Question:

Let $\vec{u} = \begin{pmatrix} 8 \\ 2 \\ 3 \end{pmatrix}$ and $A = \begin{pmatrix} 4 & 3 & 5 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{pmatrix}$.Is $\vec{u}$ in the subset of $\mathbb{R}^3$ spanned by the columns of A? Why or why not?

Solution:

Denoting the column vectors of matrix A as follows;

$\vec{u_1} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$, $\vec{u_2} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ and $\vec{u_3} = \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix}$.

If $\vec{u}$ will be in the subset of $\mathbb{R}^3$spanned by the columns of A

then,$\Longrightarrow \vec{u}$ can be expressed as a linear combination of the column vectors of A and these vectors are in fact the elements of $\mathbb{R}^3$.

$\Longrightarrow$ the vector equation:$x\vec{u_1} + y\vec{u_2} + z\vec{u_3} = \vec{u}$ has the solution, where x, y and z are the unknowns.

$\therefore$from $x\vec{u_1} + y\vec{u_2} + z\vec{u_3} = \vec{u}$

$$\Longrightarrow x\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} + y\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + z\begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ 3 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 4x \\ 0 \\ x \end{pmatrix} + \begin{pmatrix} 3y \\ y \\ 2y \end{pmatrix} + \begin{pmatrix} 5z \\ -z \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ 3 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 4x + 3y + 5z \\ y - z \\ x + 2y \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ 3 \end{pmatrix}$$

$$\Longrightarrow \begin{matrix} 4x + 3y + 5z = 8 \\ 0x + y - z = 2 \\ x + 2y + 0z = 3 \end{matrix}$$ is the associated system of Linear equations whose augmented matrix is: $\begin{pmatrix} 4 & 3 & 5 & 8 \\ 0 & 1 & -1 & 2 \\ 1 & 2 & 0 & 3 \end{pmatrix}$.

Now we reduce this into reduced Echelon form as follows;

$$\begin{pmatrix} 4 & 3 & 5 & 8 \\ 0 & 1 & -1 & 2 \\ 1 & 2 & 0 & 3 \end{pmatrix}$$

By $R_1 \longleftrightarrow R_3$(Interchanging 1st and 3rd rows)

$$\sim \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 2 \\ 4 & 3 & 5 & 8 \end{pmatrix}$$

By $R_3' \longleftrightarrow R_3 - 4R_1$

$$\sim \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 2 \\ 4-4(1) & 3-4(2) & 5-4(0) & 8-4(3) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & -5 & 5 & -4 \end{pmatrix}$$

By $R_3' \longleftrightarrow R_3 + 5R_2$

$$\sim \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & -5+5(1) & 5+5(-1) & -4+5(2) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$ i.e. the reduced Echelon form.

Now the last row implies $0z = 6$, which is impossible $\forall z \in \mathbb{R}$, which means that there does not exist a real number which on multiplying with *zero* can given 6.

$\implies$ the system(ultimately the vector equation) does not have the solution.

$\implies \vec{u}$ cant be lie in the subset of $\mathbb{R}^3$spanned by the columns of A.