Question:

Determine if the columns of the given matrix: $\begin{pmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & 1 & 3 \end{pmatrix}$ form a linearly dependent set.

Solution:

$\because$we now that the set of column vectors $\vec{u}, \vec{v}$, $\vec{w}$ and $\vec{r}$ are Linearly Independent$\Longleftrightarrow$the vector equation: $x\vec{u}+y\vec{v}+z\vec{w}+t\vec{r} = \vec{0}$, has the trivial($zero$) solution, where x, y, z and t are unknowns.

Now $x\vec{u} + y\vec{v} + z\vec{w} + t\vec{r} = \vec{0}$

$$\Longrightarrow x \begin{pmatrix} 1 \\ 0 \\ -1 \\ 3 \end{pmatrix} + y \begin{pmatrix} -1 \\ 1 \\ 2 \\ -1 \end{pmatrix} + z \begin{pmatrix} -3 \\ 5 \\ 8 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 4 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} x - y - 3z \\ 4t + y + 5z \\ 5t - x + 2y + 8z \\ 3t + 3x - y + z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \begin{matrix} x - y - 3z = 0 \\ 4t + y + 5z = 0 \\ 5t - x + 2y + 8z = 0 \\ 3t + 3x - y + z = 0 \end{matrix}$$ $\Longrightarrow$ is the associated system of Linear Equations, whose Corresponding augmented matrix: $\begin{pmatrix} 1 & -1 & -3 & 0 & 0 \\ 0 & 1 & 5 & 4 & 0 \\ -1 & 2 & 8 & 5 & 0 \\ 3 & -1 & 1 & 3 & 0 \end{pmatrix}$

Now reducing this by applying Elementary row operations to have its reduce Echelon form.

$$\begin{pmatrix} 1 & -1 & -3 & 0 & 0 \\ 0 & 1 & 5 & 4 & 0 \\ -1 & 2 & 8 & 5 & 0 \\ 3 & -1 & 1 & 3 & 0 \end{pmatrix}$$

By $R_3' \to R_3 + R_1, R_4' \to R_4 - 3R_1$

$$\sim \begin{pmatrix} 1 & -1 & -3 & 0 & 0 \\ 0 & 1 & 5 & 4 & 0 \\ -1+1 & 2+(-1) & 8+(-3) & 5+0 & 0+0 \\ 3-3(1) & -1-3(-1) & 1-3(-3) & 3-3(0) & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -3 & 0 & 0 \\ 0 & 1 & 5 & 4 & 0 \\ 0 & 1 & 5 & 5 & 0 \\ 0 & 2 & 10 & 3 & 0 \end{pmatrix}$$

By $R_3' \to R_3 - R_2, R_4' \to R_4 - 2R_2$

$$\sim \begin{pmatrix} 1 & -1 & -3 & 0 & 0 \\ 0 & 1 & 5 & 4 & 0 \\ 0 & 1-1 & 5-5 & 5-4 & 0-0 \\ 0 & 2-2(1) & 10-2(5) & 3-2(4) & 0-2(0) \end{pmatrix} = \begin{pmatrix} 1 & -1 & -3 & 0 & 0 \\ 0 & 1 & 5 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{pmatrix}$$

By $R_4' \to R_4 + 5R_3$

$$\sim \begin{pmatrix} 1 & -1 & -3 & 0 & 0 \\ 0 & 1 & 5 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -5+5(1) & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -3 & 0 & 0 \\ 0 & 1 & 5 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$ i.e. the Echelon form.

Now last row $0t = 0$,which is true $\forall t \in \mathbb{R}$

$\Longrightarrow$ the system has infinite many solution.

$\Longrightarrow$ the vector equation does not have the trivial solution:$(x, y, z) \neq (0, 0, 0)$

$\therefore$ the given vectors are Linearly dependent.

Alternate method:

$\because$ we know that three vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ and $\vec{r}$ are Linearly Independent$\Longleftrightarrow$the matrix $\begin{pmatrix} \vec{u} & \vec{v} & \vec{w} & \vec{r} \end{pmatrix}$ is a square matrix and its determinant is non-zero.

$\therefore$ taking $\det \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} & \vec{r} \end{pmatrix}$

$$= \det \begin{pmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & 1 & 3 \end{pmatrix}$$

By elementary row operations same as above, we get

$$= \det \begin{pmatrix} 1 & -1 & -3 & 0 & 0 \\ 0 & 1 & 5 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 0$$ (since the last column is zero and hence the determinant is zero)

$\Longrightarrow$ the given vectors are Linearly dependent.