Question:

Consider the partitioned system: $\begin{pmatrix} 5 & 2 & 2 & 3 \\ 2 & 1 & -3 & 1 \\ 1 & 0 & 4 & 1 \\ 0 & 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 0 \\ 0 \end{pmatrix}$.Solve this system by 1st expressing it as $\begin{pmatrix} A & B \\ I & D \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} b \\ \mathbf{O} \end{pmatrix}$ or equivalently $\begin{matrix} Au + Bv = b \\ u + Dv = \mathbf{O} \end{matrix}$ .Next solving the second equation for u in terms of v, and then substituting in the first equation. Check your answer by solving the system directly.

Solution:

Since the given system $\begin{pmatrix} 5 & 2 & 2 & 3 \\ 2 & 1 & -3 & 1 \\ 1 & 0 & 4 & 1 \\ 0 & 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 0 \\ 0 \end{pmatrix}$ is complex with 4 equations in 4 variables. So we partition this into blocks as follows:

Expressing $\begin{pmatrix} 5 & 2 & 2 & 3 \\ 2 & 1 & -3 & 1 \\ 1 & 0 & 4 & 1 \\ 0 & 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} A & B \\ I & D \end{pmatrix}$,where $A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 \\ -3 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}$,

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ into $\begin{pmatrix} u \\ v \end{pmatrix}$,where $u = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $v = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$ and finally $\begin{pmatrix} 2 \\ 6 \\ 0 \\ 0 \end{pmatrix}$ into $\begin{pmatrix} b \\ \mathbf{O} \end{pmatrix}$,where $b = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ and $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$\therefore$ the given system will be reduced to: $\begin{pmatrix} A & B \\ I & D \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} b \\ \mathbf{O} \end{pmatrix}$ or equivalently $\begin{matrix} Au + Bv = b \\ u + Dv = \mathbf{O} \end{matrix}$.

Now 2nd equation$\Longrightarrow u = -Dv$(Put this in 1st)

$\therefore$2nd equation$\Longrightarrow A(-Dv) + Bv = b$

$\Longrightarrow (-AD + B)v = b$

$$\Longrightarrow \left\{ \begin{pmatrix} -5 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ -3 & 1 \end{pmatrix} \right\} v = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

$$\Longrightarrow \left\{ \begin{pmatrix} -20 & -9 \\ -8 & -4 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ -3 & 1 \end{pmatrix} \right\} v = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} -18 & -6 \\ -11 & -3 \end{pmatrix} v = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -18 & -6 \\ -11 & -3 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} \\ -\frac{11}{12} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$
$$\Longrightarrow v = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\frac{5}{2} \\ \frac{43}{6} \end{pmatrix}$$
Now the 1st equation$\Longrightarrow Au + Bv = b$
$$\Longrightarrow \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} -\frac{5}{2} \\ \frac{43}{6} \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$
$$\Longrightarrow \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{33}{2} \\ \frac{44}{3} \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$
$$\Longrightarrow \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} \frac{33}{2} \\ \frac{44}{3} \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$
$$\Longrightarrow \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{29}{2} \\ -\frac{26}{3} \end{pmatrix}$$
$$\Longrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -\frac{29}{2} \\ -\frac{26}{3} \end{pmatrix}$$
$$\Longrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} -\frac{29}{2} \\ -\frac{26}{3} \end{pmatrix}$$
$$\Longrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{17}{6} \\ -\frac{43}{3} \end{pmatrix}$$
$\therefore$ the final solution: $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{17}{6} \\ -\frac{43}{3} \\ -\frac{5}{2} \\ \frac{43}{6} \end{pmatrix}$