GRAEFFE'S Root Squaring method:-

We will describe this method for a cubic polynomial but it can be illustrated for higher degree polynomial too.

Let a cubic polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \longrightarrow (1)$$

on putting $x = -x$, we have.

$$f(-x) = a_0 - a_1 x + a_2 x^2 - a_3 x^3 \longrightarrow (2)$$

(1) × (2); we have.

$$f(x)\,f(-x) = \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right)\left(a_0 - a_1 x + a_2 x^2 - a_3 x^3\right).$$

$$= a_0^2 - \cancel{a_0 a_1 x} + a_0 a_2 x^2 - a_0 a_3 x^3 + \cancel{a_1 a_0 x} - a_1^2 x^2 + a_1 a_2 x^3 - a_1 a_3 x^4 + a_0 a_2 x^2 - a_1 a_2 x^3 + a_2^2 x^4 - a_2 a_3 x^2 x^3 + a_0 a_3 x^3 - a_1 a_3 x^4 + a_2 a_3 x^5 - a_3^2 x^6$$

$$= a_0^2 + \left(a_0 a_2 - a_1^2 + a_0 a_2\right)x^2 + \left(-a_0 a_3 + a_1 a_2 - a_1 a_2 + a_0 a_3\right)x^3 + \left(-a_1 a_3 + a_2^2 - a_1 a_3\right)x^4 + \left(-a_2 a_3 + a_2 a_3\right)x^5 - a_3^2 x^6.$$

$$= a_0^2 + \left(2a_0 a_2 - a_1^2\right)x^2 + \left(a_2^2 - 2a_1 a_3\right)x^4 - a_3^2 x^6.$$

(2)

$$f(x)f(-x) = a_0^2 + (2a_0a_2 - a_1^2)x^2 + (a_2^2 - 2a_1a_3)x^4 - a_3^2x^6.$$

Put $x^2 = t$

$$f(x)f(-x) = a_0^2 + (2a_0a_2 - a_1^2)t + (a_2^2 - 2a_1a_3)t^2 - a_3^2t^3$$

$$= -a_3^2t^3 + (a_2^2 - 2a_1a_3)t^2 + (2a_0a_2 - a_1^2)t - a_0^2 \rightarrow (3)$$

$$= a_3^2t^3 - (-2a_1a_3 + a_2^2)t^2 + (a_1^2 - 2a_0a_2)t + a_0^2.$$

Note:- The roots of equation (3) are powers of the original roots (ie powers of x).

Equation (3) can again be squared and this squaring process is repeated as many times as required. After each squaring the coefficients become large and overflow is possible as i increases.

Suppose we have squared the given polynomial i times, then we can estimate the values of the roots by evaluating 2^i root of

$$\left| \frac{a_i}{a_{i-1}} \right|, \quad i = 1, 2, \cdots, n$$

where n is the degree of the given polynomial.

EXAMPLE:-

Sol:- $x^3 - 6x^2 + 11x - 6 = 0 \longrightarrow$ (A).

Here

$a_0 = -6,\ a_1 = 11,\ a_2 = -6,\ a_3 = 1.$

using (3), we can find the squared polynomial of (A) as; For $i=1$

$x^3 - (36-22)x^2 + (121-72)x - 36$

$= x^3 - 14x^2 + 49x - 36 \longrightarrow$ (B)

Now for $i=2$; we have

Here $a_3 = 1,\ a_2 = -14,\ a_1 = 49,\ a_0 = -36.$

$x^3 - (196 - 98)x^2 + (2401 - 1008)x - 1296.$

$= x^3 - 98x^2 + 1393x - 1296. \rightarrow$ (C).

similarly for $i = 3$, the polynomial is

$x^3 - (9604 - 2786)x^2 + (1940449 - 254016)x$

$- 1679616$

$= x^3 - 6818x^2 + 1686433x - 1679616 \rightarrow$ (D)

~~Now the roots of (B)~~ are.

$\cancel{\left|\frac{a_1}{a_0}\right| = 49}$

The roots of Polynomial (B) are

$\sqrt{\frac{36}{49}} = 0.85714$; $\sqrt{\frac{49}{14}} = 1.8708$; $\sqrt{\frac{14}{1}} = 3.7417$

Similarly for (C) are.

$\sqrt[4]{\frac{1296}{1393}} = 0.9821$, $\sqrt[4]{\frac{1393}{98}} = 1.9417$, $\sqrt[4]{\frac{98}{1}} = 3.1464$

For (D) are.

$\sqrt[8]{\frac{1679616}{1686433}} = 0.99949$, $\sqrt[8]{\frac{1686433}{6818}} = 1.99143$

$\sqrt[8]{\frac{6818}{1}} = 3.0144.$