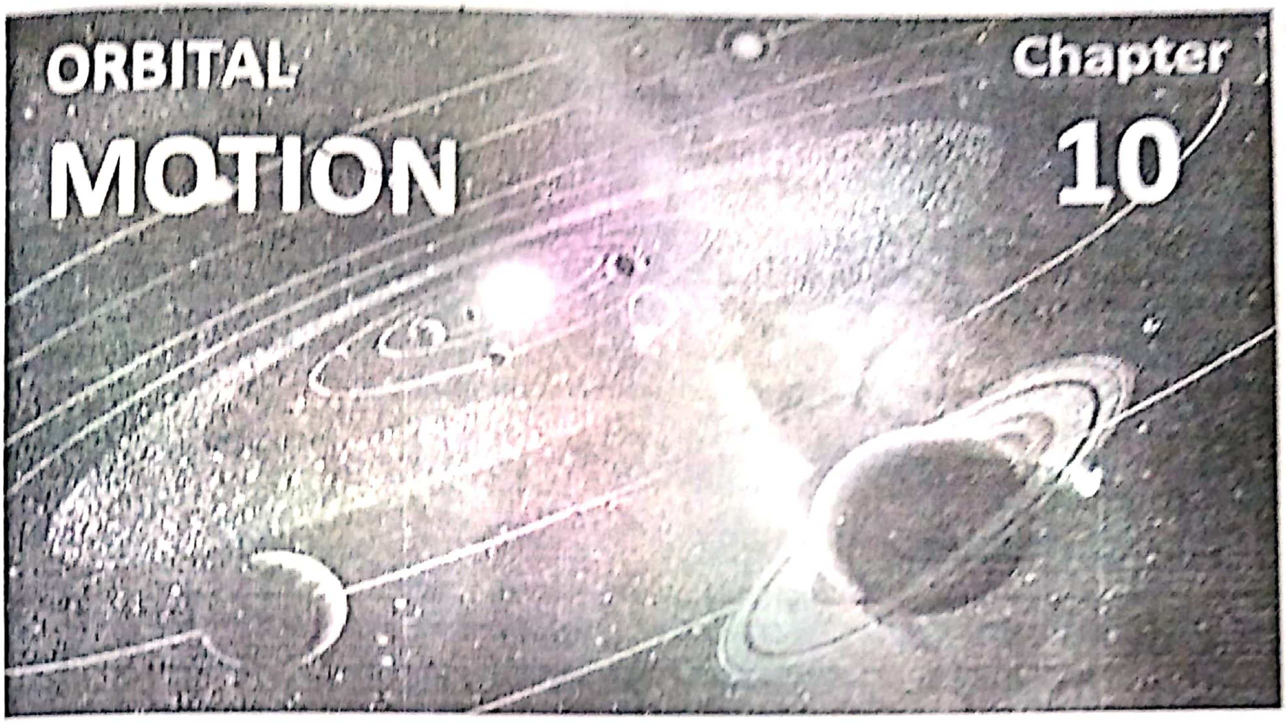


10-1 Motion under a Central Force

Central Force: If the force $\vec{F}$ on a particle always passes through a fixed point O, the force is called a *central force* and O is called the *centre of force*. A central force is *repulsive* or *attractive* according as it is directed away from or towards the centre of force.

Theorem-1: *The orbit of a particle moving under a central force is necessarily a plane curve.*

PU, 2011 (B.A/B.Sc.)

Proof: If $\vec{F}$ is the central force acting on a particle of mass m and the origin is taken at the centre of force, then since $\vec{r}$ and $\vec{F}$ along the same line, so

$$\vec{r}\times\vec{F}=0$$

$$\vec{r}\times\left(m\frac{d\vec{v}}{dt}\right)=0$$

$$\vec{r}\times\frac{d\vec{v}}{dt}=0$$

$$\vec{v}\times\vec{v}+\vec{r}\times\frac{d\vec{v}}{dt}=0 \qquad \because \vec{v}\times\vec{v}=0$$

$$\frac{d\vec{r}}{dt}\times\vec{v}+\vec{r}\times\frac{d\vec{v}}{dt}=0 \qquad \because \frac{d\vec{r}}{dt}=\vec{v}$$

$$\frac{d}{dt}(\vec{r}\times\vec{v})=0$$

$$\vec{r}\times\vec{v}=\text{constant}$$

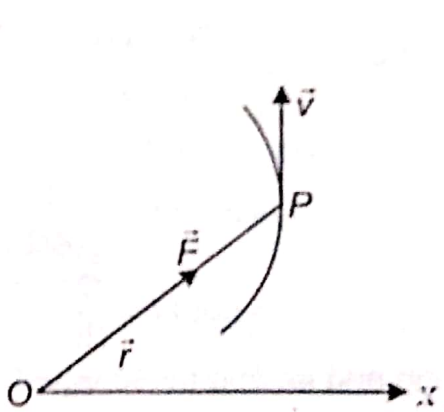


Fig.10.1

Therefore, the normal to the plane determined by the radius vector $\vec{r}$ and the velocity vector $\vec{v}$ (which is along the tangent to the orbit) has a constant direction. This is possible only in the case of a plane curve (Fig.10.1). This completes proof.

10-2 Elliptic Orbit under a Central Force

Theorem-2: *The orbit described under a central attractive force varying directly as the distance, is an ellipse having the centre at the centre of the force.*

P.U. 2008 (B.A/B.Sc)

Proof: Let xy-plane be the plane of the orbit. The central attractive force per unit mass is given by

$$\vec{F} = -k^2\vec{r} \qquad \text{...(1)}$$

where k^2 is constant of proportionality and the –ive sign is due to attractive force directed towards origin O. Using Cartesian coordinates, (1) takes the form

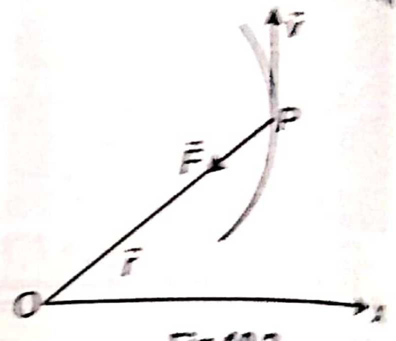


Fig.10.2

$$\ddot{x}\hat{i} + \ddot{y}\hat{j} = -k^2(x\hat{i} + y\hat{j})$$

$$\Rightarrow \ddot{x} = -k^2x, \qquad \ddot{y} = -k^2y$$

$$\Rightarrow \ddot{x} + k^2x = 0, \qquad \ddot{y} + k^2y = 0$$

These are second order differential equations and their solutions are

$$x = A\cos kt + B\sin kt \qquad \text{...(2)}$$

$$y = C\cos kt + D\sin kt \qquad \text{...(3)}$$

We can rewrite these equations as

$$A\cos kt + B\sin kt - x = 0 \qquad \text{...(4)}$$

$$C\cos kt + D\sin kt - y = 0 \qquad \text{...(5)}$$

Solving (4) and (5) simultaneously, we have

$$\frac{\cos kt}{-By + Dx} = \frac{\sin kt}{-Ay + Cx} = -\frac{1}{AD - BC}$$

$$\Rightarrow \frac{\cos kt}{-By + Dx} = -\frac{1}{AD - BC}, \qquad \frac{\sin kt}{-Ay + Cx} = -\frac{1}{AD - BC}$$

$$\Rightarrow \cos kt = -\frac{-By + Dx}{AD - BC}, \qquad \sin kt = -\frac{-Ay + Cx}{AD - BC}$$

$$\Rightarrow \cos kt = \frac{Dx - By}{BC - AD}, \quad \sin kt = \frac{Cx - Ay}{BC - AD}$$

Squaring and adding these equations, we have

$$\cos^2 kt + \sin^2 kt = \left(\frac{Dx - By}{BC - AD}\right)^2 + \left(\frac{Cx - Ay}{BC - AD}\right)^2$$

$$\Rightarrow (Cx - Ay)^2 + (Dx - By)^2 = (BC - AD)^2$$

This represents an ellipse, where constants A, B, C, D are calculated from boundary (initial) conditions.

