Question:

Do the columns of the matrix $A = \begin{pmatrix} 3 & 5 \\ 1 & 1 \\ 2 & 8 \end{pmatrix}$ span $\mathbb{R}^3$?

Solution:

Denoting the column vectors of A as:

$\vec{u} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 5 \\ 1 \\ 8 \end{pmatrix}$.

If column vectors of A span $\mathbb{R}^3$, then

$\Longrightarrow$ any arbitrary vector say $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbb{R}^3$ can be expressed as a linear combination of column vectors $\vec{u}$ and $\vec{v}$ of matrix A.

$\Longrightarrow$vector equation: $\alpha\vec{u} + \beta\ \vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ has the solution, where of course α and β are unknowns, but the role of x, y and z is also very critical.

Now vector equation:$\alpha\vec{u} + \beta\ \vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\Longrightarrow \alpha \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 5 \\ 1 \\ 8 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 3\alpha \\ \alpha \\ 2\alpha \end{pmatrix} + \begin{pmatrix} 5\beta \\ \beta \\ 8\beta \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 3\alpha + 5\beta \\ \alpha + \beta \\ 2\alpha + 8\beta \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\Longrightarrow \begin{matrix} 3\alpha + 5\beta = x \\ \alpha + \beta = y \\ 2\alpha + 8\beta = z \end{matrix}$, i.e. the associated system of equations in TWO variables α and β and its corresponding augmented matrix is: $\begin{pmatrix} 3 & 5 & x \\ 1 & 1 & y \\ 2 & 8 & z \end{pmatrix}$.Now we reduce this into reduced Echelon form as follows;

$$\begin{pmatrix} 3 & 5 & x \\ 1 & 1 & y \\ 2 & 8 & z \end{pmatrix}$$

By $R_1 \longleftrightarrow R_2$

$$\sim \begin{pmatrix} 1 & 1 & y \\ 3 & 5 & x \\ 2 & 8 & z \end{pmatrix}$$

By $R_1^{'} \longrightarrow R_2 - 3R_1, R_2^{'} \longrightarrow R_2 - 2R_1$

$$\sim \begin{pmatrix} 1 & 1 & y \\ 3-3(1) & 5-3(1) & x-3(y) \\ 2-2(1) & 8-2(1) & z-2(y) \end{pmatrix} = \begin{pmatrix} 1 & 1 & y \\ 0 & 2 & x-3y \\ 0 & 6 & z-2y \end{pmatrix}$$

By $R_3^{'} \longrightarrow R_3 - 3R_2$

$$\sim \begin{pmatrix} 1 & 1 & y \\ 0 & 2 & x-3y \\ 0 & 6-3(2) & z-2y-3(x-3y) \end{pmatrix} = \begin{pmatrix} 1 & 1 & y \\ 0 & 2 & x-3y \\ 0 & 0 & -3x+7y+z \end{pmatrix}$$

Now the last row$\Longrightarrow$ two cases i.e. either $-3x+7y+z \neq 0$ or $-3x+7y+z = 0$

Case 1:

if $-3x+7y+z \neq 0$ then last row$\Longrightarrow 0\beta = -3x+7y+z \neq 0 \Longrightarrow 0\beta \neq 0$, which is impossible $\forall \beta \in \mathbb{R}$, as $\nexists$ a real number which on multiplying with *zero* can give a $non-zero$ real number.

$\Longrightarrow$solution does not exist whenever $-3x+7y+z \neq 0$

$\therefore$columns of matrix A can not span $\mathbb{R}^3$.

Case 2:

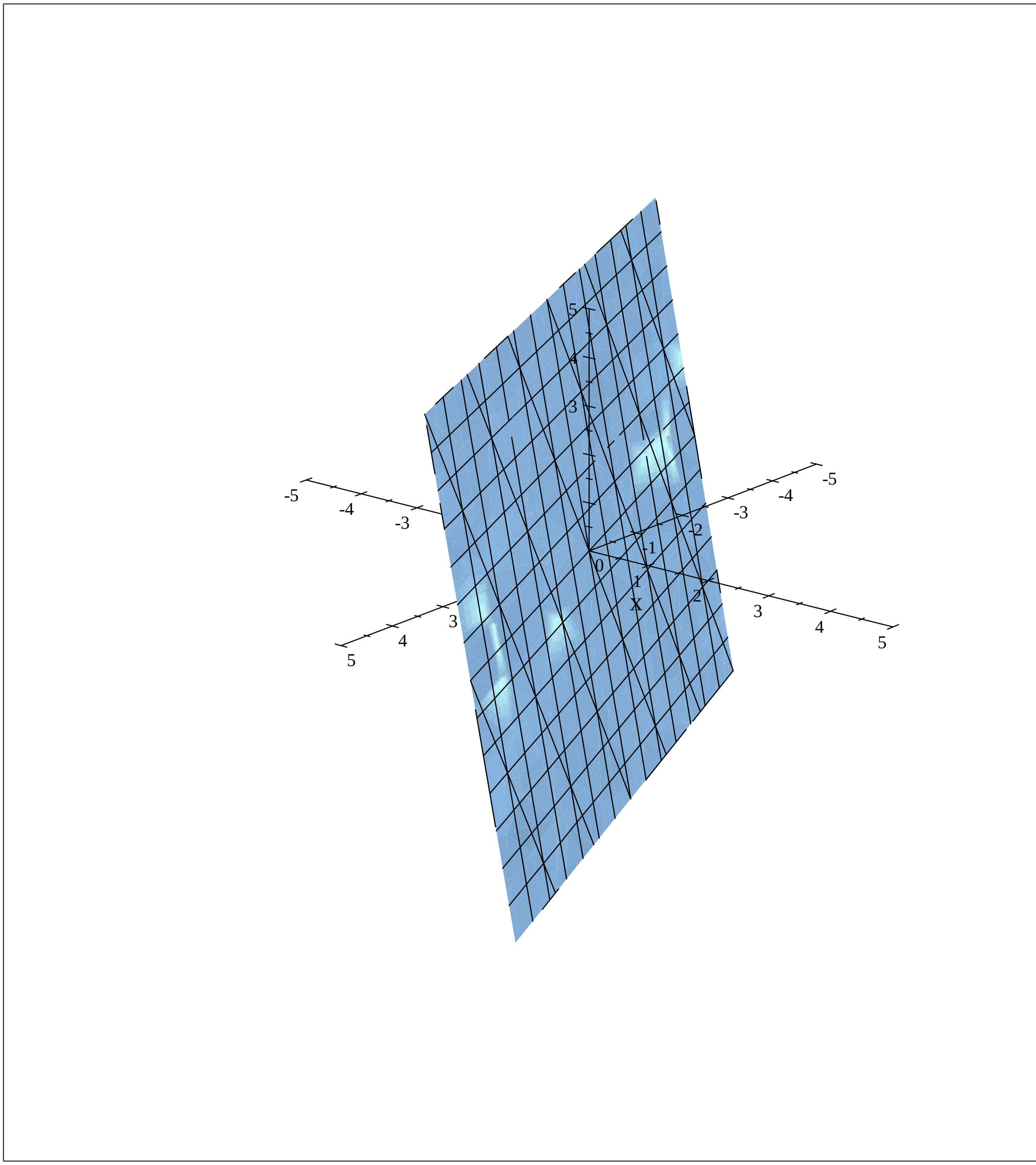
if $-3x+7y+z = 0$ then last row$\Longrightarrow 0\beta = -3x+7y+z = 0 \Longrightarrow 0\beta = 0$, which is true $\forall \beta \in \mathbb{R}$ because every real number on multiplying with *zero* gives a *zero*.

$\Longrightarrow$solution does exist whenever $-3x+7y+z = 0$, but it implies geometrically in this case that all vectors of the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$should lie in the plane through origin as $-3x+7y+z = 0$ represents the plane which is 2-dimensional(equivalent to $\mathbb{R}^2$), as shown in the following figure.

$\Longrightarrow$choice of taking the arbitrary vectors from $\mathbb{R}^3$ is *restricted* to those vectors lie only on the plane: $-3x+7y+z = 0$

$\Longrightarrow$columns of matrix A can span $\mathbb{R}^2$only!

$\therefore$**columns of matrix A can not span** $\mathbb{R}^3$.



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Alternative Solution:

Applying the theorem-2 from lecture-6 on this problem.

Theorem 2:

Let A be an $m \times n$matrix. Then the following statements are logically equivalent.

(a) For each $\vec{b}$ in $\mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.
(b) The columns of A Span $\mathbb{R}^m$.
(c) A has a pivot position in every row.

(b) and (c) are important to note which say "*columns of A Span $\mathbb{R}^m \Longleftrightarrow A$ has a pivot position in every row*".

Now we reduce matrix $A = \begin{pmatrix} 3 & 5 \\ 1 & 1 \\ 2 & 8 \end{pmatrix}$ into Echelon form.

By $R_1 \longleftrightarrow R_2$

$\sim \begin{pmatrix} 1 & 1 \\ 3 & 5 \\ 2 & 8 \end{pmatrix}$

By $R_1' \longrightarrow R_2 - 3R_1, R_2' \longrightarrow R_2 - 2R_1$

$\sim \begin{pmatrix} 1 & 1 \\ 3-3(1) & 5-3(1) \\ 2-2(1) & 8-2(1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 6 \end{pmatrix}$

By $R_3' \longrightarrow R_3 - 3R_2$

$\sim \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 6-3(2) \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 1 \\ 0 & \mathbf{2} \\ 0 & 0 \end{pmatrix}$ i.e the Echelon form, where it can be seen that 3rd row does not contain any Pivot.

$\Longrightarrow$columns of A does not span $\mathbb{R}^3$.