Question:

Determine whether $\overrightarrow{b} = \begin{pmatrix} 3 \\ -7 \\ -3 \end{pmatrix}$ is a linear combination of $\overrightarrow{a}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$, $\overrightarrow{a}_2 = \begin{pmatrix} -4 \\ 3 \\ 8 \end{pmatrix}$ and $\overrightarrow{a}_3 = \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix}$.

Solution:

Since we know that *a vector* $\overrightarrow{b}$ *is a linear combination of the vectors* $\overrightarrow{a}_1$, $\overrightarrow{a}_2$ *and* $\overrightarrow{a}_3 \iff$ *the following vector equation(in fact a system of equations) has the solution.*

Vector equation: $\overrightarrow{b} = x\overrightarrow{a}_1 + y\overrightarrow{a}_2 + z\overrightarrow{a}_3$, where x, y and z are unknowns.

$$\Longrightarrow \begin{pmatrix} 3 \\ -7 \\ -3 \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + y\begin{pmatrix} -4 \\ 3 \\ 8 \end{pmatrix} + z\begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 3 \\ -7 \\ -3 \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ -2x \end{pmatrix} + \begin{pmatrix} -4y \\ 3y \\ 8y \end{pmatrix} + \begin{pmatrix} 2z \\ 5z \\ -4z \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 3 \\ -7 \\ -3 \end{pmatrix} = \begin{pmatrix} x - 4y + 2z \\ 0x + 3y + 5z \\ -2x + 8y - 4z \end{pmatrix}$$

$$\Longrightarrow \begin{matrix} x - 4y + 2z = 3 \\ 0x + 3y + 5z = -7 \\ -2x + 8y - 4z = -3 \end{matrix} \quad \text{and its Corresponding augmented matrix: } \begin{pmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ -2 & 8 & -4 & -3 \end{pmatrix}.$$

Now we reduce it into Reduced Echelon form to have its solution.

$$\begin{pmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ -2 & 8 & -4 & -3 \end{pmatrix}$$

By $R_3' \longrightarrow R_3 + 2R_1$

$$\sim \begin{pmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ -2+2(1) & 8+2(-4) & -4+2(2) & -3+2(3) \end{pmatrix} = \begin{pmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$\therefore$ last row implies that $0z = 3$, which is impossible as $\nexists$ a $z \in \mathbb{R}$ which on multiplying with *zero* can give 3.

$\Longrightarrow$ given system does not have any solution.

$\therefore$ $\overrightarrow{b}$ can never be written as a linear combination of the vectors $\overrightarrow{a}_1$, $\overrightarrow{a}_2$ and $\overrightarrow{a}_3$.