Question:

Use as many columns of $A = \begin{pmatrix} 8 & -3 & 0 & -7 & 2 \\ -9 & 4 & 5 & 11 & -7 \\ 6 & -2 & 2 & -4 & 4 \\ 5 & -1 & 7 & 0 & 10 \end{pmatrix}$ as possible to construct a matrix B with the property that equation $B\vec{x} = 0$ has only the trivial solution.

Solution:

To construct B, we firstly reduce A into Echelon form, then the columns corresponding to Pivots will form B.

$A = \begin{pmatrix} 8 & -3 & 0 & -7 & 2 \\ -9 & 4 & 5 & 11 & -7 \\ 6 & -2 & 2 & -4 & 4 \\ 5 & -1 & 7 & 0 & 10 \end{pmatrix}$

By $R_1' \rightarrow (\frac{1}{8})R_1$

$$\sim \begin{pmatrix} \frac{8}{8} & \frac{-3}{8} & \frac{0}{8} & \frac{-7}{8} & \frac{2}{8} \\ -9 & 4 & 5 & 11 & -7 \\ 6 & -2 & 2 & -4 & 4 \\ 5 & -1 & 7 & 0 & 10 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{3}{8} & 0 & -\frac{7}{8} & \frac{1}{4} \\ -9 & 4 & 5 & 11 & -7 \\ 6 & -2 & 2 & -4 & 4 \\ 5 & -1 & 7 & 0 & 10 \end{pmatrix}$$

By $R_2' \rightarrow R_2 + 9R_1, R_3' \rightarrow R_3 - 6R_1$ and $R_4' \rightarrow R_4 - 5R_1$

$$\sim \begin{pmatrix} 1 & -\frac{3}{8} & 0 & -\frac{7}{8} & \frac{1}{4} \\ -9+9(1) & 4+9(-\frac{3}{8}) & 5+9(0) & 11+9(-\frac{7}{8}) & -7+9(\frac{1}{4}) \\ 6-6(1) & -2-6(-\frac{3}{8}) & 2-6(0) & -4-6(-\frac{7}{8}) & 4-6(\frac{1}{4}) \\ 5-5(1) & -1-5(-\frac{3}{8}) & 7-5(0) & 0-5(-\frac{7}{8}) & 10-5(\frac{1}{4}) \end{pmatrix} = \begin{pmatrix} 1 & -\frac{3}{8} & 0 & -\frac{7}{8} & \frac{1}{4} \\ 0 & \frac{5}{8} & 5 & \frac{25}{8} & -\frac{19}{4} \\ 0 & \frac{1}{4} & 2 & \frac{5}{4} & \frac{5}{2} \\ 0 & \frac{7}{8} & 7 & \frac{35}{8} & \frac{35}{4} \end{pmatrix}$$

By $R_2' \rightarrow (\frac{8}{5})R_2$

$$\sim \begin{pmatrix} 1 & -\frac{3}{8} & 0 & -\frac{7}{8} & \frac{1}{4} \\ 0(\frac{8}{5}) & \frac{5}{8}(\frac{8}{5}) & 5(\frac{8}{5}) & \frac{25}{8}(\frac{8}{5}) & -\frac{19}{4}(\frac{8}{5}) \\ 0 & \frac{1}{4} & 2 & \frac{5}{4} & \frac{5}{2} \\ 0 & \frac{7}{8} & 7 & \frac{35}{8} & \frac{35}{4} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{3}{8} & 0 & -\frac{7}{8} & \frac{1}{4} \\ 0 & 1 & 8 & 5 & -\frac{38}{5} \\ 0 & \frac{1}{4} & 2 & \frac{5}{4} & \frac{5}{2} \\ 0 & \frac{7}{8} & 7 & \frac{35}{8} & \frac{35}{4} \end{pmatrix}$$

