Lecture 3

Systems of Linear Equations

In this lecture we will discuss some ways in which systems of linear equations arise, how to solve them, and how their solutions can be interpreted geometrically.

Linear equations:

A line in $\boldsymbol{R^2}$ (2-dimensions) can be represented by an equation of the form $a_1x + a_2y = b$ (where a_1, a_2 not both zero). Similarly a plane in $\boldsymbol{R^3}$ (3-dimensional space) can be represented by an equation of the form $a_1x + a_2y + a_3z = b$ (where a_1, a_2, a_3 not all zero).

A linear equation in *n* variables $x_1, x_2, \cdots, x_n$ can be expressed in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \qquad (1)$$

where $a_1, a_2, \cdots, a_n$ and *b* are constants and the "*a's*" are not all zero.

Homogeneous linear equation:

In the special case if $b = 0$, Equation (1) has the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ (2)
This equation is called homogeneous linear equation.

Note: A linear equation does not involve any products or square roots of variables. All variables occur only to the first power and do not appear, as arguments of trigonometric, logarithmic, or exponential functions.

Examples of Linear Equations:

(1) The equations

$$2x_1 + 3x_2 + 2 = x_3 \quad \textit{and} \quad x_2 = 2\left(\sqrt{5} + x_1\right) + 2x_3$$ are both linear

(2) The following equations are also linear

$$x + 3y = 7 \qquad x_1 - 2x_2 - 3x_3 + x_4 = 0$$

$$\tfrac{1}{2}x - y + 3z = -1 \qquad x_1 + x_2 + \cdots + x_n = 1$$

(3) The equations $3x_1 - 2x_2 = x_1x_2$ *and* $x_2 = 4\sqrt{x_1} - 6$
are ***not linear*** because of the presence of x_1x_2 in the first equation and $\sqrt{x_1}$ in the second.

System of linear equations:

A finite set of linear equations is called a system of linear equations or ***linear system***. The variables in a linear system are called the ***unknowns***.

For example,

$$4x_1 - x_2 + 3x_3 = -1$$
$$3x_1 + x_2 + 9x_3 = -4$$

is a linear system of two equations in three unknowns x_1, x_2, and x_3.

General system of linear equations:

A general linear system of *m* equations in n-unknowns $x_1, x_2, \cdots, x_n$ can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots \quad\quad &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \qquad (3)$$

Solution of a system of linear equations:

A solution of a linear system in the unknowns $x_1, x_2, \cdots, x_n$ is a sequence of *n* numbers that when substituted for $x_1, x_2, \cdots, x_n$ respectively, makes every equation in the system a true statement. The set of all solutions of a linear system is called its ***solution set***.

Linear System with Two Unknowns:

When two lines intersect in $\boldsymbol{R^2}$, we get system of linear equations with two unknowns

For example, consider the linear system

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

The graphs of these equations are straight lines in the *xy-plane*, so a solution (*x*, *y*) of this system is infect a point of intersection of these lines.

Thus, there are three possibilities:

1. The lines may be parallel and distinct, in which case there is no intersection and consequently ***no solution***.
2. The lines may intersect at only one point, in which case the system has exactly ***one solution***.
3. The lines may coincide, in which case there are infinitely many points of intersection (the points on the common line) and consequently ***infinitely many solutions.***

Consistent and inconsistent system:

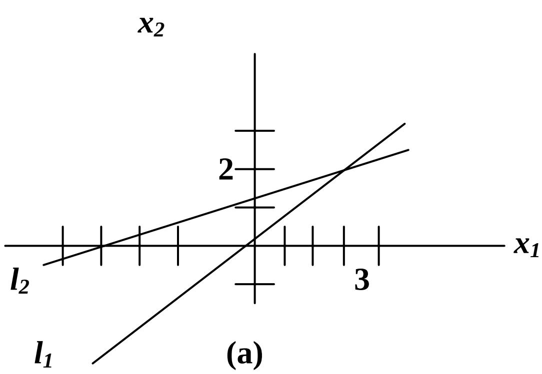
A linear system is said to be **consistent** if it has at least one solution and it is called **inconsistent** if it has no solutions.