10-3 Polar Form of the Orbit

Theorem-3: *When a particle moves under a central force, the areal velocity is constant.*

PU, 2010 (B.A/B.Sc.); University of Sargodha, 2014 (B.A/B.Sc.)

Proof: Let the particle, moving under a central force have polar coordinates (r,θ). Using radial and transverse directions, we have

$$\vec{F} = F_r(-\hat{r}) + F_\theta \hat{s}$$

or

$$\vec{F} = -F_r\hat{r} + F_\theta \hat{s}$$

or

$$\vec{F} = -ma_r\hat{r} + ma_\theta \hat{s} \qquad ...(1)$$

where $a_r = \ddot{r} - r\dot{\theta}^2$, $a_\theta = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$ are radial and transverse components of acceleration. Since $\vec{F}$ is central force, so its transverse component is zero, i.e.

$$F_\theta = 0$$

$$\Rightarrow ma_\theta = 0$$

$$\Rightarrow \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$$

$$\Rightarrow \frac{d}{dt}(r^2\dot{\theta}) = 0$$

$$\Rightarrow (r^2\dot{\theta}) = \text{constant} = h \text{ (say)}$$

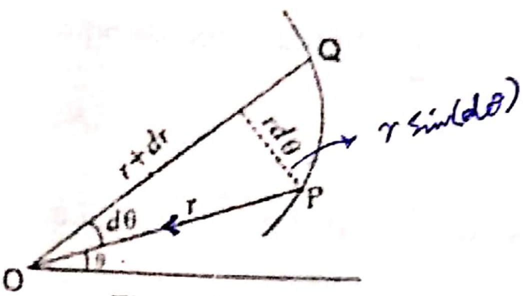


Fig.10.3

If radius vector OP sweeps an infinitesimal area POQ, dA, in time dt, then

$$dA = \frac{1}{2}r(r+dr)\sin d\theta$$

$$dA = \frac{1}{2}r^2\theta \quad \text{in the limit } \left(\begin{smallmatrix}\Delta\theta\to 0\\ \Delta r\to 0\end{smallmatrix}\right)$$

$$\Delta A = \frac{1}{2}(r+\Delta r)\, r\sin\Delta\theta.$$

$$= \frac{1}{2}(r^2\sin\Delta\theta + r\Delta r.\sin\Delta\theta)$$

Defining the areal velocity of the particle as $\frac{dA}{dt}$, we have

$$\frac{dA}{dt} = r^2\dot{\theta} = h$$

$$\frac{\Delta A}{\Delta t} = \frac{1}{2}r^2\frac{\Delta\theta}{\Delta t} + r.\Delta r.\frac{\Delta\theta}{\Delta t}$$

$$\left\{\Delta t\to 0,\ \Delta r\to 0,\ \therefore \Delta\theta\to 0,\ \sin\Delta\theta\to\Delta\theta\right\}$$

This shows that the areal velocity is constant.

Areal speed $= \frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{1}{2}h = \text{const}$

Differential Equation of the Orbit in Polar Coordinates:

Let the particle, moving under a central force have polar coordinates (r,θ). Using radial and transverse directions, we have

$$\vec{F} = F_r(-\hat{r}) + F_\theta \hat{s}$$

or

$$\vec{F} = -F_r\hat{r} + F_\theta \hat{s}$$

or

$$\vec{F} = -ma_r\hat{r} + ma_\theta \hat{s} \qquad ...(1)$$

where $a_r = \ddot{r} - r\dot{\theta}^2$, $a_\theta = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$ are radial and transverse components of acceleration. Since $\vec{F}$ is central force, so its transverse component is zero, i.e.

$$F_\theta = 0$$
$$\Rightarrow ma_\theta = 0$$
$$\Rightarrow a_\theta = 0$$
$$\Rightarrow \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$$
$$\Rightarrow \frac{d}{dt}(r^2\dot{\theta}) = 0$$
$$\Rightarrow (r^2\dot{\theta}) = \text{constant} = h\ (\text{say}) \quad ...(2)$$

Putting $a_\theta = 0$, $a_r = \ddot{r} - r\dot{\theta}^2$ in (1), we have $\vec{F} = -m(\ddot{r} - r\dot{\theta}^2)\hat{r}$

$$\Rightarrow m(\ddot{r} - r\dot{\theta}^2)\hat{r} = -\vec{F}$$

Taking magnitudes on both sides, we have

$$m(\ddot{r} - r\dot{\theta}^2) = -F \quad ...(3)$$

Let $r = \frac{1}{u}$, then differentiating w.r.t. t, we have

$$\dot{r} = \frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} \quad ...(4)$$

Since $r^2\dot{\theta} = h$, so $\frac{1}{u^2}\dot{\theta} = h$, i.e. $\dot{\theta} = hu^2$. Putting this value in (4), we have

$$\dot{r} = -\frac{1}{u^2}\frac{du}{d\theta}(hu^2) = -h\frac{du}{d\theta} \quad ...(5)$$

Once again differentiating w.r.t. t, we have

$$\ddot{r} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right) = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{d\theta}{dt} = -h\frac{d^2u}{d\theta^2}(hu^2)$$

$$\ddot{r} = -h^2u^2\frac{d^2u}{d\theta^2} \quad ...(6)$$

Putting the values of $r, \ddot{r}, \dot{\theta}$ in (3), we have

$$m\left(-h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}(hu^2)^2\right) = -F$$

$$m\left(h^2u^2\frac{d^2u}{d\theta^2} + h^2u^3\right) = mf$$

$$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{f}{h^2u^2}} \quad ...(10)$$

which is differential equation of the orbit in polar coordinates.

Differential Equation of the Orbit in Pedal Form:

PU, 2008 (B.A/B.Sc.)

Differential equation of the orbit in polar form is

$$\frac{d^2u}{d\theta^2}+u=\frac{f}{h^2u^2} \qquad ...(1)$$

If p is the perpendicular distance from origin to the tangent to the point of the orbit, then from calculus, we have

$$\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \qquad ...(2)$$

Since $r=\frac{1}{u}$, so using chain rule, we have

$$\frac{dr}{d\theta}=\frac{d}{d\theta}\left(\frac{1}{u}\right)=\frac{d}{du}\left(\frac{1}{u}\right)\frac{du}{d\theta}=-\frac{1}{u^2}\frac{du}{d\theta}$$

Putting these values in (2), we have

$$\frac{1}{p^2}=u^2+u^4\left(-\frac{1}{u^2}\frac{du}{d\theta}\right)^2$$

$$\frac{1}{p^2}=u^2+\left(\frac{du}{d\theta}\right)^2$$

Differentiating both sides w.r.t. θ, we have

$$\frac{d}{d\theta}\left(\frac{1}{p^2}\right)=\frac{d}{d\theta}(u^2)+\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)^2$$

$$-\frac{2}{p^3}\frac{dp}{d\theta}=2u\frac{du}{d\theta}+2\frac{du}{d\theta}\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)$$

$$-\frac{1}{p^3}\frac{dp}{d\theta}=u\frac{du}{d\theta}+\frac{du}{d\theta}\frac{d^2u}{d\theta^2}$$

$$-\frac{1}{p^3}\frac{dp}{du}\frac{du}{d\theta}=\frac{du}{d\theta}\left(u+\frac{d^2u}{d\theta^2}\right)$$

$$-\frac{1}{p^3}\frac{dp}{du}=u+\frac{d^2u}{d\theta^2}$$

$$-\frac{1}{p^3}\frac{dp}{du}=\frac{f}{h^2u^2} \qquad \text{(using (1))}$$

$$-\frac{1}{p^3}\frac{dp}{dr}\frac{dr}{du}=\frac{f}{h^2u^2}$$

$$-\frac{1}{p^3}\frac{dp}{dr}\left(-\frac{1}{u^2}\right)=\frac{f}{h^2u^2}$$

$$\frac{h^2}{p^3}\frac{dp}{dr} = f \tag{10.2}$$

which is differential equation of the orbit in pedal form.