By $R_3' \rightarrow R_3 - \frac{1}{4}R_2$ and $R_4' \rightarrow R_4 - \frac{7}{8}R_2$

$$\sim \begin{pmatrix} 1 & -\frac{3}{8} & 0 & -\frac{7}{8} & \frac{1}{4} \\ 0 & 1 & 8 & 5 & -\frac{38}{5} \\ 0 & \frac{1}{4}-\frac{1}{4}(1) & 2-\frac{1}{4}(8) & \frac{5}{4}-\frac{1}{4}(5) & \frac{5}{2}-\frac{1}{4}(-\frac{38}{5}) \\ 0 & \frac{7}{8}-\frac{7}{8}(1) & 7-\frac{7}{8}(8) & \frac{35}{8}-\frac{7}{8}(5) & \frac{35}{4}-\frac{7}{8}(-\frac{38}{5}) \end{pmatrix} = \begin{pmatrix} 1 & -\frac{3}{8} & 0 & -\frac{7}{8} & \frac{1}{4} \\ 0 & 1 & 8 & 5 & -\frac{38}{5} \\ 0 & 0 & 0 & 0 & \frac{22}{5} \\ 0 & 0 & 0 & 0 & \frac{77}{5} \end{pmatrix}$$

By $R_3' \rightarrow (\frac{5}{22})R_3$

$$\sim \begin{pmatrix} 1 & -\frac{3}{8} & 0 & -\frac{7}{8} & \frac{1}{4} \\ 0 & 1 & 8 & 5 & -\frac{38}{5} \\ 0(\frac{5}{22}) & 0(\frac{5}{22}) & 0(\frac{5}{22}) & 0(\frac{5}{22}) & \frac{22}{5}(\frac{5}{22}) \\ 0 & 0 & 0 & 0 & \frac{77}{5} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{3}{8} & 0 & -\frac{7}{8} & \frac{1}{4} \\ 0 & 1 & 8 & 5 & -\frac{38}{5} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{77}{5} \end{pmatrix}$$

By $R_4' \rightarrow R_4 - \frac{77}{5}R_3$

$$\sim \begin{pmatrix} 1 & -\frac{3}{8} & 0 & -\frac{7}{8} & \frac{1}{4} \\ 0 & 1 & 8 & 5 & -\frac{38}{5} \\ 0 & 0 & 0 & 0 & 1 \\ 0-\frac{77}{5}(0) & 0-\frac{77}{5}(0) & 0-\frac{77}{5}(0) & 0-\frac{77}{5}(0) & \frac{77}{5}-\frac{77}{5}(1) \end{pmatrix} = \begin{pmatrix} \mathbf{1} & -\frac{3}{8} & 0 & -\frac{7}{8} & \frac{1}{4} \\ 0 & \mathbf{1} & 8 & 5 & -\frac{38}{5} \\ 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Here we can see that the possible Pivot Columns' combination to form B are (1^{st},2^{nd},5^{th}),(1^{st},3^{rd},5^{th}) and (1^{st},4^{th},5^{th}) from the GIVEN matrix A.

Now we discuss and verify all the above combinations from columns of A.

Case 1: for ($\mathbf{1^{st}}$,$\mathbf{2^{nd}}$,$\mathbf{5^{th}}$)

$$B = \begin{pmatrix} 8 & -3 & 2 \\ -9 & 4 & -7 \\ 6 & -2 & 4 \\ 5 & -1 & 10 \end{pmatrix}.$$

Observe that for $B\vec{x} = \vec{0}$,the order of $A = 4 \times 3$

$\Longrightarrow$no of columns of $A = 3$

$\Longrightarrow$the product will only be confirmable$\Longleftrightarrow$no. of rows in vector $\vec{x}$ are$= 3$

$\therefore$vector $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ say! where x, y, z are unknowns.

now $B\vec{x} = \vec{0}$

$$\Longrightarrow \begin{pmatrix} 8 & -3 & 2 \\ -9 & 4 & -7 \\ 6 & -2 & 4 \\ 5 & -1 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 8x-3y+2z \\ 4y-9x-7z \\ 6x-2y+4z \\ 5x-y+10z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \begin{matrix} 8x-3y+2z=0 \\ 4y-9x-7z=0 \\ 6x-2y+4z=0 \\ 5x-y+10z=0 \end{matrix}$$ is the associated system of Linear Equations, whose corresponding augmented matrix is: $\begin{pmatrix} 8 & -3 & 2 & 0 \\ -9 & 4 & -7 & 0 \\ 6 & -2 & 4 & 0 \\ 5 & -1 & 10 & 0 \end{pmatrix}$.Now reducing this into Echelon form as follows;

$$\begin{pmatrix} 8 & -3 & 2 & 0 \\ -9 & 4 & -7 & 0 \\ 6 & -2 & 4 & 0 \\ 5 & -1 & 10 & 0 \end{pmatrix}$$

