right|^2 \\
&= \left|\frac{\mathbf{y}-\mathbf{x}}{2}\right|^2 + 2\mathbf{w}\cdot\frac{\mathbf{x}-\mathbf{y}}{2} + |\mathbf{w}|^2 \\
&= \frac{d^2}{4} + 0 + r^2 - \frac{d^2}{4} \\
&= r^2,
\end{aligned}$$

and a similar relation holds for $|\mathbf{z}-\mathbf{y}|^2$.

($b$) The proof of the triangle inequality shows that equality can hold in this inequality only if it holds in the Schwarz inequality, i.e., one of the two vectors is a scalar multiple of the other. Further examination of the proof shows that the scalar must be nonnegative. Now the conditions of this part of the problem show that

$$|\mathbf{x}-\mathbf{y}| = d = |\mathbf{x}-\mathbf{z}| + |\mathbf{z}-\mathbf{y}|.$$

Hence it follows that there is a nonnegative scalar $t$ such that

$$\mathbf{x}-\mathbf{z} = t(\mathbf{z}-\mathbf{y}).$$

However, the hypothesis also shows immediately that $t = 1$, and so $\mathbf{z}$ is uniquely determined as

$$\mathbf{z} = \frac{\mathbf{x}+\mathbf{y}}{2}.$$

($c$) If $\mathbf{z}$ were to satisfy this condition, the triangle inequality would be violated, i.e., we would have

$$|\mathbf{x}-\mathbf{y}| = d > 2r = |\mathbf{x}-\mathbf{z}| + |\mathbf{z}-\mathbf{y}|.$$

When $k = 2$, there are precisely 2 solutions in case ($a$). When $k = 1$, there are no solutions in case ($a$). The conclusions in cases ($b$) and ($c$) do not require modification.

**Exercise 1.17** Prove that

$$|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

if $\mathbf{x} \in R^k$ and $\mathbf{y} \in R^k$. Interpret this geometrically as a statement about parallelograms.

*Solution.* The proof is a routine computation, using the relation

$$|\mathbf{x}\pm\mathbf{y}|^2 = (\mathbf{x}\pm\mathbf{y})\cdot(\mathbf{x}\pm\mathbf{y}) = |\mathbf{x}|^2 \pm 2\mathbf{x}\cdot\mathbf{y} + |\mathbf{y}|^2.$$

If $\mathbf{x}$ and $\mathbf{y}$ are the sides of a parallelogram, then $\mathbf{x}+\mathbf{y}$ and $\mathbf{x}-\mathbf{y}$ are its diagonals. Hence this result says that the sum of the squares on the diagonals of a parallelogram equals the sum of the squares on the sides.

**Exercise 1.18** If $k \geq 2$ and $\mathbf{x} \in R^k$, prove that there exists $\mathbf{y} \in R^k$ such that $\mathbf{y} \neq \mathbf{0}$ but $\mathbf{x} \cdot \mathbf{y} = 0$. Is this also true if $k = 1$?

*Solution.* If $\mathbf{x}$ has any components equal to 0, then $\mathbf{y}$ can be taken to have the corresponding components equal to 1 and all others equal to 0. If all the components of $\mathbf{x}$ are nonzero, $\mathbf{y}$ can be taken as $(-x_2, x_1, 0, \ldots, 0)$. This is, of course, not true when $k = 1$, since the product of two nonzero real numbers is nonzero.

**Exercise 1.19** Suppose $\mathbf{a} \in R^k$, $\mathbf{b} \in R^k$. Find $\mathbf{c} \in R^k$ and $r > 0$ such that

$$|\mathbf{x}-\mathbf{a}| = 2|\mathbf{x}-\mathbf{b}|$$

if and only if $|\mathbf{x}-\mathbf{c}| = r$. (*Solution:* $3\mathbf{c} = 4\mathbf{b} - \mathbf{a}$, $3r = 2|\mathbf{b}-\mathbf{a}|$.)

*Solution.* Since the solution is given to us, all we have to do is verify it, i.e., we need to show that the equation

$$|\mathbf{x}-\mathbf{a}| = 2|\mathbf{x}-\mathbf{b}|$$

is equivalent to $|\mathbf{x}-\mathbf{c}| = r$, which says

$$\left|\mathbf{x} - \frac{4}{3}\mathbf{b} + \frac{1}{3}\mathbf{a}\right| = \frac{2}{3}|\mathbf{b}-\mathbf{a}|.$$

If we square both sides of both equations, we an equivalent pair of equations, the first of which reduces to

$$3|\mathbf{x}|^2 + 2\mathbf{a}\cdot\mathbf{x} - 8\mathbf{b}\cdot\mathbf{x} - |\mathbf{a}|^2 + 4|\mathbf{b}|^2 = 0,$$

and the second of which reduces to this equation divided by 3. Hence these equations are indeed equivalent.

**Exercise 1.20** With reference to the Appendix, suppose that property (III) were omitted from the definition of a cut. Keep the same definitions of order and addition. Show that the resulting ordered set has the least-upper-bound property, that addition satisfies axioms (A1) to (A4) (with a slightly different zero element!) but that (A5) fails.

*Solution.* We are now defining a cut to be a proper subset of the rational numbers that contains, along with each of its elements, all smaller rational numbers. Order is defined by containment. Now given a set $A$ of cuts having an upper bound $\beta$, let $\alpha$ be the union of all the cuts in $A$. Obviously $\alpha$ is properly contained in $\beta$, and so is a proper subset of the rationals. It also obviously satisfies the property that if $p \in \alpha$ and $q < p$, then $q \in \alpha$; hence $\alpha$ is a cut. It is further obvious that $\alpha$ contains each elements of $A$, and so is an upper bound for $A$. It remains to prove that there is no smaller upper bound.

To that end, suppose, $\gamma < \alpha$, then $\alpha$ contains an element $x$ not in $\gamma$. By definition of $\alpha$, $x$ must belong to some cut $\delta$ in $A$. But then $\gamma < \delta$, and so $\gamma$ is not an upper bound for $A$. Thus $\alpha$ is the least upper bound.

The proof given in the text goes over without any change to show that (A1), (A2), and (A3) hold. As for (A4) let $O = \{r : r \leq 0\}$. We claim $O + \alpha = \alpha$. The proof is easy. First, we obviously have $O + \alpha \subseteq \alpha$. For $r + s \leq s$ if $r \leq 0$. Hence $r + s \in \alpha$ if $s \in \alpha$. Conversely $\alpha \subseteq O + \alpha$, since each $s$ in $\alpha$ can be written as $0 + s$.

Unfortunately, if $O' = \{r : r < 0\}$, there is no element $\alpha$ such that $\alpha + O' = O$. For $\alpha + O'$ has no largest element. If $x = r + s \in \alpha + O'$, where $r \in \alpha$ and $s \in O'$, there is an element $t \in O'$ with $t > s$, and so $r + t \in \alpha + O'$ and $r + t > s$. Since $O$ has a largest element (namely 0), these two sets cannot be equal.