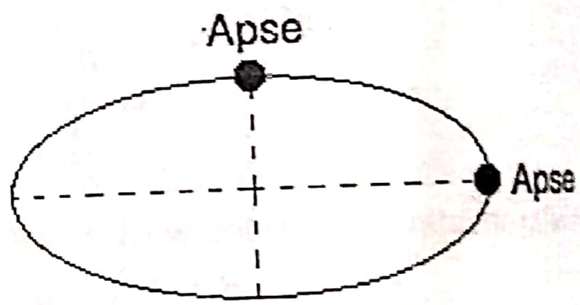
10-4 Apse and Apsidal Distance

PU, 2012; 2011 (B.A/B.Sc.); PU, 2013; 2011 (BS Math)

An *apse* is a point on an orbit at which the radius vector drawn from the centre of force is a maximum or minimum.

The length of the radius vector at such a point is known as the *apsidal distance*. The line joining an apse to the centre of force is an *apse line*. The angle between two consecutive apse lines is an *apsidal angle*.

The analytical condition for a point P with radius vector r (referred to the centre of force as origin) to be an apse, is



$$\frac{dr}{d\theta} = 0$$

or $$\frac{du}{d\theta} = 0$$

If ϕ is the angle between the radius vector and the tangent at P, then from calculus, we have

$$\tan\phi = r\frac{d\theta}{dr}$$

$$\Rightarrow \cot\phi = \frac{1}{r}\frac{dr}{d\theta}$$

$$\Rightarrow \cot\phi = 0 \text{ if and only if } \frac{dr}{d\theta} = 0$$

$$\Rightarrow \phi = \frac{\pi}{2} \text{ if and only if } \frac{dr}{d\theta} = 0$$

Thus

The radius vector through an apse and the tangent at it to the orbit are perpendicular.

10-5 Classification of Orbits in Terms of Energy

Theorem-4: *The planet will describe an ellipse, a parabola or a hyperbola according as total energy per unit mass, E, is negative, zero or positive.*

Proof: If M and m are the masses of the sun and the planet, then by Newton's law of gravitation, they attract each other with a force $G\dfrac{Mm}{r^2}$.

Referred to the sun as the pole, the differential equation of the orbit is

$$mh^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)=G\frac{Mm}{r^2}$$

$$mh^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)=GMmu^2$$

$$\frac{d^2u}{d\theta^2}+u=\frac{GM}{h^2}$$

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2} \qquad ...(1)$$

Where $\mu=GM$. Equation (1) can be written as

$$\frac{d^2u}{d\theta^2}+u-\frac{\mu}{h^2}=0$$

$$\Rightarrow \frac{d^2}{d\theta^2}\left(u-\frac{\mu}{h^2}\right)+\left(u-\frac{\mu}{h^2}\right)=0 \qquad ...(2)$$

Its general solution is

$$u-\frac{\mu}{h^2}=C\cos(\theta-\theta_0) \qquad ...(3)$$

where C and θ_0 are constants of integration. Equation (3) is the equation of the most general orbit described under a central force varying as the inverse square of the distance. Constant θ_0 can be made zero by rotating the base line $\theta=0$, so that equation (3) takes the form

$$u=\frac{\mu}{h^2}+C\cos\theta \qquad ...(4)$$

or

$$\frac{1}{r}=\frac{\mu}{h^2}+C\cos\theta$$

or

$$\frac{\frac{h^2}{\mu}}{r}=1+\frac{h^2C}{\mu}\cos\theta \qquad ...(5)$$

Thus the orbit is a conic with one focus at the centre of force, semi-latus rectum equal to $\dfrac{h^2}{\mu}$, and eccentricity $\dfrac{h^2C}{\mu}$. To determine C, we may proceed as follows:

Let V be the potential energy per unit mass. Then

$$-\frac{\mu}{r^2}=-\frac{dV}{dr}$$

$$E = -\frac{\mu}{2a} \quad \text{...(13)}$$

$$h = \sqrt{\mu l} = \sqrt{\mu a(1-e^2)} \quad \text{...(14)}$$

$$e = \sqrt{1+\frac{2Eh^2}{\mu^2}} \quad \text{...(15)}$$

These formulae give a, e in terms of E, h and vice versa.

Velocity: From the equation of energy, for a unit mass,

$$T + V = E$$

$$\frac{1}{2}v^2 - \mu u = -\frac{\mu}{2a}$$

$$\frac{1}{2}v^2 = \frac{\mu}{r} - \frac{\mu}{2a}$$

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

This is the velocity at any point of the orbit.

10-6 Kepler's Laws of Planetary Motion

Kepler's First Law: *Each planet describes an ellipse with the sun at one focus.*

PU, 2013; 2011 (BS Math)

Proof: If M and m are the masses of the sun and the planet, then by Newton's law of gravitation, they attract each other with a force $G\frac{Mm}{r^2}$.

Referred to the sun as the pole, the differential equation of the orbit is

$$mh^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = G\frac{Mm}{r^2}$$

$$mh^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = GMmu^2$$

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2}$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} \quad \text{...(1)}$$

Where $\mu = GM$. Equation (1) can be written as

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{h^2} = 0$$

$$\Rightarrow \frac{d^2}{d\theta^2}\left(u-\frac{\mu}{h^2}\right)+\left(u-\frac{\mu}{h^2}\right)=0 \qquad ...(2)$$

Its general solution is

$$u-\frac{\mu}{h^2}=C\cos(\theta-\theta_0) \qquad ...(3)$$

where C and θ_0 are constants of integration. Equation (3) is the equation of the most general orbit described under a central force varying as the inverse square of the distance. Constant θ_0 can be made zero by rotating the base line $\theta=0$, so that equation (3) takes the form

$$u=\frac{\mu}{h^2}+C\cos\theta \qquad ...(4)$$

or $$\frac{1}{r}=\frac{\mu}{h^2}+C\cos\theta \text{ or } \frac{\frac{h^2}{\mu}}{r}=1+\frac{h^2C}{\mu}\cos\theta$$

or $$\frac{l}{r}=1+e\cos\theta \qquad ...(5)$$

Thus the orbit is a conic with one focus at the centre of force, semi-latus rectum equal to $l=\frac{h^2}{\mu}$, and eccentricity $e=\frac{h^2C}{\mu}$. This completes the proof.

Kepler's Second Law: *The radius vector drawn from the sun to the planet sweeps out equal areas in equal times.*

PU, 2013; 2011 (BS Math)

Proof: Let the planet, moving under a central force due to sun have polar coordinates (r,θ). Let P and Q be the neighbouring positions of the planet. Let sun be at origin O. Using radial and transverse directions, we have

$$\vec{F}=F_r(-\hat{r})+F_\theta\hat{s}=-F_r\hat{r}+F_\theta\hat{s}$$

or $$\vec{F}=-ma_r\hat{r}+ma_\theta\hat{s} \qquad ...(1)$$

where $a_r=\ddot{r}-r\dot{\theta}^2$, $a_\theta=\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$ are radial and transverse components of acceleration. Since $\vec{F}$ is central force, so its transverse component is zero, i.e.

$$F_\theta=0,$$

$$\Rightarrow ma_\theta=0$$

$$\Rightarrow \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})=0$$

$$\Rightarrow \frac{d}{dt}(r^2\dot{\theta})=0$$

$$\Rightarrow (r^2\dot{\theta})=\text{constant}=h\text{ (say)}$$

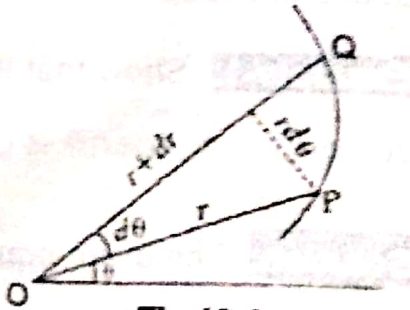


Fig.10.4

If radius vector OP sweeps an infinitesimal area POQ, dA, in time dt, then

$$dA = \frac{1}{2} r(r+dr)\sin d\theta$$

$$dA = \frac{1}{2} r^2 \theta \quad \text{in the limit}$$

Defining the areal velocity of the particle as $\frac{dA}{dt}$, we have

$$\frac{dA}{dt} = r^2\dot{\theta} = h$$

This shows that the areal velocity is constant, i.e. the radius vector drawn from the sun to the planet sweeps out equal areas in equal times. This completes the proof.

