Solution of Practice Questions Lecture # 6

Question # 1:

Express $\vec{b}=\begin{bmatrix}2\\4\end{bmatrix}$ as a linear combination of $\vec{s}=\begin{bmatrix}1\\2\end{bmatrix}$ and $\vec{t}=\begin{bmatrix}3\\4\end{bmatrix}$

Solution:

To express $\vec{b}$ as a linear combination of $\vec{s}$ and $\vec{t}$, the scalars c_1 and c_2 should be determined as following:

$$\begin{aligned}\vec{b} &= c_1\,\vec{s}+c_2\,\vec{t}\\ (2,4) &= c_1\,(1,2)+c_2\,(3,4)\\ (2,4) &= (c_1+3c_2\,,2c_1+4\,c_2)\end{aligned}$$

The following system of equations is obtained by equating the corresponding entries:

$$c_1+3c_2=2$$

$$2c_1+4c_2=4$$

multiply first equation by 2 *and subtract from* second

$$\begin{array}{l} 2c_1+6c_2=4\\ 2c_1+4c_2=4\\ -\quad -\qquad -\\ \hline \quad 2c_2=0\\ \qquad c_2=0\end{array}$$

Put in first equation

$$\therefore \quad c_1=2$$

Hence $(2,4)=2(1,2)+0(3,4)$

Question # 2

Determine whether the set of vectors $\vec{v}_1=(1,2,-1)$, $\vec{v}_2=(3,-3,4)$ and $\vec{v}_3=(2,-1,-2)$ will span R^3?

Solution:

To show that the given vectors span R^3, choose a general vector from R^3, let $\vec{u}=(u_1,u_2,u_3)\in R^3$ and determine if we can find scalars c_1 , c_2 , c_3 so that $\vec{u}\in R^3$ can be written as a linear combination of the given vectors. That is ,

$$\vec{u}=c_1\vec{v}_1+c_2\vec{v}_2+c_3\vec{v}_3$$

$$(u_1\ ,\ u_2\ ,\ u_3)=c_1(1\ ,2\ ,-1)+c_2(3\ ,-3\ ,\ 4)+c_3(2\ ,-1\ ,-2)$$

The following system of equations is obtained by doing some vector algebra:

$$u_1 = c_1+3c_2+2c_3$$
$$u_2 = 2c_1-3c_2-c_3$$
$$u_3 = -c_1+4c_2-2c_3$$

Writing in matrix form

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & -3 & -1 \\ -1 & 4 & -2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}=\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Now we determine if this system is consistent (i.e have at least one solution) for every possible choice of $\vec{u}=(u_1,u_2,u_3)\in R^3$. We know that the system will be consistent for every possible choice of $\vec{u}=(u_1,u_2,u_3)$ provided the coefficient matrix is invertible and that will be checked by computing the determinate of the coefficient matrix. So that

$$\det(A)=45\neq 0$$

Therefore the coefficient matrix is invertible and so this system will have solution for every choice of $\vec{u}=(u_1,u_2,u_3)$ which in turns determine that the given set of vectors span R^3.

Question # 3

Determine whether the set of vectors $\vec{v}_1=(1,3,1,1)$, $\vec{v}_2=(1,2,1,0)$ and $\vec{v}_3=(1,1,0,0)$ will span R^3?

Solution:

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\rightarrow\begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\rightarrow\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\rightarrow\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}\rightarrow\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

At this point, it is clear the rank of the matrix is 3, so the vectors span a subspace of dimension 3, hence they span R^3.

Question # 4

Determine whether the vectors $v_1 = (1, -1, 4)$, $v_2 = (-2, 1, 3)$, and $v_3 = (4, -3, 5)$ span R^3.

Solution:

$$\begin{bmatrix} 1 & -2 & 4 \\ -1 & 1 & -3 \\ 4 & 3 & 5 \end{bmatrix} \xrightarrow{R_2+R_1,\, R_3-4R_1} \begin{bmatrix} 1 & -2 & 4 \\ 0 & -1 & 1 \\ 0 & 11 & -11 \end{bmatrix} \xrightarrow{-R_1,\, 1/11R_3} \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

It is in Echelon form, where it can be seen that 3rd row does not contain any Pivot . so it cannot span R^3.

Question # 5

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$ and $\vec{z} = \begin{bmatrix} h \\ 2 \\ -3 \end{bmatrix}$. If $\vec{z}$ can be generated by $\vec{v}_1$ and $\vec{v}_2$, then find value of 'h'.

Solution:

Consider $\vec{c}_1\vec{v}_1 + \vec{c}_2\vec{v}_2 = \vec{z}$.

Therefore, $\vec{c}_1 \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \vec{c}_2 \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} h \\ 2 \\ -3 \end{bmatrix}$

$$\begin{aligned} \vec{c}_1 - \vec{c}_2 &= h \quad ....e.q.(1) \\ 2\vec{c}_1 + c_2 &= 2 \quad ....e.q.(2) \\ -3\vec{c}_1 + 4\vec{c}_2 &= -3 \quad ....e.q.(3) \end{aligned}$$

Multiply equation 2 by 3 and multiply equation 3 by 2 and then add both equations

$$\begin{aligned} 6\vec{c}_1 + 3c_2 &= 6 \\ -6\vec{c}_1 + 8\vec{c}_2 &= -6 \\ \hline 11\vec{c}_2 &= 0 \\ \vec{c}_2 &= 0 \end{aligned}$$

Put the value of $\vec{c}_2$ in equation 3 we get

$\vec{c}_1 = 1$

Now put the value of $\vec{c}_1$ *and* $\vec{c}_2$

$1 - 0 = h$

$h = 1$