By $R_1^{'} \rightarrow (\frac{1}{8})R_1$

$$\sim \begin{pmatrix} \frac{8}{8} & \frac{-3}{8} & \frac{2}{8} \\ -9 & 4 & -7 \\ 6 & -2 & 4 \\ 5 & -1 & 10 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{3}{8} & \frac{1}{4} \\ -9 & 4 & -7 \\ 6 & -2 & 4 \\ 5 & -1 & 10 \end{pmatrix}$$ **Note:**(Omitting last zero column as it does not suffer any change for Elementary row operation)

By $R_2^{'} \to R_2 + 9R_1, R_3^{'} \to R_3 - 6R_1$ and $R_4^{'} \to R_4 - 5R_1$

$$\sim \begin{pmatrix} 1 & -\frac{3}{8} & \frac{1}{4} \\ -9+9(1) & 4+9(-\frac{3}{8}) & -7+9(\frac{1}{4}) \\ 6-6(1) & -2-6(-\frac{3}{8}) & 4-6(\frac{1}{4}) \\ 5-5(1) & -1-5(-\frac{3}{8}) & 10-5(\frac{1}{4}) \end{pmatrix} = \begin{pmatrix} 1 & -\frac{3}{8} & \frac{1}{4} \\ 0 & \frac{5}{8} & -\frac{19}{4} \\ 0 & \frac{1}{4} & \frac{5}{2} \\ 0 & \frac{7}{8} & \frac{35}{4} \end{pmatrix}$$

By $R_2^{'} \to (\frac{8}{5})R_2$

$$\sim \begin{pmatrix} 1 & -\frac{3}{8} & \frac{1}{4} \\ 0(\frac{8}{5}) & \frac{5}{8}(\frac{8}{5}) & -\frac{19}{4}(\frac{8}{5}) \\ 0 & \frac{1}{4} & \frac{5}{2} \\ 0 & \frac{7}{8} & \frac{35}{4} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{3}{8} & \frac{1}{4} \\ 0 & 1 & -\frac{38}{5} \\ 0 & \frac{1}{4} & \frac{5}{2} \\ 0 & \frac{7}{8} & \frac{35}{4} \end{pmatrix}$$

By $R_3^{'} \to R_3 - \frac{1}{4}R_2$ and $R_4^{'} \to R_4 - \frac{7}{8}R_2$

$$\sim \begin{pmatrix} 1 & -\frac{3}{8} & \frac{1}{4} \\ 0 & 1 & -\frac{38}{5} \\ 0 & \frac{1}{4}-\frac{1}{4}(1) & \frac{5}{2}-\frac{1}{4}(-\frac{38}{5}) \\ 0 & \frac{7}{8}-\frac{7}{8}(1) & \frac{35}{4}-\frac{7}{8}(-\frac{38}{5}) \end{pmatrix} = \begin{pmatrix} 1 & -\frac{3}{8} & \frac{1}{4} \\ 0 & 1 & -\frac{38}{5} \\ 0 & 0 & \frac{22}{5} \\ 0 & 0 & \frac{77}{5} \end{pmatrix}$$

By $R_3^{'} \to (\frac{5}{22})R_3$

$$\sim \begin{pmatrix} 1 & -\frac{3}{8} & \frac{1}{4} \\ 0 & 1 & -\frac{38}{5} \\ 0(\frac{5}{22}) & 0(\frac{5}{22}) & \frac{22}{5}(\frac{5}{22}) \\ 0 & 0 & \frac{77}{5} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{3}{8} & \frac{1}{4} \\ 0 & 1 & -\frac{38}{5} \\ 0 & 0 & 1 \\ 0 & 0 & \frac{77}{5} \end{pmatrix}$$

