Question # 1

Determine whether the following system has a trivial solution or non-trivial solution:

\[ \begin{align*}
    x_1 - 2x_2 + x_3 &= 0 \\
    3x_2 - 3x_3 &= 0 \\
    x_1 - 3x_2 &= 0
\end{align*} \]

Solution:

The coefficient matrix is

\[
\begin{bmatrix}
    1 & -2 & 1 & 0 \\
    0 & 3 & -3 & 0 \\
    1 & -3 & 0 & 0 \\
\end{bmatrix}
\]

The corresponding system of equations will be

\[ \begin{align*}
    x_1 - 2x_2 + x_3 &= 0 \\
    x_2 - x_3 &= 0 \\
    -2x_3 &= 0
\end{align*} \]

By backward substitution, we get \( x_1 = 0, \ x_2 = 0, \ x_3 = 0 \). Thus, the given system has only the trivial solution.

Question # 2

Solve the following system using the reduced echelon form:

\[ \begin{align*}
    x_1 + 4x_2 + 6x_3 &= 0 \\
    -2x_1 - 5x_3 &= 0 \\
    3x_2 - 7x_3 &= 0
\end{align*} \]

Solution:

Given system of equation could be translated into matrix form as
\[
\begin{bmatrix}
1 & 4 & 6 \\
-2 & 0 & -5 \\
0 & 3 & -7 \\
\end{bmatrix}
\]

add 2 times the 1st row to the 2nd row

\[
\begin{bmatrix}
1 & 4 & 6 \\
0 & 8 & 7 \\
0 & 3 & -7 \\
\end{bmatrix}
\]

Multiply the 2nd row by 1/8

\[
\begin{bmatrix}
1 & 4 & 6 \\
0 & 1 & \frac{7}{8} \\
0 & 3 & -7 \\
\end{bmatrix}
\]

add -3 times the 2nd row to the 3rd row

\[
\begin{bmatrix}
1 & 4 & 6 \\
0 & 1 & \frac{7}{8} \\
0 & 0 & \frac{-77}{8} \\
\end{bmatrix}
\]

Multiply the 3rd row by -8/77

\[
\begin{bmatrix}
1 & 4 & 6 \\
0 & 1 & \frac{7}{8} \\
0 & 0 & \frac{8}{8} \\
\end{bmatrix}
\]
$\begin{bmatrix} 7 \\ 8 \\ 1 \end{bmatrix}$

add $-7/8$ times the 3rd row to the 2nd row

$\begin{bmatrix} 1 & 4 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

add $-6$ times the 3rd row to the 1st row

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

add $-4$ times the 2nd row to the 1st row

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Question # 3

Check whether $\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ is linearly dependent or not?

\[
\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} , \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}
\]

Solution:

The set $S = \{v_1, v_2, v_3\}$ of vectors in $\mathbb{R}^3$ is \textbf{linearly independent} if the only solution of

\[c_1 v_1 + c_2 v_2 + c_3 v_3 = 0\]

is \[c_1, c_2, c_3 = 0\]

Otherwise (i.e., if a solution with at least some nonzero values exists), $S$ is \textbf{linearly dependent}.
With our vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, becomes:

\[
\begin{align*}
\begin{bmatrix}
-2 & 1 & 2 \\
1 & 0 & -2 \\
3 & 2 & 1 \\
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\end{align*}
\]

Rearranging the left hand side yields

\[
\begin{align*}
-2c_1 + c_2 + 2c_3 &= 0 \\
1c_1 + 0c_2 - 2c_3 &= 0 \\
3c_1 + 2c_2 + 1c_3 &= 0
\end{align*}
\]

The matrix equation above is equivalent to the following **homogeneous system of equations**

\[
\begin{align*}
-2c_1 + c_2 + 2c_3 &= 0 \\
1c_1 + 0c_2 - 2c_3 &= 0 \\
3c_1 + 2c_2 + 1c_3 &= 0
\end{align*}
\]

We now transform the coefficient matrix of the homogeneous system above to the reduced row echelon form to determine whether the system has

- the trivial solution only (meaning that $S$ is linearly independent), or
- the trivial solution as well as nontrivial ones ($S$ is linearly dependent).

\[
\begin{bmatrix}
-2 & 1 & 2 \\
1 & 0 & -2 \\
3 & 2 & 1 \\
\end{bmatrix}
\]

can be transformed by a sequence of elementary row operations to the matrix

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

The reduced row echelon form of the coefficient matrix of the homogeneous system is
which corresponds to the system

\[
\begin{align*}
1 \ c_1 &= 0 \\
1 \ c_2 &= 0 \\
1 \ c_3 &= 0
\end{align*}
\]

Since each column contains a leading entry (highlighted in yellow), then the system has only the trivial solution, so that the only solution is \( c_1, c_2, c_3 = 0 \).

