Question:

$T : \mathbb{R}^2 \to \mathbb{R}^2$ first rotates points counterclockwise through $\frac{\pi}{4}$ radians and then reflects the result in the x_2-axis.

Solution:

the transformation that rotates each point in R_2 through an angle θ, with counterclockwise rotation for a positive angle is given by the matrix:$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

$\therefore$for the given angle $\theta = \frac{\pi}{4}$,

$A = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$\because$the standard basis vectors for $\mathbb{R}^2$are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\therefore$matrix transformation of rotation counterclockwise through $\frac{\pi}{4}$ radians is given by the followin columns:

$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}$

and

$A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}$

$\therefore$ the matrix transformation of rotaion is:$\begin{pmatrix} A\begin{pmatrix} 1 \\ 0 \end{pmatrix} & A\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}$

But the given transformation also reflect on x_2-axis, where $x_1 = 0$ and this is given by the transformation:$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$

$\Longrightarrow T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Longrightarrow$the reflection matrix on x_2-axis:$\begin{pmatrix} T\begin{pmatrix} 1 \\ 0 \end{pmatrix} & T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore$ the required matrix transformation first rotates points counterclockwise through $\frac{\pi}{4}$ radians and then reflects the result in the x_2-axis, is given by:

$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\left(A\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2}\sqrt{2} \end{pmatrix}$

$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\left(A\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2}\sqrt{2} \end{pmatrix}$

$\Longrightarrow$required matrix:$\begin{pmatrix} 0 & 0 \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}$

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Alternate Solution:

$\because$the transformation that rotates each point in R_2 through an angle θ, with counterclockwise rotation for a positive angle is given by the matrix:$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

$\therefore$for the given angle $\theta = \frac{\pi}{4}$,

$A = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

And,the reflection matrix on x_2-axis:$B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore$ the required matrix transformation first rotates points counterclockwise through $\frac{\pi}{4}$ radians and then reflects the result in the x_2-axis, is given by:$BA\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$
$B\left\{A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} = B\left\{\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} = B\begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}$
$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}$