Question:

Given that $A = \begin{pmatrix} 2 & 3 & 5 \\ 5 & 1 & 4 \\ 3 & 1 & 4 \\ 1 & 0 & 1 \end{pmatrix}$, observe that the third column is the sum of the first two columns. Find a nontrivial solution of $A\vec{x} = \vec{0}$ without performing row operations.

Solution:

Observe that for $A\vec{x} = \vec{0}$,the order of $A = 4 \times 3$

$\Longrightarrow$no of columns of $A = 3$

$\Longrightarrow$the product will only be confirmable$\Longleftrightarrow$no. of rows in vector $\vec{x}$ are= 3

$\therefore$vector $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ say! where x, y, z are unknowns.

now $A\vec{x} = \vec{0} \Longrightarrow \begin{pmatrix} 2 & 3 & 5 \\ 5 & 1 & 4 \\ 3 & 1 & 4 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Longrightarrow \begin{pmatrix} 2x+3y+5z \\ 5x+y+4z \\ 3x+y+4z \\ x+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 2x \\ 5x \\ 3x \\ x \end{pmatrix} + \begin{pmatrix} 3y \\ y \\ y \\ 0y \end{pmatrix} + \begin{pmatrix} 5z \\ 4z \\ 4z \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow x \begin{pmatrix} 2 \\ 5 \\ 3 \\ 1 \end{pmatrix} + y \begin{pmatrix} 3 \\ 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 5 \\ 4 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} --------(1)$$

$\because$it is given that, " *the third column of A is the sum of the first two columns*"

$$\Longrightarrow \begin{pmatrix} 2 \\ 5 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 4 \\ 1 \end{pmatrix}$$

By adding additive inverse of $\begin{pmatrix} 5 \\ 4 \\ 4 \\ 1 \end{pmatrix}$ on both sides.

$$\therefore\Longrightarrow \begin{pmatrix} 2 \\ 5 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 4 \\ 1 \end{pmatrix}$$

$$\implies \begin{pmatrix} 2 \\ 5 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and quite obviously above can be expressed also as;

$$\implies (1) \begin{pmatrix} 2 \\ 5 \\ 3 \\ 1 \end{pmatrix} + (1) \begin{pmatrix} 3 \\ 1 \\ 1 \\ 0 \end{pmatrix} + (-1) \begin{pmatrix} 5 \\ 4 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - - - - - (2)$$

Now comparing left hand sides of (1) and (2), we can see that

$x = 1, y = 1$ and $z = -1$

$\therefore$the required solution is: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$