Kepler's Third Law: *The squares of the periods of the planets are proportional to the cubes of the semi-major axes of their orbits.*

PU, 2013;.2011 (BS Math)

Proof: Since $\frac{1}{2}h$ is equal to the area described in a unit time, it follows that if T be the time that the particle takes to describe the whole ellipse, then

$$\frac{1}{2}hT = \text{area of the ellipse} = \pi ab$$

$$\Rightarrow T = \frac{2\pi ab}{h} = \frac{2\pi ab}{\sqrt{\mu l}} \qquad \because l = \frac{h^2}{\mu}$$

$$= \frac{2\pi}{\sqrt{\mu}}\frac{ab}{\sqrt{l}} = \frac{2\pi}{\sqrt{\mu}}\frac{ab}{\sqrt{\frac{b^2}{a}}} \qquad \because l = \frac{b^2}{a}$$

$$= \frac{2\pi}{\sqrt{\mu}}\frac{a^{\frac{3}{2}}b}{b} = \frac{2\pi}{\sqrt{\mu}}a^{\frac{3}{2}}$$

$$\Rightarrow T^2 = \frac{4\pi^2}{\mu}a^3$$

$$\Rightarrow T^2 \propto a^3$$

This shows that the squares of the periods of the planets are proportional to the cubes of the semi-major axes of their orbits.

Example-1: Show that the law of force towards the pole, of a particle describing the curve $r^n = a^n \cos n\theta$ is given by $f = \frac{(n+1)h^2 a^{2n}}{r^{2n+3}}$.

PU, 2013; 2010 (B.A./B.Sc.); PU, 2011 (BS Math)

Solution: The given equation $r^n = a^n \cos n\theta$ can be written as

$$u^n = \frac{1}{a^n \cos n\theta} \qquad \because u = \frac{1}{r} \qquad \text{...(1)}$$

$$\Rightarrow nu^{n-1}\frac{du}{d\theta}=\frac{1}{a^n}\left(-\frac{1}{\cos^2 n\theta}\right)(-n\sin n\theta)$$

$$\Rightarrow nu^{n-1}\frac{du}{d\theta}=\frac{1\cdot n\sin n\theta}{a^n\cos^2 n\theta}$$

Multiplying both sides by u, we have

$$nu^n\frac{du}{d\theta}=\frac{1}{a^n}u\frac{n\sin n\theta}{\cos^2 n\theta}$$

$$n\left(\frac{1}{a^n\cos n\theta}\right)\frac{du}{d\theta}=\frac{1}{a^n}u\frac{n\sin n\theta}{\cos^2 n\theta} \qquad \text{(using (1))}$$

$$\frac{du}{d\theta}=u\tan n\theta \qquad ...(2)$$

$$\frac{d^2u}{d\theta^2}=\frac{d}{d\theta}(u\tan n\theta)=\tan n\theta\frac{du}{d\theta}+u\frac{d}{d\theta}(\tan n\theta)$$

$$=\tan n\theta(u\tan n\theta)+u(n\sec^2 n\theta)$$

$$=u\tan^2 n\theta+nu\sec^2 n\theta$$

$$\frac{d^2u}{d\theta^2}=u(\tan^2 n\theta+n\sec^2 n\theta) \qquad ...(3)$$

Differential equation of the orbit is

$$\frac{f}{h^2u^2}=\frac{d^2u}{d\theta^2}+u$$

$$\frac{f}{h^2u^2}=u(\tan^2 n\theta+n\sec^2 n\theta)+u \qquad \text{(using (3))}$$

$$\frac{f}{h^2u^2}=\tan^2 n\theta+n\sec^2 n\theta+1$$

$$\frac{f}{h^2u^3}=n\sec^2 n\theta+\sec^2 n\theta$$

$$\frac{f}{h^2u^3}=(n+1)\sec^2 n\theta$$

$$\frac{f}{h^2u^3}=\frac{n+1}{\cos^2 n\theta}$$

$$\frac{f}{h^2u^3}=(n+1)(a^nu^n)^2 \qquad \text{(using (1))}$$

$$\frac{f}{h^2u^3}=(n+1)a^{2n}u^{2n}$$

$$f=(n+1)h^2a^{2n}u^{2n+3}$$

$$f=\frac{(n+1)h^2a^{2n}}{r^{2n+3}}$$

Example-2: A particle describes the curve $\frac{a}{r} = e^{n\theta}$ under force f to the pole, show that the force is stated as $f \propto \frac{1}{r^3}$.

PU, 2009 (B.A/B.Sc)

Solution:

$$\frac{a}{r} = e^{n\theta}$$

$$\Rightarrow au = e^{n\theta} \qquad \because u = \frac{1}{r}$$

$$\Rightarrow a\frac{du}{d\theta} = ne^{n\theta} = n(au)$$

$$\Rightarrow \frac{du}{d\theta} = nu$$

$$\Rightarrow \frac{d^2u}{d\theta^2} = n\frac{du}{d\theta} = n(nu) = n^2u \qquad ...(1)$$

Differential equation of the orbit is

$$\frac{f}{h^2u^2} = \frac{d^2u}{d\theta^2} + u$$

$$\frac{f}{h^2u^2} = n^2u + u \qquad \text{(using (1))}$$

$$\frac{f}{h^2u^2} = (n^2+1)u$$

$$f = (n^2+1)h^2u^3$$

$$f = \frac{(n^2+1)h^2}{r^3}$$

$$f \propto \frac{1}{r^3} \qquad \because (n^2+1)h^2 = \text{constant}$$

Example-3: A particle describes the curve $\frac{a}{r} = n\theta$ under force f to the pole, show that the force is stated as $f \propto \frac{1}{r^3}$.

Solution:

$$\frac{a}{r} = n\theta$$

$$\Rightarrow au = n\theta \qquad \because u = \frac{1}{r}$$

$$\Rightarrow a\frac{du}{d\theta} = n$$

$$\Rightarrow \frac{d^2u}{d\theta^2} = 0 \qquad \ldots(1)$$

Differential equation of the orbit is

$$\frac{f}{h^2u^2} = \frac{d^2u}{d\theta^2} + u$$

$$\frac{f}{h^2u^2} = 0 + u \qquad \text{(using (1))}$$

$$\frac{f}{h^2u^2} = u$$

$$f = h^2u^3$$

$$f = \frac{h^2}{r^3}$$

$$f \propto \frac{1}{r^3} \qquad \because h^2 = \text{constant}$$

Example-[illegible] *A particle describes the curve* $\frac{a}{r} = \cosh n\theta$ *under force* f *to the pole, show that the force is stated as* $f \propto \frac{1}{r^3}$.

Solution:

$$\frac{a}{r} = \cosh n\theta$$

$$\Rightarrow au = \cosh n\theta \qquad \because u = \frac{1}{r}$$

$$\Rightarrow a\frac{du}{d\theta} = n\sinh n\theta$$

$$\Rightarrow a\frac{d^2u}{d\theta^2} = n^2\cosh n\theta = n^2(au) = an^2u$$

$$\Rightarrow \frac{d^2u}{d\theta^2} = n^2u \qquad \ldots(1)$$

Differential equation of the orbit is

$$\frac{f}{h^2u^2} = \frac{d^2u}{d\theta^2} + u$$

$$\frac{f}{h^2u^2} = n^2u + u \qquad \text{(using (1))}$$

$$\frac{f}{h^2u^2} = (n^2+1)u$$

$$f = (n^2+1)h^2u^3$$

$$f = \frac{(n^2+1)h^2}{r^3}$$

$$f \propto \frac{1}{r^3} \qquad \because (n^2+1)h^2 = \text{constant}$$

Example-5: A particle describes the curve $\frac{a}{r} = \sin n\theta$ under force f to the pole, show that the force is stated as $f \propto \frac{1}{r^3}$.

