Question:

Determine if the system has a nontrivial solution. Try to use as few row operations as possible.

$3x+6y-4z-t=0$
$-5x+8z+3t=0$
$8x-y+7t=0$

Solution:

The given system is $\begin{matrix} 3x+6y-4z-t=0 \\ -5x+8z+3t=0 \\ 8x-y+7t=0 \end{matrix}$ $\Longrightarrow$ its Corresponding augmented matrix: $\begin{pmatrix} 3 & 6 & -4 & -1 & 0 \\ -5 & 0 & 8 & 3 & 0 \\ 8 & -1 & 0 & 7 & 0 \end{pmatrix}$.Now we apply elementary row operations to reduce it into reduced Echelon form.

$$\begin{pmatrix} 3 & 6 & -4 & -1 & 0 \\ -5 & 0 & 8 & 3 & 0 \\ 8 & -1 & 0 & 7 & 0 \end{pmatrix}$$

By $R_1^{'} \longrightarrow (\frac{1}{3})R_1$,

$$\sim \begin{pmatrix} \frac{3}{3} & \frac{6}{3} & \frac{-4}{3} & \frac{-1}{3} & 0 \\ -5 & 0 & 8 & 3 & 0 \\ 8 & -1 & 0 & 7 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -\frac{4}{3} & -\frac{1}{3} & 0 \\ -5 & 0 & 8 & 3 & 0 \\ 8 & -1 & 0 & 7 & 0 \end{pmatrix}$$

By $R_2^{'} \longrightarrow R_2+5R_1, R_3^{'} \longrightarrow R_3-8R_1$

$$\sim \begin{pmatrix} 1 & 2 & -\frac{4}{3} & -\frac{1}{3} & 0 \\ -5+5(1) & 0+5(2) & 8+5(-\frac{4}{3}) & 3+5(-\frac{1}{3}) & 0+5(0) \\ 8-8(1) & -1-8(2) & 0-8(-\frac{4}{3}) & 7-8(-\frac{1}{3}) & 0-8(0) \end{pmatrix} = \begin{pmatrix} 1 & 2 & -\frac{4}{3} & -\frac{1}{3} & 0 \\ 0 & 10 & \frac{4}{3} & \frac{4}{3} & 0 \\ 0 & -17 & \frac{32}{3} & \frac{29}{3} & 0 \end{pmatrix}$$

By $R_2^{'} \longrightarrow (\frac{1}{10})R_1$,

$$\sim \begin{pmatrix} 1 & 2 & -\frac{4}{3} & -\frac{1}{3} & 0 \\ 0 & \frac{10}{10} & \frac{4}{3}\left(\frac{1}{10}\right) & \frac{4}{3}\left(\frac{1}{10}\right) & 0\left(\frac{1}{10}\right) \\ 0 & -17 & \frac{32}{3} & \frac{29}{3} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -\frac{4}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{15} & \frac{2}{15} & 0 \\ 0 & -17 & \frac{32}{3} & \frac{29}{3} & 0 \end{pmatrix}$$

By $R_1^{'} \longrightarrow R_1-2R_2, R_3^{'} \longrightarrow R_3+17R_2$

$$\sim \begin{pmatrix} 1-2(0) & 2-2(1) & -\frac{4}{3}-2(\frac{2}{15}) & -\frac{1}{3}-2(\frac{2}{15}) & 0 \\ 0 & 1 & \frac{2}{15} & \frac{2}{15} & 0 \\ 0+17(0) & -17+17(1) & \frac{32}{3}+17\left(\frac{2}{15}\right) & \frac{29}{3}+17\left(\frac{2}{15}\right) & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{8}{5} & -\frac{3}{5} & 0 \\ 0 & 1 & \frac{2}{15} & \frac{2}{15} & 0 \\ 0 & 0 & \frac{194}{15} & \frac{179}{15} & 0 \end{pmatrix}$$

By $R_3^{'} \longrightarrow (\frac{15}{194})R_3$

$$\sim \begin{pmatrix} 1 & 0 & -\frac{8}{5} & -\frac{3}{5} & 0 \\ 0 & 1 & \frac{2}{15} & \frac{2}{15} & 0 \\ 0 & 0 & \frac{194}{15}(\frac{15}{194}) & \frac{179}{15}(\frac{15}{194}) & 0(\frac{15}{194}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{8}{5} & -\frac{3}{5} & 0 \\ 0 & 1 & \frac{2}{15} & \frac{2}{15} & 0 \\ 0 & 0 & 1 & \frac{179}{194} & 0 \end{pmatrix}$$

By $R_2^{'} \longrightarrow R_2-\frac{2}{15}R_3, R_1^{'} \longrightarrow R_1+\frac{8}{5}R_3$

$$\sim \begin{pmatrix} 1 & 0 & -\frac{8}{5}+\frac{8}{5}(1) & -\frac{3}{5}+\frac{8}{5}\left(\frac{179}{194}\right) & 0 \\ 0 & 1 & \frac{2}{15}-\frac{2}{15}(1) & \frac{2}{15}-\frac{2}{15}\left(\frac{179}{194}\right) & 0 \\ 0 & 0 & 1 & \frac{179}{194} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \frac{85}{97} & 0 \\ 0 & 1 & 0 & \frac{1}{97} & 0 \\ 0 & 0 & 1 & \frac{179}{194} & 0 \end{pmatrix}$$

Now the last row$\Longrightarrow z+\frac{179}{194}t=0 \Longrightarrow z=-\frac{179}{194}t$, so it can be seen that $\forall$ infinite many values of $t \in \mathbb{R}, \exists$ infinite many values of z.

2nd row$\Longrightarrow y+\frac{1}{97}t=0 \Longrightarrow y=-\frac{1}{97}t$
1st row $\Longrightarrow x+\frac{85}{97}t=0 \Longrightarrow x=-\frac{85}{97}t$

Hence the given system has infinite many solutions other than than the trivial solution: $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

Now the vector form of the non-trivial solution is;
$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -\frac{85}{97}t \\ -\frac{1}{97}t \\ -\frac{179}{194}t \\ t \end{pmatrix} = t \begin{pmatrix} -\frac{85}{97} \\ -\frac{1}{97} \\ -\frac{179}{194} \\ 1 \end{pmatrix}$, which is the required non-trivial solution spanned by the vector $\begin{pmatrix} -\frac{85}{97} \\ -\frac{1}{97} \\ -\frac{179}{194} \\ 1 \end{pmatrix}$ and here t is the free variable.