By $R_4^{'} \to R_4 - \frac{77}{5}R_3$

$$\sim \begin{pmatrix} 1 & -\frac{3}{8} & \frac{1}{4} \\ 0 & 1 & -\frac{38}{5} \\ 0 & 0 & 1 \\ 0-\frac{77}{5}(0) & 0-\frac{77}{5}(0) & \frac{77}{5}-\frac{77}{5}(1) \end{pmatrix} = \begin{pmatrix} \mathbf{1} & -\frac{3}{8} & \frac{1}{4} \\ 0 & \mathbf{1} & -\frac{38}{5} \\ 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 \end{pmatrix}$$

Now the final form $\begin{pmatrix} 1 & -\frac{3}{8} & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{38}{5} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is the trivial solution.

Case 2: for (1^{st},3^{rd},5^{th})

$$B = \begin{pmatrix} 8 & 0 & 2 \\ -9 & 5 & -7 \\ 6 & 2 & 4 \\ 5 & 7 & 10 \end{pmatrix}$$

Observe that for $B\vec{x} = \vec{0}$,the order of $A = 4 \times 3$

$\Longrightarrow$no of columns of $A = 3$

$\Longrightarrow$the product will only be confirmable$\Longleftrightarrow$no. of rows in vector $\vec{x}$ are$= 3$

$\therefore$vector $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ say! where x, y, z are unknowns.

now $B\vec{x} = \vec{0}$

$$\Longrightarrow \begin{pmatrix} 8 & 0 & 2 \\ -9 & 5 & -7 \\ 6 & 2 & 4 \\ 5 & 7 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Longrightarrow \begin{pmatrix} 8x+2z \\ 5y-9x-7z \\ 6x+2y+4z \\ 5x+7y+10z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{matrix} 8x+2z=0 \\ 5y-9x-7z=0 \\ 6x+2y+4z=0 \\ 5x+7y+10z=0 \end{matrix}$, is the associated system of Linear Equations, whose augmented matrix is: $\begin{pmatrix} 8 & 0 & 2 & 0 \\ -9 & 5 & -7 & 0 \\ 6 & 2 & 4 & 0 \\ 5 & 7 & 10 & 0 \end{pmatrix}$.Now reducing this into Echelon form.

By $R_1^{'} \rightarrow (\frac{1}{8})R_1$

$\sim \begin{pmatrix} \frac{8}{8} & \frac{0}{8} & \frac{2}{8} \\ -9 & 5 & -7 \\ 6 & 2 & 4 \\ 5 & 7 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{1}{4} \\ -9 & 5 & -7 \\ 6 & 2 & 4 \\ 5 & 7 & 10 \end{pmatrix}$ **Note:**(Omitting last zero column as it does not suffer any change for Elementary row operation)

By $R_2^{'} \rightarrow R_2 + 9R_1, R_3^{'} \rightarrow R_3 - 6R_1$ and $R_4^{'} \rightarrow R_4 - 5R_1$

$$\sim \begin{pmatrix} 1 & 0 & \frac{1}{4} \\ -9+9(1) & 5+9(0) & -7+9(\frac{1}{4}) \\ 6-6(1) & 2-6(0) & 4-6(\frac{1}{4}) \\ 5-5(1) & 7-5(0) & 10-5(\frac{1}{4}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 5 & -\frac{19}{4} \\ 0 & 2 & \frac{5}{2} \\ 0 & 7 & \frac{35}{4} \end{pmatrix}$$

By $R_2^{'} \rightarrow (\frac{8}{5})R_2$

$$\sim \begin{pmatrix} 1 & 0 & \frac{1}{4} \\ 0(\frac{8}{5}) & 5(\frac{8}{5}) & -\frac{19}{4}(\frac{8}{5}) \\ 0 & 2 & \frac{5}{2} \\ 0 & 7 & \frac{35}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 8 & -\frac{38}{5} \\ 0 & 2 & \frac{5}{2} \\ 0 & 7 & \frac{35}{4} \end{pmatrix}$$

