show that $\begin{vmatrix} a_1 & b_1+ta_1 & c_1+rb_1+sa_1 \\ a_2 & b_2+ta_2 & c_2+rb_2+sa_2 \\ a_3 & b_3+ta_2 & c_3+rb_3+sa_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

LHS

$$= \begin{vmatrix} a_1 & b_1+ta1 & c_1+rb_1+sa_1 \\ a_2 & b_2+ta_2 & c_2+rb_2+sa_2 \\ a_3 & b_3+ta_2 & c_3+rb_3+sa_3 \end{vmatrix}$$

By taking the transpose

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1+a_1t & b_2+ta_2 & b_3+ta_3 \\ c_1+rb_1+sa_1 & c_2+rb_2+sa_2 & c_3+rb_3+sa_3 \end{vmatrix} \because |A|^t = |A|$$

By $R_2' \rightarrow R_2 - tR_1$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ (b_1+a_1t)-a_1t & (b_2+ta_2)-ta_2 & (b_3+ta_3)-ta_3 \\ c_1+rb_1+sa_1 & c_2+rb_2+sa_2 & c_3+rb_3+sa_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1+rb_1+sa_1 & c_2+rb_2+sa_2 & c_3+rb_3+sa_3 \end{vmatrix}$$

By $R_3' \rightarrow R_3 - (rR_2 + sR_1)$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ (c_1+rb_1+sa_1)-(rb_1+sa_1) & (c_2+rb_2+sa_2)-(rb_2+sa_2) & (c_3+rb_3+sa_3)-(rb_3+sa_3) \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \text{RHS}$$