Dear Student,

We try to understand this example of Jacobi method

Find all the eigen values of the matrix by Jacobi's method.

$$A=\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Solution

This matrix is real symmetric matrix as the $A^t = A$. Now we see that all the off-diagonal elements are of the same order of magnitude. Therefore, we can choose any one of them. Suppose, we choose a_{12} as the largest element and compute. First we shall compute the angle whose formula is

$$\tan 2\theta = \frac{2a_{12}}{2-2} = \frac{2(-1)}{0} = \infty$$

Which gives,

$$2\theta = \tan^{-1}(\infty) = \frac{\pi}{2}$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

Now we construct orthogonal matrix S as

$$S = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Remember here that cos and sine functions are placed from where we take the angle. Here we computed the angle from a_{12}, a_{11} and a_{22} So we took the cos and sine function there.

Inverse of S is

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now The first rotation gives

$$D_1 = S_1^{-1} A S_1$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

After matrix multiplication of the above 3 matrices we get

$$= \begin{bmatrix} 1 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 3 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 2 \end{bmatrix}$$

Here we have completed our first iteration now we shall start 2^{nd} iteration similarly as we did above.

Now, we choose $d_{13} = -1/\sqrt{2}$

As the largest element of D1 and compute

$$\tan 2\theta = \frac{2d_{13}}{d_{11} - d_{33}} = \frac{-\sqrt{2}}{1-2}$$

$$\theta = 27^o 22' 41''.$$

Now we construct another orthogonal matrix S2, such that

$$S_2 = \begin{bmatrix} 0.888 & 0 & -0.459 \\ 0 & 1 & 0 \\ 0.459 & 0 & 0.888 \end{bmatrix}$$

At the end of second rotation, we obtain

$$D_2 = S_2^{-1} D_1 S_2 = \begin{bmatrix} 0.634 & -0.325 & 0 \\ 0.325 & 3 & -0.628 \\ 0 & -0.628 & 2.365 \end{bmatrix}$$

Now, the numerically largest off-diagonal element of *D*2 is found to be $d_{23} = -0.628$ and compute.

$$\tan 2\theta = \frac{-2 \times 0.628}{3 - 2.365}$$

$$\theta = -31^o 35' 24''.$$

Thus, the orthogonal matrix is

$$S_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.852 & 0.524 \\ 0 & -0.524 & 0.852 \end{bmatrix}$$

At the end of third rotation, we get

$$D_3 = S_3^{-1} D_2 S_3 = \begin{bmatrix} 0.634 & -0.277 & 0 \\ 0.277 & 3.386 & 0 \\ 0 & 0 & 1.979 \end{bmatrix}$$

To reduce *D3* to a diagonal form, some more rotations are required. However, we may take 0.634, 3.386 and 1.979 as Eigen values of the given matrix.