Question

Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$. Does $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$span $\mathbb{R}^4$?

Solution:

If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$span $\mathbb{R}^4$, then for any arbitrary vector say $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ in $\mathbb{R}^4$, the vector equation:$\alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3 = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ has the solution, where α, β and γ are unknowns but the roles of x, y, z and t is also critical and noteworthy.

Now from vector equation:

$$\alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3 = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

$$\Longrightarrow \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} \alpha \\ 0 \\ 0 \\ -\alpha \end{pmatrix} + \begin{pmatrix} 0 \\ \beta \\ 0 \\ -\beta \end{pmatrix} + \begin{pmatrix} \gamma \\ 0 \\ -\gamma \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} \alpha + \gamma \\ \beta \\ -\gamma \\ -\alpha - \beta \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

$\Longrightarrow \begin{matrix} \alpha + 0\beta + \gamma = x \\ 0\alpha + \beta + 0\gamma = y \\ 0\alpha + \beta - \gamma = z \\ -\alpha - \beta + 0\gamma = t \end{matrix}$ i.e. the associated system of linear equations in THREE variable α, β and γ.And its corresponding augmented matrix: $\begin{pmatrix} 1 & 0 & 1 & x \\ 0 & 1 & 0 & y \\ 0 & 1 & -1 & z \\ -1 & -1 & 0 & t \end{pmatrix}$.

Now its reduced Echelon form is as follows;

$$\begin{pmatrix} 1 & 0 & 1 & x \\ 0 & 1 & 0 & y \\ 0 & 1 & -1 & z \\ -1 & -1 & 0 & t \end{pmatrix}$$

By $R_4^{'} \longrightarrow R_4 + R_1$

$$\begin{pmatrix} 1 & 0 & 1 & x \\ 0 & 1 & 0 & y \\ 0 & 1 & -1 & z \\ -1+1 & -1+0 & 0+1 & t+x \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & x \\ 0 & 1 & 0 & y \\ 0 & 1 & -1 & z \\ 0 & -1 & 1 & t+x \end{pmatrix}$$

By $R_3^{'} \longrightarrow R_3 - R_2, R_4^{'} \longrightarrow R_4 + R_2$

$$\sim \begin{pmatrix} 1 & 0 & 1 & x \\ 0 & 1 & 0 & y \\ 0 & 1-1 & -1-0 & z-y \\ 0 & -1+1 & 1+0 & t+x+y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & -1 & z-y \\ 0 & 0 & 1 & t+x+y \end{pmatrix}$$

By $R_4^{'} \longrightarrow R_4 + R_3$

$$\sim \begin{pmatrix} 1 & 0 & 1 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & -1 & z-y \\ 0 & 0 & 1+1 & t+x+y+z-y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & -1 & z-y \\ 0 & 0 & 0 & t+x+z \end{pmatrix}$$

Now the last row$\Longrightarrow$ two cases i.e. either $t+x+0y+z \neq 0$ or $t+x+0y+z = 0$

Case 1:

if $t+x+0y+z \neq 0$ then last row$\Longrightarrow 0\gamma = t+x+0y+z \neq 0 \implies 0\gamma \neq 0$, which is impossible $\forall \gamma \in \mathbb{R}$, as $\nexists$ a real number which on multiplying with *zero* can give a $non-zero$ real number.

$\Longrightarrow$solution does not exist whenever $t+x+0y+z \neq 0$

$\therefore$ the given vectors do not span $\mathbb{R}^4$.

Case 2:

if $t+x+0y+z = 0$ then last row$\Longrightarrow 0\gamma = t+x+0y+z = 0 \implies 0\gamma = 0$, which is true $\forall \gamma \in \mathbb{R}$ because every real number on multiplying with *zero* gives a *zero*.

$\Longrightarrow$solution does exist whenever $t+x+0y+z = 0$, but it implies geometrically in this case that all vectors of the form $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ should lie in the hyper-plane through origin as $t+x+0y+z = 0$ represents the hyper-plane in $\mathbb{R}^4$ which is 3-dimensional(equivalent to $\mathbb{R}^3$).

$\Longrightarrow$choice of taking the arbitrary vectors from $\mathbb{R}^4$ is *restricted* to those vectors lie only on one of its hyper-plane:$t+x+0y+z = 0$

$\Longrightarrow$the given columns vectors can span $\mathbb{R}^3$only!

$\therefore$**columns of matrix A can not span** $\mathbb{R}^4$.

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Alternative Solution:

By applying theorem-2 from lecture-6 on this problem.

Theorem 2:

Let A be an $m \times n$matrix. Then the following statements are logically equivalent.

(a) For each $\vec{b}$ in $\mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.
(b) The columns of A Span $\mathbb{R}^m$.
(c) A has a pivot position in every row.

(b) and (c) are important to note which say "*columns of A Span $\mathbb{R}^m \Longleftrightarrow A$ has a pivot position in every row*".

Here $A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$.Now we reduce this into Echelon form as follows;

By $R_4' \longrightarrow R_4 + R_1$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ -1+1 & -1+0 & 0+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

By $R_3' \longrightarrow R_3 - R_2, R_4' \longrightarrow R_4 + R_2$

$$\sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1-1 & -1-0 \\ 0 & -1+1 & 1+0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

By $R_4' \longrightarrow R_4 + R_3$

$$\sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1+1 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 & 1 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{-1} \\ 0 & 0 & 0 \end{pmatrix}$$ i.e. the reduced Echelon form, where it can be seen that 4th row does not contain Pivot.

$\Longrightarrow$columns of A does not span $\mathbb{R}^4$.

$\therefore$ given vectors also does not span $\mathbb{R}^4$.