Solution:

$$\frac{a}{r} = \sin n\theta$$

$$\Rightarrow au = \sin n\theta \qquad \because u = \frac{1}{r}$$

$$\Rightarrow a\frac{du}{d\theta} = n\cos n\theta$$

$$\Rightarrow a\frac{d^2u}{d\theta^2} = -n^2 \sin n\theta$$

$$\Rightarrow a\frac{d^2u}{d\theta^2} = -n^2(au)$$

$$\Rightarrow \frac{d^2u}{d\theta^2} = -n^2u \qquad ...(1)$$

Differential equation of the orbit is

$$\frac{f}{h^2u^2} = \frac{d^2u}{d\theta^2} + u$$

$$\frac{f}{h^2u^2} = -n^2u + u \qquad \text{(using (1))}$$

$$\frac{f}{h^2u^2} = (1-n^2)u$$

$$f = (1-n^2)h^2u^3$$

$$f = \frac{(1-n^2)h^2}{r^3}$$

$$f \propto \frac{1}{r^3} \qquad \because (1-n^2)h^2 = \text{constant}$$

Example-6: A particle describes the curve $r^n \cos n\theta = a^n$ under force f to the pole show that the force is stated as $f \propto r^{2n-3}$.

Solution:

$$r^n \cos n\theta = a^n$$

$$\frac{1}{r^n} = \frac{\cos n\theta}{a^n}$$

$$u^n = \frac{\cos n\theta}{a^n} \qquad ...(1)$$

$$\Rightarrow \ln u^n = \ln \frac{\cos n\theta}{a^n}$$

$$\Rightarrow n \ln u = \ln \cos n\theta - \ln a^n$$

$$\Rightarrow n \ln u = \ln \cos n\theta - n \ln a$$

$$\Rightarrow \frac{n}{u}\frac{du}{d\theta} = \frac{1}{\cos n\theta}(-n \sin n\theta) - 0$$

$$\Rightarrow \frac{1}{u}\frac{du}{d\theta} = -\frac{\sin n\theta}{\cos n\theta}$$

$$\Rightarrow \frac{du}{d\theta} = -u \tan n\theta$$

$$\Rightarrow \frac{d^2u}{d\theta^2} = -\frac{du}{d\theta}\tan n\theta - nu\sec^2 n\theta = -(-u\tan n\theta)\tan n\theta - nu\sec^2 n\theta$$

$$= u\tan^2 n\theta - nu\sec^2 n\theta = u(\sec^2 n\theta - 1) - nu\sec^2 n\theta$$

$$= u\sec^2 n\theta - u - nu\sec^2 n\theta = (1-n)u\sec^2 n\theta - u$$

$$= \frac{(1-n)u}{\cos^2 n\theta} - u = \frac{(1-n)u}{(a^n u^n)^2} - u = \frac{(1-n)u}{a^{2n}u^{2n}} - u$$

$$\Rightarrow \frac{d^2u}{d\theta^2} + u = \frac{1-n}{a^{2n}u^{2n-1}} \qquad ...(2)$$

Differential equation of the orbit is

$$\frac{f}{h^2u^2} = \frac{d^2u}{d\theta^2} + u$$

$$\frac{f}{h^2u^2} = \frac{1-n}{a^{2n}u^{2n-1}} \qquad \text{(using (2))}$$

$$\frac{f}{h^2} = \frac{1-n}{a^{2n}u^{2n-3}}$$

$$f = \frac{(1-n)h^2}{a^{2n}u^{2n-3}}$$

$$f = \frac{(1-n)h^2 r^{2n-3}}{a^{2n}}$$

$$f \propto r^{2n-3} \qquad \because (1-n)h^2 = \text{constant}$$

Example-7: *A particle describes the curve $r^n = A\cos n\theta + B\sin n\theta$ under force f to the pole, show that the force is stated as $f \propto \frac{1}{r^{2n+3}}$.*

Solution:

$$r^n = A\cos n\theta + B\sin n\theta \qquad ...(1)$$

Let $A = a\cos\alpha$, $B = a\sin\alpha$, then putting these values in (1), we have

$$r^n = a\cos\alpha\cos n\theta + a\sin\alpha\sin n\theta = a\cos(n\theta-\alpha)$$

$$\Rightarrow \frac{1}{u^n} = a\cos(n\theta-\alpha)$$

$$\Rightarrow u^{-n} = a\cos(n\theta-\alpha) \quad \ldots(2)$$

$$\Rightarrow -nu^{-n-1}\frac{du}{d\theta} = -na\sin(n\theta-\alpha)$$

$$\Rightarrow u^{-n-1}\frac{du}{d\theta} = a\sin(n\theta-\alpha)$$

$$\Rightarrow \frac{du}{d\theta} = au^{n+1}\sin(n\theta-\alpha) \quad \ldots(3)$$

$$\Rightarrow \frac{d^2u}{d\theta^2} = a(n+1)u^n\frac{du}{d\theta}\sin(n\theta-\alpha) + anu^{n+1}\cos(n\theta-\alpha)$$

$$= a(n+1)u^n[au^{n+1}\sin(n\theta-\alpha)]\sin(n\theta-\alpha) + anu^{n+1}\cos(n\theta-\alpha) \text{ (by (3))}$$

$$= a^2(n+1)u^{2n+1}\sin^2(n\theta-\alpha) + anu^{n+1}\cos(n\theta-\alpha)$$

$$= a^2(n+1)u^{2n+1}[1-\cos^2(n\theta-\alpha)] + anu^{n+1}\cos(n\theta-\alpha)$$

Putting the value of $\cos(n\theta-\alpha)$ from (2), we have

$$\frac{d^2u}{d\theta^2} = a^2(n+1)u^{2n+1}\left[1-\left(\frac{1}{au^n}\right)^2\right] + anu^{n+1}\left(\frac{1}{au^n}\right)$$

$$= a^2(n+1)u^{2n+1}\left[1-\frac{1}{a^2u^{2n}}\right] + nu$$

$$= a^2(n+1)u^{2n+1} - \frac{a^2(n+1)u^{2n+1}}{a^2u^{2n}} + nu$$

$$= a^2(n+1)u^{2n+1} - (n+1)u + nu$$

$$= a^2(n+1)u^{2n+1} - u$$

$$\Rightarrow \frac{d^2u}{d\theta^2} + u = a^2(n+1)u^{2n+1} \quad \ldots(4)$$

Differential equation of the orbit is

$$\frac{f}{h^2u^2} = \frac{d^2u}{d\theta^2} + u$$

$$\frac{f}{h^2u^2} = a^2(n+1)u^{2n+1} \quad \text{(using (2))}$$

$$f = a^2(n+1)h^2u^{2n+3}$$

$$f = \frac{a^2(n+1)h^2}{r^{2n+3}}$$

$$f \propto \frac{1}{r^{2n+3}} \qquad \because a^2(n+1)h^2 = \text{constant}$$

Example-8: *Prove that the speed at any point of a central orbit is given by vp = h, where h is the areal speed and p is the perpendicular distance from the centre of force, of the tangent at that point.*

Hence find the expression for v when a particle, subject to the inverse square law of force, describes (i) an elliptic, (ii) a parabolic, (iii) hyperbolic orbit.

Solution: We know that

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = h^2\left(\frac{du}{d\theta}\right)^2 + \frac{1}{u^2}(h^2u^4) = h^2\left(\frac{du}{d\theta}\right)^2 + h^2u^2$$

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]$$

$$v^2 = h^2\frac{1}{p^2} \qquad \because u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{1}{p^2}$$

$$v^2p^2 = h^2$$

$$vp = h$$

When the force (per unit mass) is $\frac{\mu}{r^2} = \mu u^2$, then $h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = \mu u^2$

$$\Rightarrow h^2\left(u + \frac{d^2u}{d\theta^2}\right) = \mu$$

Multiplying both sides by $2\frac{du}{d\theta}$, we have

$$h^2\left(2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}\right) = 2\mu\frac{du}{d\theta}$$

$$\Rightarrow h^2\left(2u\frac{du}{d\theta} + \frac{d}{d\theta}\left(\frac{du}{d\theta}\right)^2\right) = 2\mu\frac{du}{d\theta}$$

$$\Rightarrow h^2\left(2udu + d\left(\frac{du}{d\theta}\right)^2\right) = 2\mu du$$

$$\Rightarrow h^2\left(\int 2udu + \int d\left(\frac{du}{d\theta}\right)^2\right) = \int 2\mu du$$

$$\Rightarrow h^2\left(u^2 + \left(\frac{du}{d\theta}\right)^2\right) = 2\mu u + C$$

$$\Rightarrow \frac{h^2}{p^2} = \frac{2\mu}{r} + C \quad ...(1)$$

which is (r,p) equation of the orbit.

