Find a standard basis vector that can be added to the set $\{v_1, v_2\}$ to produce a basis for $\mathbb{R}^4$, $v_1 = (1, -4, 2, -3), v_2 = (-3, 8, -4, 6)$.

Solution:

$\mathbb{R}^4$ can be generated only by least four linearly independent vectors.

$$A = \begin{pmatrix} 1 & -3 \\ -4 & 8 \\ 2 & -4 \\ -3 & 6 \end{pmatrix}$$

By $R_2' \to R_2 + 4R_1, R_3' \to R_3 - 2R_1, R_4' \to R_4 + 3R_1$

$$\sim \begin{pmatrix} 1 & -3 \\ -4+4(1) & 8+4(-3) \\ 2-2(1) & -4-2(-3) \\ -3+3(1) & 6+3(-3) \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 0 & -4 \\ 0 & 2 \\ 0 & -3 \end{pmatrix}$$

By $R_2' \to (\frac{-1}{4})R_2$

$$\sim \begin{pmatrix} 1 & -3 \\ 0 & -4(\frac{-1}{4}) \\ 0 & 2 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \\ 0 & 2 \\ 0 & -3 \end{pmatrix}$$

By $R_3' \to R_3 - 2R_2, R_4' \to R_4 + 3R_2$

$$\sim \begin{pmatrix} 1 & -3 \\ 0 & 1 \\ 0 & 2-2(1) \\ 0 & -3+3(1) \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \text{i.e. in the Echelon form.}$$

$\Longrightarrow v_1$and v_2 are linearly independent vectors in $\mathbb{R}^4$.

$\therefore$to from the basis of $\mathbb{R}^4$, we need least two more vectors which are linearly independent to $v_1$and v_2.

$\because v_1$and v_2 have the pivots in 1st and 2nd rows.

$\Longrightarrow$another vectors must have pivots in 3rd and 4th rows respectively.

$\therefore$another more vectors can be taken $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, which are linearly independent to $v_1$and v_2 as well and all these four vectors form the basis for $\mathbb{R}^4$.