Practice Question Lecture # 35

Question:

Find the dominant Eigen pair (i.e. the Eigen value and Eigen vector) by using the Power Method for the following matrix.

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solution:

$$Ax_o = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$u_o = 4$$

$$x_1 = \frac{1}{u_o} Ax_o = \frac{1}{4}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/4 \end{bmatrix}$$

$$Ax_1 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 17/4 \\ 7/4 \end{bmatrix} = \begin{bmatrix} 4.25 \\ 1.75 \end{bmatrix}$$

$$u_1 = 4.25$$

Repeat *the process until you get the repeated value of u.*

Question:

Perform next iteration for power method, where $Ax_o = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$,where A= $\begin{bmatrix} 3 & 1 \\ 2 & 7 \end{bmatrix}$

Solution:

$$\mu_o = 2$$

$$x_1 = \frac{1}{\mu_o} Ax_o$$

$$x_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} .5 \\ 1 \end{bmatrix}$$

$$Ax_1 = \begin{bmatrix} 3 & 1 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

Questions:

Perform next iteration for power method, where $Ax_1 = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$, where A= $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

Solution:

Same process as above.

Question

Check whether the matrix $\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ has orthonormal columns or not?

Solution:

$$u = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$

$$U^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$U^T U = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \boldsymbol{I}$$

The columns of U are orthonormal.

Question:

Determine whether the vectors $\boldsymbol{y} = \begin{bmatrix} -2 \\ -3 \\ 4 \\ 1 \end{bmatrix}, \boldsymbol{z} = \begin{bmatrix} 7 \\ -2 \\ 1 \\ 4 \end{bmatrix}$ are orthogonal.

Solution:

y. z = -14+6+4+4 = 0
$\therefore$ **y** and **z** are orthogonal.

Question

Find the distance between $x = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$ *and* $y = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$.

Solution:

$$dis(x, y) = \|x - y\|^2 = |7-(-1)|^2 + |-3-(-2)|^2 = 8^2 + 1 = 65$$

Question:

Let $u = \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix}, v = \begin{bmatrix} 2 \\ -5 \\ 7 \end{bmatrix}$ .Compute and compare $u.v, \|u\|^2, \|v\|^2$ *and* $\|u+v\|^2$.

Solution:

$$u.v = 3(2) + (-4)(-5) + (-2)(7)$$

$$\|u\|^2 = u.u$$

$$\|v\|^2 = v.\mathrm{v}$$

$$\|u+v\|^2 = (u+v).(u+v)$$

Question:

Let $u = \begin{bmatrix} -3 \\ 4 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, w = \begin{bmatrix} 4 \\ -4 \\ -2 \end{bmatrix}$ .Find

$(a)\ \dfrac{v.u}{u.u}$

$(b)\ \|w\|$

(c) $(\dfrac{u.v}{v.v})v$

Same process as in above question.

Question:

Express the vector v in terms of the orthogonal basis B = {u_1, u_2, u_3}, where

$$v = \begin{bmatrix} -2 \\ 3 \\ 5 \\ -1 \end{bmatrix}, u_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}, u_2 = \begin{bmatrix} -3 \\ -1 \\ -1 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Solution:

Use the following formula:

$$v = \frac{v.u_1}{u_1.u_1}.u_1 + \frac{v.u_2}{u_2.u_2}.u_2 + \frac{v.u_3}{u_3.u_3}.u_3$$

Question:

Determine whether the set S= {u_1, u_2, u_3} is an orthogonal set?

Where $u_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $u_3 = \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$

Solution:

If u1.u2, u1.u3 and u2.u3 are equal to zero then the S is an orthogonal set.

Question:

Compute the orthogonal projection of $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ onto the line through $\begin{bmatrix} 2 \\ -5 \end{bmatrix}$ and the origin.

Solution:

$$\hat{y} = \frac{y.u}{u.u}.u$$

Then compute

$$\|\hat{y} - y\|$$

Here y= $y = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, and \quad u = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

Question:

Let $y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $u = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$. Compute the distance from y to the line through u and the origin.

Solution:

Since we know that the distance from a vector ***y*** to a line through the line from ***u*** and origin is $\|y - projection\ of\ \ y\ on\ u\|$ also we know that

$projection\ of\ \ y\ on\ u$

$$\hat{y} = \frac{y.u}{u.u}u$$

Then compute

$\|\hat{y} - y\|$

Here y= $y = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, and \quad u = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

Question:

Find the orthogonal projection of **y** onto $Span\{u_1, u_2\}$.

$$y = \begin{bmatrix} -8 \\ -5 \\ 4 \end{bmatrix}, u_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, u_2 = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix}$$

Solution:

Same as above

Question:

Find a least square solution for the system ***Ax = b***

Where $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 4 & 3 \end{bmatrix}, b = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

Solution:

First Compute $A^T A\ and\ A^T b$

Then using the formula

$$A^T Ax = A^T b$$

Compute the value of x.

Question

Apply the Gram-Schmidt process to transform the vectors $u_1 = (1, 0, 0), u_2 = (0,1,0), u_3 = (0,0,1)$ into an orthonormal basis.

Solution:

Let $v_1 = u_1$

Now

$$v_2 = u_2 - \frac{u_1.v_1}{v_1.v_1} v_1$$

$$v_2 = u_3 - \left[\frac{u_3.v_1}{v_1.v_1} v_1 + \frac{u_3.v_2}{v_2.v_2} v_2 \right]$$

Thus v_1, v_2, v_3 are orthonormal basis.

Question

Let W = Span {x_1, x_2}, where $x_1 = \begin{bmatrix} 6 \\ 0 \\ -2 \end{bmatrix}, x_2 = \begin{bmatrix} -4 \\ 3 \\ -2 \end{bmatrix}$. Construct an orthogonal basis {v_1, v_2} for W.

Solution:

Same as above.

Question

Let W be the subspace of R^2 spanned by $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$. Find a unit vector that is a basis for W.

Solution:

Let

$$y = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$\|y\|^2 = 4^2 + 6^2 = 16 + 36 = 52$$

Now compute $\|y\| = \sqrt{52}$

So

$$z = \frac{1}{2\sqrt{13}} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{13} \\ 3/\sqrt{13} \end{bmatrix}$$