Thus, a consistent linear system of two equations in two unknowns has either one solution or infinitely many solutions – there is no other possibility.

Example: consider the system of linear equations in two variables $x_1 - 2x_2 = -1,\ -x_1 + 3x_2 = 3$

Solve the equation simultaneously:

Adding both equations we get $x_2 = 2$, Put $x_2 = 2$ in any one of the above equation we get $x_1 = 3$. So the solution is the single point **(3, 2)**. See the graph of this linear system

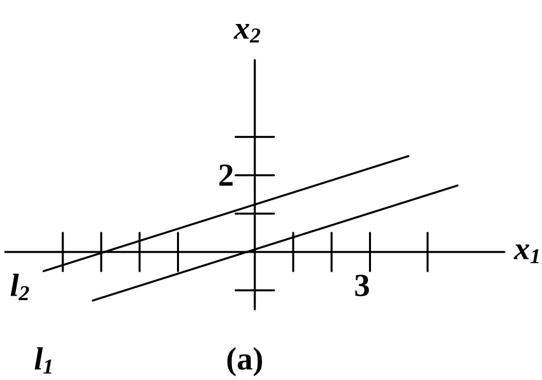


(a)

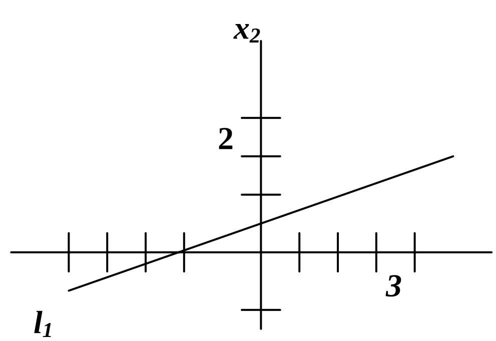
This system has exactly one solution

See the graphs to the following linear systems:

$$(a)\quad \begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= \ \ 3 \end{aligned} \qquad\qquad (b)\quad \begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= \ \ 1 \end{aligned}$$



(a)



(b)

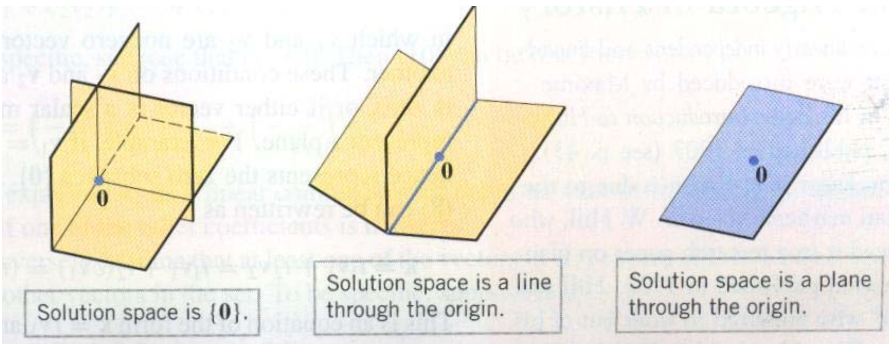
(a) No solution. (b) Infinitely many solutions.

Linear System with Three Unknowns:

Consider a linear system of three equations in three unknowns:

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$
$$a_3x+b_3y+c_3z=d_3$$

In this case, the graph of each equation is a plane, so the solutions of the system, If any correspond to points where all three planes intersect; and again we see that there are only three possibilities – no solutions, one solution, or infinitely many solutions as shown in figure.



Theorem 1: Every system of linear equations has zero, one or infinitely many solutions; there are no other possibilities.

Example 1: Solve the linear system

$$x-y=1$$
$$2x+y=6$$

Solution:

Adding both equations, we get $x=\frac{7}{3}$. Putting this value of x in 1st equation, we get $y=\frac{4}{3}$. Thus, the system has the ***unique solution*** $x=\frac{7}{3}, y=\frac{4}{3}$.

