Question:

For what value of h, $\overrightarrow{y} = \begin{pmatrix} -4 \\ 3 \\ h \end{pmatrix}$ belongs to the $span\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\}$ where $\overrightarrow{v_1} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \overrightarrow{v_2} = \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix}, \overrightarrow{v_3} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$.

Solution:

Since we know that a vector say $\overrightarrow{y}$ is in the $span\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\} \iff \exists$the scalars(real numbers $\in \mathbb{R}$in this case) such that the following vector equation(system of equations in three variables in this case) is satisfied(mean that it has the solution.)

vector equation: $\alpha_1\overrightarrow{v_1} + \alpha_2\overrightarrow{v_2} + \alpha_3 \overrightarrow{v_3} = \overrightarrow{y}$or

$$\Longrightarrow \alpha_1 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} + \alpha_3 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ h \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} \alpha_1 \\ -\alpha_1 \\ -2\alpha_1 \end{pmatrix} + \begin{pmatrix} 5\alpha_2 \\ -4\alpha_2 \\ -7\alpha_2 \end{pmatrix} + \begin{pmatrix} -3\alpha_3 \\ \alpha_3 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ h \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} \alpha_1 + 5\alpha_2 - 3\alpha_3 \\ \alpha_3 - 4\alpha_2 - \alpha_1 \\ -2\alpha_1 - 7\alpha_2 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ h \end{pmatrix}$$

which gives the system of equations: $\begin{matrix} \alpha_1 + 5\alpha_2 - 3\alpha_3 = -4 \\ -\alpha_1 - 4\alpha_2 + \alpha_3 = 3 \\ -2\alpha_1 - 7\alpha_2 = h \end{matrix}$ whose corresponding augmented matrix is: $\begin{pmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{pmatrix}$.Now we reduce this into the Echelon form as follows;

$$A_b = \begin{pmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{pmatrix}$$

By $R_2' \longrightarrow R_2 + R_1, R_3' \longrightarrow R_3 + 2R_1$

$$\sim \begin{pmatrix} 1 & 5 & -3 & -4 \\ -1+1 & -4+5 & 1+(-3) & 3+(-4) \\ -2+2(1) & -7+2(5) & 0+2(-3) & h+2(-4) \end{pmatrix} = \begin{pmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{pmatrix}$$

By $R_3' \longrightarrow R_3 - 3R_1$

$$\sim \begin{pmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3-3(1) & -6-3(-2) & h-8-3(-1) \end{pmatrix} = \begin{pmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{pmatrix}$$

Now last row$\Longrightarrow$either $h - 5 \neq 0$ or $h - 5 = 0$

Case1:

If $h - 5 \neq 0$ then last row$\Longrightarrow 0\alpha_3 = h - 5 \neq 0 \Longrightarrow 0\alpha_3 \neq 0$ which is impossible as $\nexists \alpha_3 \in \mathbb{R}$which on multiplying with zero will give a non-zero. $\Longrightarrow$

the system has no solution whenever $h-5 \neq 0$ or $h \neq 5 \Longrightarrow$ the given vector $\overrightarrow{y} \notin span\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\} \Longleftrightarrow h-5 \neq 0$ or $h \neq 5$

Case-2:

If $h-5=0$ then last row$\Longrightarrow 0\alpha_3 = h-5 = 0 \Longrightarrow 0\alpha_3 = 0$ which is true $\forall \alpha_3 \in \mathbb{R} \Longrightarrow$ the system has the solution(infect infinite many solutions). $\Longrightarrow$ the given vector $\overrightarrow{y} \in span\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\} \Longleftrightarrow h-5=0$ or $h=5$