By $R_3^{'} \rightarrow R_3 - \frac{1}{4}R_2$ and $R_4^{'} \rightarrow R_4 - \frac{7}{8}R_2$

$$\sim \begin{pmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 8 & -\frac{38}{5} \\ 0 & 2-\frac{1}{4}(8) & \frac{5}{2}-\frac{1}{4}(-\frac{38}{5}) \\ 0 & 7-\frac{7}{8}(8) & \frac{35}{4}-\frac{7}{8}(-\frac{38}{5}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 8 & -\frac{38}{5} \\ 0 & 0 & \frac{22}{5} \\ 0 & 0 & \frac{77}{5} \end{pmatrix}$$

By $R_3^{'} \rightarrow (\frac{5}{22})R_3$

$$\sim \begin{pmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 8 & -\frac{38}{5} \\ 0(\frac{5}{22}) & 0(\frac{5}{22}) & \frac{22}{5}(\frac{5}{22}) \\ 0 & 0 & \frac{77}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 8 & -\frac{38}{5} \\ 0 & 0 & 1 \\ 0 & 0 & \frac{77}{5} \end{pmatrix}$$

By $R_4' \to R_4 - \frac{77}{5}R_3$

$$\sim \begin{pmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 8 & -\frac{38}{5} \\ 0 & 0 & 1 \\ 0-\frac{77}{5}(0) & 0-\frac{77}{5}(0) & \frac{77}{5}-\frac{77}{5}(1) \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 & \frac{1}{4} \\ 0 & \mathbf{8} & -\frac{38}{5} \\ 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 \end{pmatrix}.$$

Now the final form: $\begin{pmatrix} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 8 & -\frac{38}{5} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is the trivial solution.

Case: $(\mathbf{1}^{st}, \mathbf{4}^{th}, \mathbf{5}^{th})$

$$B = \begin{pmatrix} 8 & -7 & 2 \\ -9 & 11 & -7 \\ 6 & -4 & 4 \\ 5 & 0 & 10 \end{pmatrix}$$

Observe that for $B\vec{x} = \vec{0}$,the order of $A = 4 \times 3$

$\Longrightarrow$no of columns of $A = 3$

$\Longrightarrow$the product will only be confirmable$\Longleftrightarrow$no. of rows in vector $\vec{x}$ are$= 3$

$\therefore$vector $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ say! where x, y, z are unknowns.

now $B\vec{x} = \vec{0}$

$$\Longrightarrow \begin{pmatrix} 8 & -7 & 2 \\ -9 & 11 & -7 \\ 6 & -4 & 4 \\ 5 & 0 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 8x - 7y + 2z \\ 11y - 9x - 7z \\ 6x - 4y + 4z \\ 5x + 10z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Longrightarrow$

$\begin{matrix} 8x - 7y + 2z = 0 \\ 11y - 9x - 7z = 0 \\ 6x - 4y + 4z = 0 \\ 5x + 10z = 0 \end{matrix}$ is the associated system of Linear Equations, whose augmented matrix is: $\begin{pmatrix} 8 & -7 & 2 & 0 \\ -9 & 11 & -7 & 0 \\ 6 & -4 & 4 & 0 \\ 5 & 0 & 10 & 0 \end{pmatrix}$.Now reducing this into Echelon form.

$$\begin{pmatrix} 8 & -7 & 2 & 0 \\ -9 & 11 & -7 & 0 \\ 6 & -4 & 4 & 0 \\ 5 & 0 & 10 & 0 \end{pmatrix}$$

By $R_1' \to (\frac{1}{8})R_1$

$$\sim \begin{pmatrix} \frac{8}{8} & \frac{-7}{8} & \frac{2}{8} \\ -9 & 11 & -7 \\ 6 & -4 & 4 \\ 5 & 0 & 10 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{7}{8} & \frac{1}{4} \\ -9 & 11 & -7 \\ 6 & -4 & 4 \\ 5 & 0 & 10 \end{pmatrix}$$

Note:(Omitting last zero column as it does not suffer any change for Elementary row operation)

