Question:

Determine if $\vec{b} = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$ is a linear combination of the vectors formed from the columns of the matrix $A = \begin{pmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{pmatrix}$.

Solution:

$\because$we know that *a vector $\vec{b}$ can be written as a linear combination of vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ $\Longleftrightarrow$ the following vector equation has the solution.*

Vector equation: $\vec{b} = x\vec{u} + y\vec{v} + z\vec{w}$,where x,y and z are unknowns.

Firstly take columns of given matrix A as separate vectors i.e. $\vec{u} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 5 \\ -6 \\ 8 \end{pmatrix}$

Now from vector equation$\Longrightarrow \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} = x \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + z \begin{pmatrix} 5 \\ -6 \\ 8 \end{pmatrix}$

$$\Longrightarrow \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} = \begin{pmatrix} x \\ -2x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \\ 2y \end{pmatrix} + \begin{pmatrix} 5z \\ -6z \\ 8z \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} = \begin{pmatrix} x + 5z \\ y - 2x - 6z \\ 2y + 8z \end{pmatrix}$$

$\Longrightarrow$ the corresponding system of equations: $\begin{matrix} x + 0y + 5z = 2 \\ -2x + y - 6z = -1 \\ 0x + 2y + 8z = 6 \end{matrix}$, and its corresponding augmented matrix is: $\begin{pmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{pmatrix}$ .Now we reduce it into reduced Echelon form as follows;

$$\begin{pmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{pmatrix}$$

By $R_2' \longrightarrow R_2 + 2R_1$

$$\sim \begin{pmatrix} 1 & 0 & 5 & 2 \\ -2 + 2(1) & 1 + 2(0) & -6 + 2(5) & -1 + 2(2) \\ 0 & 2 & 8 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{pmatrix}$$

By $R_3' \longrightarrow R_3 - 2R_1$

$\sim \begin{pmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 - 2(1) & 8 - 2(4) & 6 - 2(3) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ i.e. the reduced Echelon form.

Now last row$\Longrightarrow$ $0z = 0$,which is true $\forall z \in \mathbb{R}$ means that we are free to choose any real value of z say $z = t$ and 2nd row$\Longrightarrow$ $y + 4z = 3$ *or*
$y = 3 - 4t$ and finally 1st row$\Longrightarrow$ $x + 5t = 2$ or $x = 2 - 5t$.

$$\therefore \quad \begin{array}{l} x = 2 - 5t \\ y = 3 - 4t \\ z = t, t \in \mathbb{R} \end{array}$$

is the solution(in fact infinite many solutions) of associated system of equations.

$\Longrightarrow$ $\overrightarrow{b}$ can be written as a linear combination of vectors $\overrightarrow{u}$, $\overrightarrow{v}$ and $\overrightarrow{w}$.