Geometrically, this means that the lines represented by the equations in the system intersect at a single point $\left(\frac{7}{3},\frac{4}{3}\right)$ and thus has a unique solution.

Example 2: Solve the linear system

$$x+\quad y=4$$
$$3x+3y=6$$

Solution:

Multiply first equation by 3 and then subtract the second equation from this. We obtain

$$0 = 6$$

This equation is contradictory.

Geometrically, this means that the lines corresponding to the equations in the original system are parallel and distinct. So the given system has ***no solution.***

Example 3: Solve the linear system

$$\begin{aligned} 4x - 2y &= 1 \\ 16x - 8y &= 4 \end{aligned}$$

Solution:

Multiply the first equation by -4 and then add in second equation.

$$\begin{array}{r} -16x + 8y = -4 \\ \underline{16x - 8y = \ \ 4} \\ 0 \qquad = 0 \end{array}$$

Thus, the solutions of the system are those values of x and y that satisfy the single equation $4x - 2y = 1$

Geometrically, this means the lines corresponding to the two equations in the original system coincide and thus the system has infinitely many solutions.

Parametric Representation:

It is very convenient to describe the solution set in this case is to ***express it parametrically.*** We can do this by letting $y = t$ and solving for x in terms of t, or by letting $x = t$ and solving for y in terms of t.

The first approach yields the following parametric equations (by taking *y=t in the equation* $4x - 2y = 1$)

$$4x - 2t = 1, \quad y = t$$

$$x = \frac{1}{4} + \frac{1}{2}t, \quad y = t$$

We can now obtain some solutions of the above system by substituting some numerical values for the parameter.

Example: For $t = 0$ the solution is $(\frac{1}{4}, 0)$. For $t = 1$, the solution is $(\frac{3}{4}, 1)$ and for $t = -1$ the solution is $(-\frac{1}{4}, -1)$ *etc.*

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Example 4: Solve the linear system

$$\begin{aligned} x - \ \ y + 2z &= 5 \\ 2x - 2y + 4z &= 10 \\ 3x - 3y + 6z &= 15 \end{aligned}$$

Solution:
Since the second and third equations are multiples of the first.

Geometrically, this means that the three planes coincide and those values of *x*, *y* and *z* that satisfy the equation $x - \ y + 2z = 5$ automatically satisfy all three equations.

We can express the solution set parametrically as

$$x = 5 + t_1 - 2t_2, y = t_1, \quad z = t_2$$

Some solutions can be obtained by choosing some numerical values for the parameters.

For example if we take $y = t_1 = 2$ and $z = t_2 = 3$ then

$$\begin{aligned} x &= 5 + t_1 - 2t_2 \\ &= 5 + 2 - 2(3) \\ &= 1 \end{aligned}$$

Put these values of x, y, and z in any equation of linear system to verify

$$\begin{aligned} x - \ y + 2z &= 5 \\ 1 - 2 + 2(3) &= 5 \\ 1 - 2 + 6 &= 5 \\ 5 &= 5 \end{aligned}$$

Hence x = 1, y = 2, z = 3 is the solution of the system. Verified.

Matrix Notation:

The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**.

Given the system

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ 2x_2 - 8x_3 &= \ 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

With the coefficients of each variable aligned in columns, the matrix $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$

is called the coefficient matrix (or matrix of coefficients) of the system.

An augmented matrix of a system consists of the coefficient matrix with an added column containing the constants from the right sides of the equations. It is always denoted by A_b

$$A_b = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

Solving a Linear System:

In order to solve a linear system, we use a number of methods. 1st of them is given below.

Successive elimination method: In this method the x_1 term in the first equation of a system is used to eliminate the x_1 terms in the other equations. Then we use the x_2 term in the second equation to eliminate the x_2 terms in the other equations, and so on, until we finally obtain a very simple equivalent system of equations.