By $R_2^{'} \to R_2 + 9R_1, R_3^{'} \to R_3 - 6R_1$ and $R_4^{'} \to R_4 - 5R_1$

$$\sim \begin{pmatrix} 1 & -\frac{7}{8} & \frac{1}{4} \\ -9+9(1) & 11+9(-\frac{7}{8}) & -7+9(\frac{1}{4}) \\ 6-6(1) & -4-6(-\frac{7}{8}) & 4-6(\frac{1}{4}) \\ 5-5(1) & 0-5(-\frac{7}{8}) & 10-5(\frac{1}{4}) \end{pmatrix} = \begin{pmatrix} 1 & -\frac{7}{8} & \frac{1}{4} \\ 0 & \frac{25}{8} & -\frac{19}{4} \\ 0 & \frac{5}{4} & \frac{5}{2} \\ 0 & \frac{35}{8} & \frac{35}{4} \end{pmatrix}$$

By $R_2^{'} \to (\frac{8}{5})R_2$

$$\sim \begin{pmatrix} 1 & -\frac{7}{8} & \frac{1}{4} \\ 0(\frac{8}{5}) & \frac{25}{8}(\frac{8}{5}) & -\frac{19}{4}(\frac{8}{5}) \\ 0 & \frac{5}{4} & \frac{5}{2} \\ 0 & \frac{35}{8} & \frac{35}{4} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{7}{8} & \frac{1}{4} \\ 0 & 5 & -\frac{38}{5} \\ 0 & \frac{5}{4} & \frac{5}{2} \\ 0 & \frac{35}{8} & \frac{35}{4} \end{pmatrix}$$

By $R_3^{'} \to R_3 - \frac{1}{4}R_2$ and $R_4^{'} \to R_4 - \frac{7}{8}R_2$

$$\sim \begin{pmatrix} 1 & -\frac{7}{8} & \frac{1}{4} \\ 0 & 5 & -\frac{38}{5} \\ 0 & \frac{5}{4}-\frac{1}{4}(5) & \frac{5}{2}-\frac{1}{4}(-\frac{38}{5}) \\ 0 & \frac{35}{8}-\frac{7}{8}(5) & \frac{35}{4}-\frac{7}{8}(-\frac{38}{5}) \end{pmatrix} = \begin{pmatrix} 1 & -\frac{7}{8} & \frac{1}{4} \\ 0 & 5 & -\frac{38}{5} \\ 0 & 0 & \frac{22}{5} \\ 0 & 0 & \frac{77}{5} \end{pmatrix}$$

By $R_3^{'} \to (\frac{5}{22})R_3$

$$\sim \begin{pmatrix} 1 & -\frac{7}{8} & \frac{1}{4} \\ 0 & 5 & -\frac{38}{5} \\ 0(\frac{5}{22}) & 0(\frac{5}{22}) & \frac{22}{5}(\frac{5}{22}) \\ 0 & 0 & \frac{77}{5} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{7}{8} & \frac{1}{4} \\ 0 & 5 & -\frac{38}{5} \\ 0 & 0 & 1 \\ 0 & 0 & \frac{77}{5} \end{pmatrix}$$

By $R_4^{'} \to R_4 - \frac{77}{5}R_3$

$$\sim \begin{pmatrix} 1 & -\frac{7}{8} & \frac{1}{4} \\ 0 & 5 & -\frac{38}{5} \\ 0 & 0 & 1 \\ 0-\frac{77}{5}(0) & 0-\frac{77}{5}(0) & \frac{77}{5}-\frac{77}{5}(1) \end{pmatrix} = \begin{pmatrix} \mathbf{1} & -\frac{7}{8} & \frac{1}{4} \\ 0 & \mathbf{5} & -\frac{38}{5} \\ 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 \end{pmatrix}.$$

Now the final form: $\begin{pmatrix} 1 & -\frac{7}{8} & \frac{1}{4} & 0 \\ 0 & 5 & -\frac{38}{5} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \implies \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is the trivial solution. Hence in all three cases, we get the Trivial solution. But one can see that in all three cases, we get 1^{st} two pivots in b/w 1^{st} and 4^{th}columns while 3^{rd}pivot arise in the 5^{th}.

$\therefore 5^{th}$ column can never be generated by 1^{st} four columns of matrix A.