Now (r,p) equations of an ellipse, parabola and hyperbola, referred to a focus as pole, are respectively

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1 \quad ...(2)$$

$$p^2 = ar \quad ...(3)$$

$$\frac{b^2}{p^2} = \frac{2a}{r} + 1 \quad ...(4)$$

Comparing equations (2), (3) and (4) with equation (1), we find that

$C = -\frac{\mu}{a}$ (i.e. C is -ve) in elliptic orbit

$C = 0$ in parabolic orbit

$C = \frac{\mu}{a}$ (i.e. C is +ve) in hyperbolic orbit

Now $\quad v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r} + C = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$ when the orbit is elliptic,

$v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r}$ when the orbit is parabolic,

and $\quad v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r} + C = \mu\left(\frac{2}{r} + \frac{1}{a}\right)$ when the orbit is hyperbolic.

Note: A circle is a special case of an ellipse in which r is equal to a. Therefore, in case of circular orbit described under the inverse square law of force,

$$v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r} + C = \mu\left(\frac{2}{a} - \frac{1}{a}\right) = \frac{\mu}{a}$$

Example-9: *A particle of unit mass describes an ellipse under the action of central force Mr. Show that the normal component of the acceleration at any instant is $\frac{abM^{\frac{3}{2}}}{v}$, where v is the velocity at that instant and a, b are the semi-axes of the ellipse.*

Solution: Differential equation of the orbit is

$$h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = Mr$$

$$h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=\frac{M}{u}$$

$$h^2\left(u+\frac{d^2u}{d\theta^2}\right)=Mu^{-3}$$

Multiplying both sides by $2\frac{du}{d\theta}$, we have

$$h^2\left(2u\frac{du}{d\theta}+2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}\right)=2Mu^{-3}\frac{du}{d\theta}$$

$$\Rightarrow h^2\left(2u\frac{du}{d\theta}+\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)^2\right)=2Mu^{-3}\frac{du}{d\theta}$$

$$\Rightarrow h^2\left(2udu+d\left(\frac{du}{d\theta}\right)^2\right)=2Mu^{-3}du$$

$$\Rightarrow h^2\left(\int 2udu+\int d\left(\frac{du}{d\theta}\right)^2\right)=\int 2Mu^{-3}du$$

$$\Rightarrow h^2\left(u^2+\left(\frac{du}{d\theta}\right)^2\right)=\frac{2Mu^{-3+1}}{-3+1}+C$$

$$\Rightarrow h^2\left(u^2+\left(\frac{du}{d\theta}\right)^2\right)=-Mu^{-2}+C$$

$$\Rightarrow \frac{h^2}{p^2}=-\frac{M}{u^2}+C$$

$$\Rightarrow \frac{h^2}{p^2}=C-Mr^2 \qquad \text{---(1)}$$

Equation of the ellipse is

$$\frac{a^2b^2}{p^2}=a^2+b^2-r^2 \qquad \text{---(2)}$$

Comparing (1) and (2), we have

$$\frac{h^2}{a^2b^2}=\frac{C}{a^2+b^2}=\frac{M}{1}$$

Therefore

$$h^2=Ma^2b^2,\quad C=M(a^2+b^2)$$

Now $v^2=\frac{h^2}{p^2}=\frac{Ma^2b^2}{p^2}$, so $p^2=\frac{Ma^2b^2}{v^2}$, i.e. $p=\frac{\sqrt{M}ab}{v}$.

The normal component of the acceleration at any instant is $\dfrac{v^2}{\rho}$.

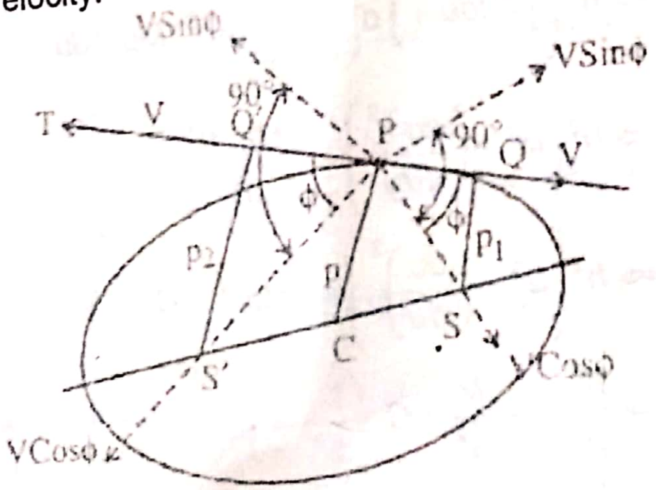
For an ellipse, $\rho = \dfrac{a^2b^2}{p^3}$, so

$$\frac{v^2}{\rho} = \frac{v^2}{\dfrac{a^2b^2}{p^3}} = \frac{v^2}{a^2b^2}p^3 = \frac{v^2}{a^2b^2}\left(\frac{\sqrt{M}ab}{v}\right)^3 = \frac{v^2}{a^2b^2}\left(\frac{M^{\frac{3}{2}}a^3b^3}{v^3}\right) = \frac{abM^{\frac{3}{2}}}{v}$$

Example-10: *If a particle be, describing an ellipse about a centre of force in the centre, show that the sum of the reciprocates of its angular velocities about the foci is constant.*

PU, 2005 (B.A./B.Sc.); University of Sargodha, 2014 (B.A./B.Sc.)

Solution: Let P be the position of the particle at time t and v be its velocity along the tangent PT. Let p be the length of the perpendicular form C on PT. Then $vp = h$, where h is the areal velocity.



Obviously $m\angle QPS = m\angle Q'PS' = \phi$ (say). If ω_1 and ω_2 are the angular velocities of the particle at P about the foci S and S' and p_1 and P_2 the perpendicular distances of the tangent from S and S', then

$$\omega_1 = \frac{v\sin\phi}{SP}, \quad \omega_2 = \frac{v\sin\phi}{S'P}$$

$$\Rightarrow \frac{1}{\omega_1} = \frac{SP}{v\sin\phi}, \quad \frac{1}{\omega_2} = \frac{S'P}{v\sin\phi}$$

Adding these, we have

$$\frac{1}{\omega_1} + \frac{1}{\omega_2} = \frac{SP}{v\sin\phi} + \frac{S'P}{v\sin\phi} = \frac{SP+S'P}{v\sin\phi} = \frac{2a}{v\sin\phi} \qquad \because SP + S'P = 2a$$

$$\Rightarrow \frac{1}{\omega_1} + \frac{1}{\omega_2} = \frac{2a}{v\sin\phi} \qquad \dots(1)$$

From right triangles ΔPQS, $\Delta PQ'S'$, we have

$$\sin\phi = \frac{p_1}{SP} = \frac{p_2}{S'P} = \frac{p_1+p_2}{SP+S'P} = \frac{2p}{2a} = \frac{p}{a}$$

Putting this value in (1), we have

$$\frac{1}{\omega_1} + \frac{1}{\omega_2} = \frac{2a}{v\left(\frac{p}{a}\right)} = \frac{2a^2}{vp} = \frac{2a^2}{h} = \text{constant} \qquad \because vp = h$$

Example-11: *A particle of mass m moves under a central force*

$$mM[3au^4 - 2(a^2-b^2)u^5], \quad (a > b)$$

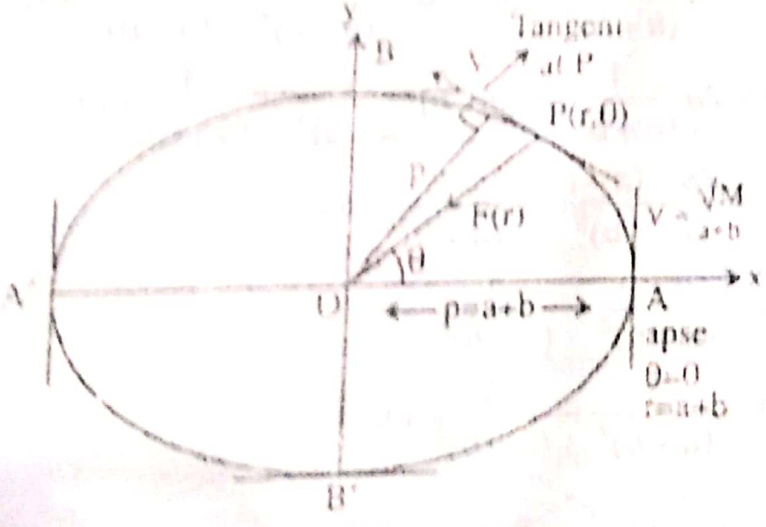
and is projected from an apse at distance $a+b$, with velocity $\frac{\sqrt{M}}{a+b}$, show that the orbit is $r = a + b\cos\theta$.

