Muller's Method

In this method, $f(x)$ is approximated by a second degree curve near the root. The roots of the quadratic are then assumed to be the approximations to the roots of the equation $f(x) = 0$. The method is iterative, converges almost quadratically, and can be used to obtain complex roots.

Let x_{i-2}, x_{i-1}, x_i be the three distinct approximations to a root of $f(x) = 0$ and let y_{i-2}, y_{i-1}, y_i be the corresponding values of $y = f(x)$.

Assuming that

$$P(x) = A(x - x_i)^2 + B(x - x_i) + y_i \qquad (1)$$

is the parabola passing through the points $(x_{i-2}, y_{i-2}), (x_{i-1}, y_{i-1})$ and (x_i, y_i), we have

$$y_{i-1} = A(x_{i-1} - x_i)^2 + B(x_{i-1} - x_i) + y_i \qquad (2)$$

$$y_{i-2} = A(x_{i-2} - x_i)^2 + B(x_{i-2} - x_i) + y_i \qquad (3)$$

From equations (2) and (3) we get

$$y_{i-1} - y_i = A(x_{i-1} - x_i)^2 + B(x_{i-1} - x_i) \qquad (4)$$

$$y_{i-2} - y_i = A(x_{i-2} - x_i)^2 + B(x_{i-2} - x_i) \qquad (5)$$

Solution of (4) and (5) gives

$$A = \frac{(x_{i-2} - x_i)(y_{i-1} - y_i) - (x_{i-1} - x_i)(y_{i-2} - y_i)}{(x_{i-1} - x_{i-2})(x_{i-1} - x_i)(x_{i-2} - x_i)}$$

$$B = \frac{(x_{i-2} - x_i)^2(y_{i-1} - y_i) - (x_{i-1} - x_i)^2(y_{i-2} - y_i)}{(x_{i-2} - x_{i-1})(x_{i-1} - x_i)(x_{i-2} - x_i)}$$

with the values of A and B the quadratic equation now gives next approximation x_{i+1}.

$$x_{i+1} - x_i = \frac{-B \pm \sqrt{B^2 - 4Ay_i}}{2A} \qquad (6)$$

A direct solution from (6) leads to inaccurate results and therefore it is usually written in the form,

$$x_{i+1} - x_i = -\frac{2y_i}{B \pm \sqrt{B^2 - 4Ay_i}} \qquad (7)$$

In (7), sign in denominator should be chosen so that the denominator will be largest in magnitude. With this choice, equation (7) gives the next approximation to the root.

Example:

Using Muller's method, find the root of the equation

$$y(x) = x^3 - 2x - 5 = 0$$

which lies between 2 and 3.

Solution:

Let $x_{i-2} = 1.9,\ x_{i-1} = 2,\ x_i = 2.1$

Then the correspondence y values are

$$y_{i-2} = -1.941, \quad y_{i-1} = -1, \quad y_i = .061$$

Now

$$A = \frac{(x_{i-2} - x_i)(y_{i-1} - y_i) - (x_{i-1} - x_i)(y_{i-2} - y_i)}{(x_{i-1} - x_{i-2})(x_{i-1} - x_i)(x_{i-2} - x_i)}$$

$$= \frac{(-0.2)(-1.061) - (-0.1)(-2.002)}{(0.1)(-0.1)(-0.2)} = \frac{0.2122 - 0.2002}{0.002} = 6$$

$$B = \frac{(x_{i-2} - x_i)^2(y_{i-1} - y_i) - (x_{i-1} - x_i)^2(y_{i-2} - y_i)}{(x_{i-2} - x_{i-1})(x_{i-1} - x_i)(x_{i-2} - x_i)}$$

$$= \frac{(-0.2)^2(-1.061) - (-0.1)^2(-2.002)}{(0.1)(-0.1)(-0.2)} = \frac{-0.04244 + 0.02002}{-0.002} = 11.21$$

The next approximation to the desired root is

$$x_{i+1} = x_i - \frac{2y_i}{B \pm \sqrt{B^2 - 4Ay_i}}$$

$$= 2.1 - \frac{2(0.061)}{11.21 \pm \sqrt{(11.21)^2 - 4(6)(0.061)}}$$

$$= 2.094542 \qquad \text{(Taking } +ve \text{ sign)}$$

The procedure can now be repeated with the three approximations as 2, 2.1, and 2.094542.

Let $x_{i-2} = 2,\ x_{i-1} = 2.1,\ x_i = 2094542$

Then the correspondence y values are

$$y_{i-2} = -1, \quad y_{i-1} = 0.061, \quad y_i = -0.0001058$$

Now

$$A = \frac{(x_{i-2} - x_i)(y_{i-1} - y_i) - (x_{i-1} - x_i)(y_{i-2} - y_i)}{(x_{i-1} - x_{i-2})(x_{i-1} - x_i)(x_{i-2} - x_i)} = 6.194492$$

$$B = \frac{(x_{i-2} - x_i)^2(y_{i-1} - y_i) - (x_{i-1} - x_i)^2(y_{i-2} - y_i)}{(x_{i-2} - x_{i-1})(x_{i-1} - x_i)(x_{i-2} - x_i)} = 11.161799$$

The next approximation to the desired root is

$$x_{i+1} = x_i - \frac{2y_i}{B \pm \sqrt{B^2 - 4Ay_i}} = 2.094551$$

Hence, the required root is 2.0945 correct up to 4 decimal places.

The procedure can be repeated with the three approximations as 2.1, 2.094542, and 2.094551