Choose h and k such that the system: $\begin{matrix} x-3y=1 \\ 2x+hy=k \end{matrix}$ has (a) no solution, (b) a unique solution, and (c) many solutions. Give separate answer for each part.

Solution:

For the given system: $\begin{matrix} x-3y=1 \\ 2x+hy=k \end{matrix}$, the corresponding augmented matrix: $\begin{pmatrix} 1 & -3 & 1 \\ 2 & h & k \end{pmatrix}$

Now we apply on the elementary row operations to get its Echelon form.

$\begin{pmatrix} 1 & -3 & 1 \\ 2 & h & k \end{pmatrix}$

By $R_2' \longrightarrow R_2 - 2R_1$

$\sim \begin{pmatrix} 1 & -3 & 1 \\ 2-2 & h-2(-3) & k-2(1) \end{pmatrix} = \begin{pmatrix} 1 & -3 & 1 \\ 0 & h+6 & k-2 \end{pmatrix}$

By $R_2' \rightarrow (\frac{1}{h+6})R_2$

$\sim \begin{pmatrix} 1 & -3 & 1 \\ 0 & (\frac{1}{h+6})(h+6) & (\frac{1}{h+6})(k-2) \end{pmatrix} = \begin{pmatrix} 1 & -3 & 1 \\ 0 & 1 & \frac{1}{h+6}(k-2) \end{pmatrix}$

$\therefore$second row$\Longrightarrow y = \frac{k-2}{h+6} - - - - - (1)$

now we have the three cases:

Case-1(No solution)

In (1), if the denominator$= h+6=0$, then y goes to infinity and hence x will also be! Hence the solution will not exist whenever $h+6=0$ or $h=-6$

Case-2(Unique solution)

In (1), if the Numerator $= k-2=0$, then$\Longrightarrow y=0$ and hence 1st row of reduced augmented matrix$\Longrightarrow 1x-3(0)=1$or $x=1$. Hence there is a unique solution:$\{(1,0)\}$ in this case whenever $k-2=0$ or $k=2$

Case-3(Infinite many solutions)

(1)$\Longrightarrow y=\frac{k-2}{h+6}$ and now if both $h+6\neq 0$ and $k-2\neq 0$, then for infinite many values of k(except $k=2$) and h(except $h=-6$), there would be infinite many values of y and hence of x.

$\therefore$the system will have infinite many solutions whenever both $h\neq 6$ and $k\neq 2$.