Question:

Let $\vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.Show that $\begin{pmatrix} h \\ k \end{pmatrix}$ is in the $Span\{\vec{u}, \vec{v}\}$ for all h and k .

Solution:

$\because$ we know that a vector $\begin{pmatrix} h \\ k \end{pmatrix}$ is in the $Span\{\vec{u}, \vec{v}\} \iff$ vector equation: $\begin{pmatrix} h \\ k \end{pmatrix} = x\vec{u} + y\vec{v}$ has the solution, where x and y are unknows.

$$\therefore \begin{pmatrix} h \\ k \end{pmatrix} = x\vec{u} + y\vec{v}$$

$$\Longrightarrow \begin{pmatrix} h \\ k \end{pmatrix} = x\begin{pmatrix} 2 \\ 1 \end{pmatrix} + y\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} 2x \\ x \end{pmatrix} + \begin{pmatrix} 2y \\ y \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} 2x + 2y \\ x + y \end{pmatrix}$$

$\Longrightarrow \begin{matrix} 2x + 2y = h \\ x + y = k \end{matrix}$ is the associated system of equations in two variables and its corresponding augmented matrix is:$\begin{pmatrix} 2 & 2 & h \\ 1 & 1 & k \end{pmatrix}$.

Now we apply elementary row operations to reduce it into Echelon forms as follows;

$$\begin{pmatrix} 2 & 2 & h \\ 1 & 1 & k \end{pmatrix}$$

By $R_1 \longleftrightarrow R_2$(interchanging 1st and 2nd rows)

$$\sim \begin{pmatrix} 1 & 1 & k \\ 2 & 2 & h \end{pmatrix}$$

By $R_2' \to R_2 - 2R_1$

$$\sim \begin{pmatrix} 1 & 1 & k \\ 2-2(1) & 2-2(1) & h-2(k) \end{pmatrix} = \begin{pmatrix} 1 & 1 & k \\ 0 & 0 & h-2k \end{pmatrix}$$

Now the last row$\Longrightarrow$ either $h - 2k = 0$ or $h - 2k \neq 0$

Case-1:

If $h - 2k = 0$, then last row implies $0y = h - 2k = 0 \Longrightarrow 0y = 0$, which is true $\forall y \in \mathbb{R}$.

$\Longrightarrow$ the system has a solution whenever $h - 2k = 0$ and it is also true $\forall h, k \in \mathbb{R}$

$$\Longrightarrow \begin{pmatrix} h \\ k \end{pmatrix} \in Span\{\vec{u}, \vec{v}\} \text{ whenever } h = 2k.$$

Case-2:

If $h - 2k \neq 0$, then last row implies $0y = h - 2k \neq 0 \Longrightarrow 0y \neq 0$, which is impossible $\forall y \in \mathbb{R}$.

$\Longrightarrow$ the system has no solution whenever $h - 2k \neq 0$ and $\begin{pmatrix} h \\ k \end{pmatrix} \notin Span\{\vec{u}, \vec{v}\}$ whenever $h \neq 2k$.