Practice Questions Lecture # 7 and 8

Question # 1

Determine whether the following system has a trivial solution or non-trivial solution**:**

$$x_1 - 2x_2 + x_3 = 0$$
$$3x_2 - 3x_3 = 0$$
$$x_1 - 3x_2 = 0$$

Solution:

The coefficient matrix is

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 3 & -3 & 0 \\ 1 & -3 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix} 1/3R_2,\, R_3 - R_1$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix} R_3 + R_2$$

The corresponding system of equations will be

$$x_1 - 2x_2 + x_3 = 0$$
$$x_2 - x_3 = 0$$
$$-2x_3 = 0$$

By backward substitution, we get $x_1 = 0,\ x_2 = 0,\ x_3 = 0$. Thus, the given system has only the trivial solution.

Question # 2

Solve the following system using the reduced echelon form**:**

$$x_1 + 4x_2 + 6x_3 = 0$$
$$-2x_1 - 5x_3 = 0$$
$$3x_2 - 7x_3 = 0$$

Solution:

Given system of equation could be translated into matrix form as

$$\begin{matrix} 1 & 4 & 6 \\ -2 & 0 & -5 \\ 0 & 3 & -7 \end{matrix}$$

add 2 times the 1st row to the 2nd row

$$\begin{matrix} 1 & 4 & 6 \\ 0 & 8 & 7 \\ 0 & 3 & -7 \end{matrix}$$

Multiply the 2nd row by 1/8

$$\begin{matrix} 1 & 4 & 6 \\ 0 & 1 & \frac{7}{8} \\ 0 & 3 & -7 \end{matrix}$$

add -3 times the 2nd row to the 3rd row

$$\begin{matrix} 1 & 4 & 6 \\ 0 & 1 & \frac{7}{8} \\ 0 & 0 & \frac{-77}{8} \end{matrix}$$

Multiply the 3rd row by -8/77

$$\begin{matrix} 1 & 4 & 6 \end{matrix}$$

$$\begin{bmatrix} 0 & 1 & \frac{7}{8} \\ 0 & 0 & 1 \end{bmatrix}$$

add -7/8 times the 3rd row to the 2nd row

$$\begin{bmatrix} 1 & 4 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

add -6 times the 3rd row to the 1st row

$$\begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

add -4 times the 2nd row to the 1st row

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question # 3

Check whether $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent or not?

$$\text{where} \quad \vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \text{ and } \vec{v}_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Solution:

The set S = {$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$} of vectors in R^3 is **linearly independent** if the only solution of

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = 0$$

is

$$c_1, c_2, c_3 = 0$$

Otherwise (i.e., if a solution with at least some nonzero values exists), S is **linearly dependent**.

With our vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, becomes:

$$c_1 \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rearranging the left hand side yields

$$\begin{bmatrix} -2\ c_1 +1\ c_2 +2\ c_3 \\ 1\ c_1 +0\ c_2 -2\ c_3 \\ 3\ c_1 +2\ c_2 +1\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix equation above is equivalent to the following **homogeneous system of equations**

$$\begin{aligned} -2\ c_1 \ +1\ c_2 \ +2\ c_3 &= 0 \\ 1\ c_1 \ +0\ c_2 \ -2\ c_3 &= 0 \\ 3\ c_1 \ +2\ c_2 \ +1\ c_3 &= 0 \end{aligned}$$

We now transform the coefficient matrix of the homogeneous system above to the reduced row echelon form to determine whether the system has

- the trivial solution only (meaning that S is linearly independent), or
- the trivial solution as well as nontrivial ones (S is linearly dependent).

$$\begin{bmatrix} -2 & 1 & 2 \\ 1 & 0 & -2 \\ 3 & 2 & 1 \end{bmatrix}$$

can be transformed by a sequence of elementary row operations to the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The reduced row echelon form of the coefficient matrix of the homogeneous system is

$$\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

which corresponds to the system

$$\begin{aligned} 1c_1 \qquad\qquad &= 0 \\ 1c_2 \qquad &= 0 \\ 1c_3 &= 0 \end{aligned}$$

Since each column contains a leading entry (highlighted in yellow), then the system has only the trivial solution, so that the only solution is $c_1, c_2, c_3 = 0$.

Therefore the set S = {$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$} is **linearly independent**.

Question # 4

Determine, without solving, whether the following set of vectors is linearly independent or dependent.

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 3 \end{bmatrix} \right\}$$

Solution:

Using the same rule in previous question, with our vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 6 \end{bmatrix} + c_4 \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Rearranging the left hand side yields

$$\begin{bmatrix} 1c_1 + 3c_2 + 4c_3 + 7c_4 \\ 2c_1 + 4c_2 + 6c_3 + 3c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The matrix equation above is equivalent to the following **homogeneous system of equations**

$$1c_1 \;\; +3c_2 \;\; +4c_3 \;\; +7c_4 = 0$$

$2c_1 + 4c_2 + 6c_3 + 3c_4 = 0$

- We now transform the coefficient matrix of the homogeneous system above to the reduced row echelon form

$$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 2 & 4 & 6 & 3 \end{bmatrix}$$

can be transformed by a sequence of elementary row operations to the matrix

$$\begin{bmatrix} 1 & 0 & 1 & \frac{-19}{2} \\ 0 & 1 & 1 & \frac{11}{2} \end{bmatrix}$$

The reduced row echelon form of the coefficient matrix of the homogeneous system is

$$\begin{bmatrix} \boxed{1} & 0 & 1 & \frac{-19}{2} \\ 0 & \boxed{1} & 1 & \frac{11}{2} \end{bmatrix}$$

which corresponds to the system

$$\begin{aligned} 1c_1 \quad\quad + 1c_3 + (-19/2)c_4 &= 0 \\ 1c_2 + 1c_3 + (11/2)c_4 &= 0 \end{aligned}$$

The leading entries have been highlighted in yellow.

Since some columns do not contain leading entries, then the system has nontrivial solutions, so that some of the values c_1, c_2, c_3, c_4 solving the system of equation may be nonzero.

Therefore the set S = {$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$} is **linearly dependent**.

Question # 5

Show that the columns of $A = \begin{bmatrix} 2 & 4 \\ 4 & -3 \end{bmatrix}$ are linearly independent.

Solution:

Vectors in R^2 is **linearly independent** if the only solution of

$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = 0$

is $c_1, c_2 = 0$

$$c_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Rearranging the left hand side yields

$$\begin{bmatrix} 2c_1 + 4c_2 \\ 4c_1 - 3c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The matrix equation above is equivalent to the following **homogeneous system of equations**

$$\begin{aligned} 2c_1 + 4c_2 &= 0 \\ 4c_1 - 3c_2 &= 0 \end{aligned}$$

- We now transform the coefficient matrix of the homogeneous system above to the reduced row echelon form

$$\begin{bmatrix} 2 & 4 \\ 4 & -3 \end{bmatrix}$$

can be transformed by a sequence of elementary row operations to the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The reduced row echelon form of the coefficient matrix of the homogeneous is

$$\begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \end{bmatrix}$$

which corresponds to the system

$$\begin{aligned} 1\,c_1 \quad\quad &= 0 \\ 1\,c_2 &= 0 \end{aligned}$$

Since each column contains a leading entry (highlighted in yellow), then the system has only the trivial solution, so that the only solution is c_1, $c_2 = 0$.

Therefore columns of A are **linearly independent**.