Question:

Decide if the vectors $\overrightarrow{u} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$, $\overrightarrow{v} = \begin{pmatrix} -3 \\ -5 \\ -2 \end{pmatrix}$ and $\overrightarrow{w} = \begin{pmatrix} 0 \\ 5 \\ -6 \end{pmatrix}$ are linearly independent. Give a reason for each answer.

Solution:

$\because$we now that the set of vectors $\overrightarrow{u}$, $\overrightarrow{v}$ and $\overrightarrow{w}$ are Linearly Independent$\Longleftrightarrow$the vector equation: $x\overrightarrow{u} + y\overrightarrow{v} + z\overrightarrow{w} = \overrightarrow{0}$, has the trivial($zero$) solution, where x, y and z are unknowns.

Now $x\overrightarrow{u} + y\overrightarrow{v} + z\overrightarrow{w} = \overrightarrow{0}$

$$\Longrightarrow x \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + y \begin{pmatrix} -3 \\ -5 \\ -2 \end{pmatrix} + z \begin{pmatrix} 0 \\ 5 \\ -6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} x - 3y \\ 3x - 5y + 5z \\ -2x - 2y - 6z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \begin{matrix} x - 3y = 0 \\ 3x - 5y + 5z = 0 \\ -2x - 2y - 6z = 0 \end{matrix}$$ is the associated system of Linear Equations, whose

Corresponding augmented matrix: $\begin{pmatrix} 1 & -3 & 0 & 0 \\ 3 & -5 & 5 & 0 \\ -2 & -2 & -6 & 0 \end{pmatrix}$.

Now reducing this by applying Elementary row operations to have its reduce Echelon form.

$$\begin{pmatrix} 1 & -3 & 0 & 0 \\ 3 & -5 & 5 & 0 \\ -2 & -2 & -6 & 0 \end{pmatrix}$$

By $R_2' \to R_2 - 3R_1, R_3' \to R_3 + 2R_1$

$$\sim \begin{pmatrix} 1 & -3 & 0 & 0 \\ 3-3(1) & -5-3(-3) & 5-3(0) & 0-3(0) \\ -2+2(1) & -2+2(-3) & -6+2(0) & 0+2(0) \end{pmatrix} = \begin{pmatrix} 1 & -3 & 0 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & -8 & -6 & 0 \end{pmatrix}$$

By $R_3' \to R_3 + 2R_2$

$$\sim \begin{pmatrix} 1 & -3 & 0 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & -8+2(4) & -6+2(5) & 0 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 0 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 4 & 0 \end{pmatrix}$$ i.e. the Echelon form.

Now last row$\Longrightarrow 4z = 0 \Longrightarrow z = 0$

2nd row$\Longrightarrow 4y + 5z = 0 \Longrightarrow y = 0$

and 3rd row$\Longrightarrow x - 3y + 0z = 0 \Longrightarrow x = 0$

$\Longrightarrow$ the vector equation has the trivial solution:$(x, y, z) = (0, 0, 0)$

$\therefore$ the given vectors are Linearly Independent.

Alternate method:

$\because$ we know that three vectors $\vec{u}, \vec{v}$ and $\vec{w}$ are Linearly Independent$\Longleftrightarrow$the matrix $\begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix}$ is a square matrix and its determinant is non-zero.

$\therefore$ taking $\det \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix}$

$$= \det \begin{pmatrix} 1 & -3 & 0 \\ 3 & -5 & 5 \\ -2 & -2 & -6 \end{pmatrix}$$

$$= 1 \begin{vmatrix} -5 & 5 \\ -2 & -6 \end{vmatrix} - (-3) \begin{vmatrix} 3 & 5 \\ -2 & -6 \end{vmatrix} + 0 \begin{vmatrix} 3 & -5 \\ -2 & -2 \end{vmatrix}$$

$$= 40 - 24 = 16 \neq 0$$

$\implies$ the given vectors are Linearly Independent.

Geometrical Method:

As can be seen that the given three vectors: $\vec{u} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \vec{v} = \begin{pmatrix} -3 \\ -5 \\ -2 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 0 \\ 5 \\ -6 \end{pmatrix}$ are belonged to $\mathbb{R}^3$ space and all these three vectors do not lie on the same plane in $\mathbb{R}^3$ and all three have different directions(because if any of two will have same direction, then these will be Linearly dependent)

$\implies$ these three vectors are linearly Independent.