Example 5: Solve

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

Solution: We perform the elimination procedure with and without matrix notation, and place the results side by side for comparison:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

To eliminate the x_1 term from third equation add 4 times equation 1 to equation 3,

$$\begin{aligned} 4x_1 - 8x_2 + 4x_3 &= 0 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \\ \hline -3x_2 + 13x_3 &= -9 \end{aligned}$$

The result of the calculation is written in place of the original third equation:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Next, multiply equation 2 by ½ in order to obtain 1 as the coefficient for x_2

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

To eliminate the x_2 term from third equation add 3 times equation 2 to equation 3,

The new system has a triangular form

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Now using 3rd equation eliminate the x_3 term from first and second equation i.e. multiply 3rd equation with 4 and add in second equation. Then subtract the third equation from first equation we get

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Adding 2 times equation 2 to equation 1, we obtain the result

$$\begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{cases} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

This completes the solution.
Our work indicates that the only solution of the original system is (29, 16, 3).

To verify that (29, 16, 3) is a solution, substitute these values into the left side of the original system for x_1, x_2 and x_3 and after computing, we get

$$\begin{gathered} (29) - 2(16) + (3) = 29 - 32 + 3 = 0 \\ 2(16) - 8(3) = 32 - 24 = 8 \\ -4(29) + 5(16) + 9(3) = -116 + 80 + 27 = -9 \end{gathered}$$

The results agree with the right side of the original system, so (29, 16, 3) is a solution of the system.

This example illustrates how operations on equations in a linear system correspond to operations on the appropriate rows of the augmented matrix. The three basic operations listed earlier correspond to the following operations on the augmented matrix.

Elementary Row Operations:

1. (Replacement) Replace one row by the sum of itself and a nonzero multiple of another row.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

Row equivalent matrices:

A matrix B is said to be row equivalent to a matrix A of the same order if B can be obtained from A by performing a finite sequence of elementary row operations of A.
If A and B are row equivalent matrices, then we write this expression mathematically as $A \sim B$.

For example $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$ are row equivalent matrices because we add 4 times of 1^{st} row in 3^{rd} row in 1^{st} matrix.

Note: If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Row operations are extremely easy to perform, but they have to be learnt and practice.

Two Fundamental Questions:

1. Is the system consistent; that is, does at least one solution exist?
2. If a solution exists is it the only one; that is, is the solution unique?

We try to answer these questions via row operations on the augmented matrix.

Example 6: Determine if the following system of linear equations is consistent

$$\begin{aligned} x_1 - 2x_2 + \quad x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

Solution:

First obtain the triangular matrix by removing x_1 and x_2 term from third equation and removing x_2 from second equation.

First divide the second equation by 2 we get

$$\begin{aligned} x_1 - 2x_2 + \quad x_3 &= 0 \\ x_2 - 4x_3 &= \ \ 4 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

Now multiply equation 1 with 4 and add in equation 3 to eliminate x_1 from third equation.

$$\begin{aligned} x_1 - 2x_2 + \quad x_3 &= 0 \\ x_2 - 4x_3 &= \ \ 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Now multiply equation 2 with 3 and add in equation 3 to eliminate x_2 from third equation.

$$\begin{aligned} & x_1 - 2x_2 + x_3 = 0 \\ & x_2 - 4x_3 = 4 \\ & x_3 = 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Put value of x_3 in second equation we get

$$x_2 - 4(3) = 4$$

$$x_2 = 16$$

Now put these values of x_2 and x_3 in first equation we get

$$x_1 - 2(16) + 3 = 0$$

$$x_1 = 29$$

So a solution exists and the system is consistent and has a unique solution.

Example 7: Solve if the following system of linear equations is consistent.