PU, 2006 (B.A./B.Sc.)

Solution: Differential equation of the orbit is

$$mh^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = mM[3au^4 - 2(a^2-b^2)u^5]$$

$$h^2\left(u + \frac{d^2u}{d\theta^2}\right) = M[3au^2 - 2(a^2-b^2)u^3] \qquad \ldots(1)$$



Since at the given apse $p = a+b$, $v = \frac{\sqrt{M}}{a+b}$, so $h = pv = (a+b)\frac{\sqrt{M}}{a+b} = \sqrt{M}$. Putting this value in (1), we have

$$M\left(u + \frac{d^2u}{d\theta^2}\right) = M[3au^2 - 2(a^2-b^2)u^3]$$

$$\Rightarrow u + \frac{d^2u}{d\theta^2} = 3au^2 - 2(a^2-b^2)u^3$$

$$\Rightarrow \frac{d^2u}{d\theta^2} = 3au^2 - 2(a^2-b^2)u^3 - u$$

Multiplying both sides by $2\frac{du}{d\theta}$, we have

$$2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}=2[3au^2-2(a^2-b^2)u^3-u]\frac{du}{d\theta}$$

$$\Rightarrow \frac{d}{d\theta}\left(\frac{du}{d\theta}\right)^2=2[3au^2-2(a^2-b^2)u^3-u]\frac{du}{d\theta}$$

$$\Rightarrow d\left(\frac{du}{d\theta}\right)^2=2[3au^2-2(a^2-b^2)u^3-u]du$$

Integrating both sides, we have

$$\int d\left(\frac{du}{d\theta}\right)^2=2\int[3au^2-2(a^2-b^2)u^3-u]du$$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2=2au^3-(a^2-b^2)u^4-u^2+C \qquad ...(2)$$

Now at the given apse, $\frac{du}{d\theta}=0,\ u=\frac{1}{r}=\frac{1}{a+b}$. Putting these values in (2), we have

$$0=2a\frac{1}{(a+b)^3}-(a^2-b^2)\frac{1}{(a+b)^4}-\frac{1}{(a+b)^2}+C$$

$$0=2a\frac{1}{(a+b)^3}-(a-b)\frac{1}{(a+b)^3}-\frac{1}{(a+b)^2}+C$$

$$0=\frac{2a-(a-b)}{(a+b)^3}-\frac{1}{(a+b)^2}+C$$

$$0=\frac{a+b}{(a+b)^3}-\frac{1}{(a+b)^2}+C$$

$$0=\frac{1}{(a+b)^2}-\frac{1}{(a+b)^2}+C$$

$$\Rightarrow C=0$$

Putting this value of C in (2), we have

$$\left(\frac{du}{d\theta}\right)^2=2au^3-(a^2-b^2)u^4-u^2$$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2=u^2[2au-(a^2-b^2)u^2-1]$$

$$\Rightarrow \left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2=\frac{1}{r^2}\left[2a\frac{1}{r}-(a^2-b^2)\frac{1}{r^2}-1\right] \quad \because u=\frac{1}{r}$$

$$\Rightarrow \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{1}{r^2}\left[2a\frac{1}{r}-(a^2-b^2)\frac{1}{r^2}-1\right]$$

$$\Rightarrow \left(\frac{dr}{d\theta}\right)^2 = 2ar - (a^2 - b^2) - r^2$$
$$= 2ar - a^2 + b^2 - r^2$$
$$= b^2 - (r^2 - 2ar + a^2)$$
$$\Rightarrow \left(\frac{dr}{d\theta}\right)^2 = b^2 - (r-a)^2$$
$$\Rightarrow \frac{dr}{d\theta} = \pm\sqrt{b^2 - (r-a)^2}$$
$$\Rightarrow \frac{dr}{\sqrt{b^2 - (r-a)^2}} = \pm d\theta$$
$$\Rightarrow \int \frac{dr}{\sqrt{b^2 - (r-a)^2}} = \pm\int d\theta$$
$$\Rightarrow \sin^{-1}\left(\frac{r-a}{b}\right) = \pm\theta + B$$
$$\Rightarrow \frac{r-a}{b} = \sin(\pm\theta + B) \qquad ...(3)$$

Now at the apse, $r = a + b$, $\theta = 0$. Putting these values in (3), we have

$$\frac{(a+b)-a}{b} = \sin(0+B)$$
$$\frac{b}{b} = \sin B$$
$$1 = \sin B$$
$$B = \sin^{-1}(1)$$
$$B = \frac{\pi}{2}$$

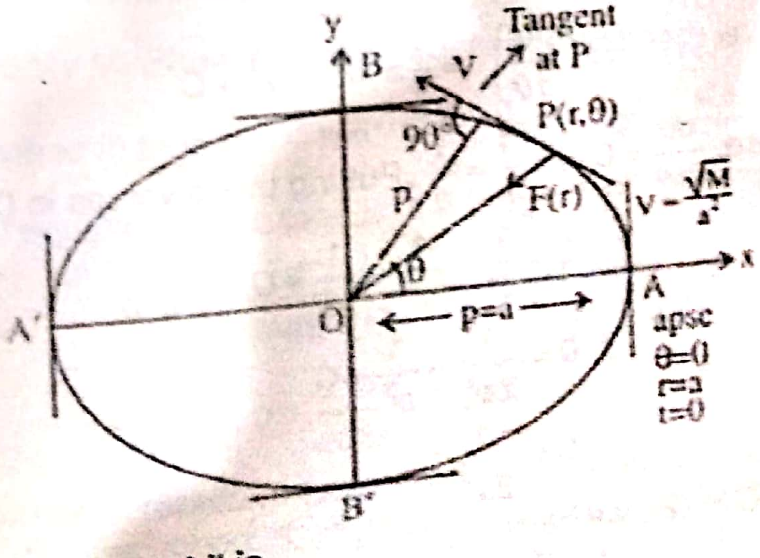
Putting this value of B in (3), we have

$$\frac{r-a}{b} = \sin\left(\frac{\pi}{2} \pm \theta\right)$$
$$\Rightarrow \frac{r-a}{b} = \cos\theta$$
$$\Rightarrow r - a = b\cos\theta$$
$$\Rightarrow r = a + b\cos\theta$$

Example-12: The law of force is Mu^5 and a particle is projected from an apse at distance a. Find the orbit when the velocity of projection is $\frac{\sqrt{M}}{a^2}$.

PU, 2012; 2008; 2005 (B.A/B.Sc.)