Therefore the set \( S = \{ v_1, v_2, v_3 \} \) is \textbf{linearly independent}.

**Question # 4**

Determine, without solving, whether the following set of vectors is linearly independent or dependent.

\[
S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 6 \\ 7 \end{bmatrix} \right\}
\]

Solution:

Using the same rule in previous question, with our vectors \( v_1, v_2, v_3, v_4 \)

\[
\begin{align*}
c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + & c_2 \begin{bmatrix} 3 \\ 4 \\ 6 \\ 7 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 6 \\ 3 \\ 3 \end{bmatrix} + c_4 \begin{bmatrix} 4 \\ 7 \\ 3 \\ 3 \end{bmatrix} = & 0 \\
\end{align*}
\]

Rearranging the left hand side yields

\[
\begin{align*}
1 \ c_1 + & 3 \ c_2 + 4 \ c_3 + 7 \ c_4 = 0 \\
2 \ c_1 + & 4 \ c_2 + 6 \ c_3 + 3 \ c_4 = 0
\end{align*}
\]

The matrix equation above is equivalent to the following \textbf{homogeneous system of equations}

\[
\begin{align*}
1 \ c_1 + & 3 \ c_2 + 4 \ c_3 + 7 \ c_4 = 0
\end{align*}
\]
\[ 2c_1 + 4c_2 + 6c_3 + 3c_4 = 0 \]

- We now transform the coefficient matrix of the homogeneous system above to the reduced row echelon form

\[
\begin{bmatrix}
1 & 3 & 4 & 7 \\
2 & 4 & 6 & 3
\end{bmatrix}
\]

can be transformed by a sequence of elementary row operations to the matrix

\[
\begin{bmatrix}
1 & 0 & 1 & -19/2 \\
0 & 1 & 1 & 11/2
\end{bmatrix}
\]

The reduced row echelon form of the coefficient matrix of the homogeneous system is

\[
\begin{bmatrix}
1 & 0 & 1 & -19/2 \\
0 & 1 & 1 & 11/2
\end{bmatrix}
\]

which corresponds to the system

\[
\begin{align*}
1c_1 & + 1c_3 + (-19/2)c_4 = 0 \\
1c_2 & + 1c_3 + (11/2)c_4 = 0
\end{align*}
\]

The leading entries have been highlighted in yellow.

Since some columns do not contain leading entries, then the system has nontrivial solutions, so that some of the values \( c_1, c_2, c_3, c_4 \) solving the system of equation may be nonzero.

Therefore the set \( S = \{v_1, v_2, v_3, v_4\} \) is **linearly dependent.**
Question # 5

Show that the columns of \( A = \begin{bmatrix} 2 & 4 \\ 4 & -3 \end{bmatrix} \) are linearly independent.

Solution:

Vectors in \( \mathbb{R}^2 \) is **linearly independent** if the only solution of

\[ c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 = 0 \]

is \( c_1, c_2 = 0 \)

\[
\begin{bmatrix}
2 & 4 \\
4 & -3
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

Rearranging the left hand side yields

\[
\begin{bmatrix}
2c_1 + 4c_2 \\
4c_1 - 3c_2
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

The matrix equation above is equivalent to the following **homogeneous system of equations**

\[
\begin{align*}
2 c_1 & + 4 c_2 = 0 \\
4 c_1 & - 3 c_2 = 0
\end{align*}
\]

- We now transform the coefficient matrix of the homogeneous system above to the reduced row echelon form

\[
\begin{bmatrix}
2 & 4 \\
4 & -3
\end{bmatrix}
\]

can be transformed by a sequence of elementary row operations to the matrix

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]
The reduced row echelon form of the coefficient matrix of the homogeneous is

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]

which corresponds to the system

\[
\begin{align*}
1 \ c_1 & = 0 \\
1 \ c_2 & = 0
\end{align*}
\]

Since each column contains a leading entry (highlighted in yellow), then the system has only the trivial solution, so that the only solution is \(c_1, c_2 = 0\).

Therefore columns of \(A\) are **linearly independent**.