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned}$$

Solution: The augmented matrix is

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

To obtain x_1 in the first equation, interchange rows 1 and 2:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

To eliminate the $5x_1$ term in the third equation, add –5/2 times row 1 to row 3:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{bmatrix}$$

Next, use the x_2 term in the second equation to eliminate the –(1/2) x_2 term from the third equation. Add ½ times row 2 to row 3:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$$

The augmented matrix is in triangular form.
To interpret it correctly, go back to equation notation:

$$\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 &= 8 \\ 0 &= 2.5 \end{aligned}$$

There are no values of x_1, x_2, x_3 that will satisfy because the equation 0 = 2.5 is never true.
Hence original ***system is inconsistent (i.e., has no solution)***.

Exercises:

1. State in words the next elementary "row" operation that should be performed on the system in order to solve it. (More than one answer is possible in (a).)

a.
$$\begin{aligned} x_1 + 4x_2 - 2x_3 + 8x_4 &= 12 \\ x_2 - 7x_3 + 2x_4 &= -4 \\ 5x_3 - x_4 &= 7 \\ x_3 + 3x_4 &= -5 \end{aligned}$$

b.
$$\begin{aligned} x_1 - 3x_2 + 5x_3 - 2x_4 &= 0 \\ x_2 + 8x_3 &= -4 \\ 2x_3 &= 7 \\ x_4 &= 1 \end{aligned}$$

2. The augmented matrix of a linear system has been transformed by row operations into the form below. Determine if the system is consistent.

$$\begin{bmatrix} 1 & 5 & 2 & -6 \\ 0 & 4 & -7 & 2 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

3. Is (3, 4, –2) a solution of the following system?

$$\begin{aligned} 5x_1 - x_2 + 2x_3 &= 7 \\ -2x_1 + 6x_2 + 9x_3 &= 0 \\ -7x_1 + 5x_2 - 3x_3 &= -7 \end{aligned}$$

4. For what values of *h* and *k* is the following system consistent?

$$\begin{aligned} 2x_1 - x_2 &= h \\ -6x_1 + 3x_2 &= k \end{aligned}$$

Solve the systems in the exercises given below;

5.
$$\begin{aligned} x_2 + 5x_3 &= -4 \\ x_1 + 4x_2 + 3x_3 &= -2 \\ 2x_1 + 7x_2 + x_3 &= -1 \end{aligned}$$

6.
$$\begin{aligned} x_1 - 5x_2 + 4x_3 &= -3 \\ 2x_1 - 7x_2 + 3x_3 &= -2 \\ 2x_1 - x_2 - 7x_3 &= 1 \end{aligned}$$

7.
$$\begin{aligned} x_1 + 2x_2 &= 4 \\ x_1 - 3x_2 - 3x_3 &= 2 \\ x_2 + x_3 &= 0 \end{aligned}$$

8.
$$\begin{aligned} 2x_1 - 4x_3 &= -10 \\ x_2 + 3x_3 &= 2 \\ 3x_1 + 5x_2 + 8x_3 &= -6 \end{aligned}$$

Determine the value(s) of h such that the matrix is augmented matrix of a consistent linear system.

9. $\begin{bmatrix} 1 & -3 & h \\ -2 & 6 & -5 \end{bmatrix}$

10. $\begin{bmatrix} 1 & h & -2 \\ -4 & 2 & 10 \end{bmatrix}$

Find an equation involving g, h, and that makes the augmented matrix correspond to a consistent system.

11. $\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix}$

12. $\begin{bmatrix} 2 & 5 & -3 & g \\ 4 & 7 & -4 & h \\ -6 & -3 & 1 & k \end{bmatrix}$

Find the elementary row operations that transform the first matrix into the second, and then find the reverse row operation that transforms the second matrix into first.

13. $\begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & -4 \\ 0 & -3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -2 \\ 0 & -3 & 4 \end{bmatrix}$

14. $\begin{bmatrix} 0 & 5 & -3 \\ 1 & 5 & -2 \\ 2 & 1 & 8 \end{bmatrix}, \begin{bmatrix} 1 & 5 & -2 \\ 0 & 5 & -3 \\ 2 & 1 & 8 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 3 & -1 & 5 \\ 0 & 1 & -4 & 2 \\ 0 & 2 & -5 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -1 & 5 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 3 & -5 \end{bmatrix}$