Solution:



Differential equation of the orbit is

$$h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = Mu^5$$

$$\Rightarrow h^2\left(u+\frac{d^2u}{d\theta^2}\right)=Mu^3 \qquad ...(1)$$

Since at the given apse $p=a,\ v=\frac{\sqrt{M}}{a^2}$, so $h=pv=a\frac{\sqrt{M}}{a^2}=\frac{\sqrt{M}}{a}$. Putting this value in (1), we have

$$\frac{M}{a^2}\left(u+\frac{d^2u}{d\theta^2}\right)=Mu^3$$

$$\Rightarrow u+\frac{d^2u}{d\theta^2}=a^2u^3$$

$$\Rightarrow \frac{d^2u}{d\theta^2}=a^2u^3-u$$

Multiplying both sides by $2\frac{du}{d\theta}$, we have

$$2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}=2(a^2u^3-u)\frac{du}{d\theta}$$

$$\Rightarrow \frac{d}{d\theta}\left(\frac{du}{d\theta}\right)^2=2(a^2u^3-u)\frac{du}{d\theta}$$

$$\Rightarrow d\left(\frac{du}{d\theta}\right)^2=(2a^2u^3-2u)du$$

Integrating both sides, we have

$$\int d\left(\frac{du}{d\theta}\right)^2=\int(2a^2u^3-2u)du$$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2=\frac{1}{2}a^2u^4-u^2+C \qquad ...(2)$$

Now at the given apse, $\frac{du}{d\theta}=0,\ u=\frac{1}{r}=\frac{1}{a}$. Putting these values in (2), we have

$$0=\frac{1}{2}a^2\frac{1}{a^4}-\frac{1}{a^2}+C$$

$$0=\frac{1}{2a^2}-\frac{1}{a^2}+C$$

$$C=\frac{1}{2a^2}$$

Putting this value of C in (2), we have

$$\left(\frac{du}{d\theta}\right)^2=\frac{1}{2}a^2u^4-u^2+\frac{1}{2a^2}$$

$$\left(\frac{du}{d\theta}\right)^2=\frac{a^4u^4-2a^2u^2+1}{2a^2}=\frac{(a^2u^2-1)^2}{2a^2}$$

$$\Rightarrow\frac{du}{d\theta}=\pm\frac{a^2u^2-1}{\sqrt{2}a}$$

$$\Rightarrow\frac{adu}{a^2u^2-1}=\pm\frac{d\theta}{\sqrt{2}}$$

$$\Rightarrow\int\frac{adu}{a^2u^2-1}=\pm\int\frac{d\theta}{\sqrt{2}}$$

$$\Rightarrow\frac{1}{2}\ln\frac{au-1}{au+1}=\pm\frac{\theta}{\sqrt{2}}+\frac{1}{2}\ln C$$

$$\Rightarrow\ln\frac{au-1}{au+1}=\pm2\frac{\theta}{\sqrt{2}}+\ln C$$

$$\Rightarrow\ln\frac{au-1}{au+1}-\ln C=\pm\sqrt{2}\theta$$

$$\Rightarrow\ln\frac{au-1}{C(au+1)}=\pm\sqrt{2}\theta$$

$$\Rightarrow\frac{au-1}{C(au+1)}=e^{\pm\sqrt{2}\theta}$$

$$\Rightarrow\frac{au-1}{au+1}=Ce^{\pm\sqrt{2}\theta}\qquad\text{...(3)}$$

Now at the apse, $u=\frac{1}{r}=\frac{1}{a}$, $\theta=0$. Putting these values in (3), we have

$$\frac{1-1}{1+1}=Ce^0$$

$$\Rightarrow C=0$$

Putting this value of C in (3), we have

$$\frac{au-1}{au+1}=0$$

$$\Rightarrow au-1=0$$

$$\Rightarrow au=1$$

$$\Rightarrow\frac{a}{r}=1$$

$$\Rightarrow r=a$$

which is the required orbit

Example-13:

A particle moves under a central repulsive force $\frac{\mu}{r^3}$ and is projected from an apse at a distance a with velocity V. Show that the equation to the path is $r\cos p\theta=a$, and that the angle θ described in time t is $\frac{1}{p}\tan^{-1}\frac{pVt}{a}$, where $p^2=\frac{\mu+a^2V^2}{a^2V^2}$.

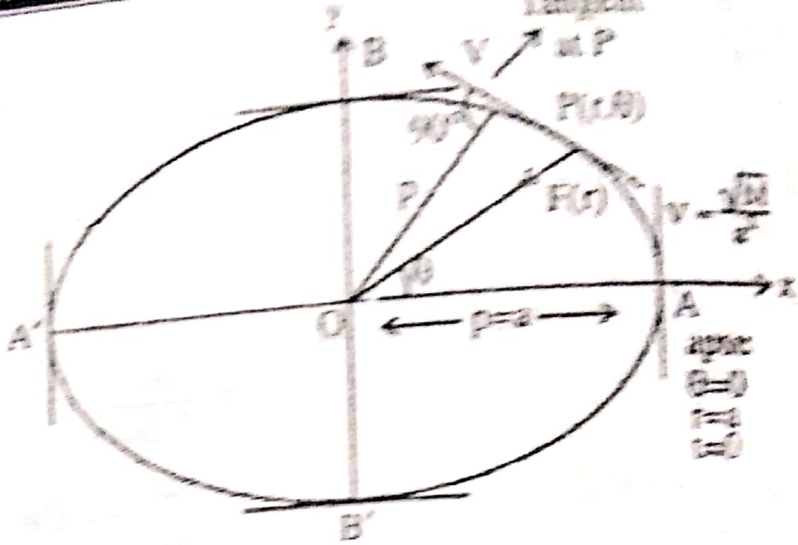
Solution: Differential equation of the orbit is

$$h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=-\frac{\mu}{r^3}=-\mu u^3$$

$$\Rightarrow h^2\left(u+\frac{d^2u}{d\theta^2}\right)=-\mu u\qquad\text{...(1)}$$

where – ive is due to central repulsive force. Now at the apse: $p=a$, $v=V$, hence we have $h=vp$, where v is the speed at any point of a central orbit, h being the areal speed, and p is the perpendicular distance from the centre of the force, of the tangent at that point. Putting $h=vp=Va$ in (1), we have

$$V^2a^2\left(u+\frac{d^2u}{d\theta^2}\right)=-\mu u$$



$$\Rightarrow u+\frac{d^2u}{d\theta^2}=-\frac{\mu u}{V^2a^2}$$

$$\Rightarrow \frac{d^2u}{d\theta^2}=-\frac{\mu u}{V^2a^2}-u$$

$$\Rightarrow \frac{d^2u}{d\theta^2}=-\frac{(\mu+V^2a^2)u}{V^2a^2}=-p^2u \quad \text{where } p^2=\frac{\mu+V^2a^2}{V^2a^2}$$

Multiplying both sides by $2\frac{du}{d\theta}$, we have

$$2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}=-2p^2u\frac{du}{d\theta}$$

$$\Rightarrow \frac{d}{d\theta}\left(\frac{du}{d\theta}\right)^2=-2p^2u\frac{du}{d\theta}$$

$$\Rightarrow d\left(\frac{du}{d\theta}\right)^2=-2p^2udu$$

$$\Rightarrow \int d\left(\frac{du}{d\theta}\right)^2=-\int 2p^2udu$$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2=-p^2u^2+C$$

At the given apse, $\frac{du}{d\theta}=0,\ u=\frac{1}{r}=\frac{1}{a}$. Putting these values in (2), we have

$$0=-\frac{p^2}{a^2}+C$$

$$\Rightarrow C=\frac{p^2}{a^2}$$

Putting this value of C in (2), we have

$$\left(\frac{du}{d\theta}\right)^2 = -p^2u^2 + \frac{p^2}{a^2}$$

$$\Rightarrow \frac{du}{d\theta} = \pm p\sqrt{\frac{1}{a^2} - u^2}$$

$$\Rightarrow \frac{du}{\sqrt{\frac{1}{a^2} - u^2}} = \pm p d\theta$$

$$\Rightarrow \int \frac{du}{\sqrt{\frac{1}{a^2} - u^2}} = \pm p \int d\theta$$

$$\Rightarrow \sin^{-1}\frac{u}{\frac{1}{a}} = \pm p\theta + B$$

$$\Rightarrow \sin^{-1} au = \pm p\theta + B \qquad \text{...(3)}$$

At the given apse, $\theta = 0$, $u = \frac{1}{r} = \frac{1}{a}$. Putting these values in (3), we have

$$\sin^{-1}\frac{a}{a} = 0 + B$$

$$\sin^{-1} 1 = B$$

$$\Rightarrow B = \frac{\pi}{2}$$

Putting this value of B in (3), we have

$$\sin^{-1} au = \pm p\theta + \frac{\pi}{2}$$

$$\Rightarrow au = \sin\left(\frac{\pi}{2} \pm p\theta\right)$$

$$\Rightarrow au = \cos p\theta$$

$$\Rightarrow \frac{a}{r} = \cos p\theta$$

$$\Rightarrow r\cos p\theta = a \qquad \text{...(4)}$$

which is the required orbit. In order to find the angle θ described in time t, we make use of the formula:

$$r^2\dot{\theta} = h$$

$$\Rightarrow r^2\frac{d\theta}{dt} = h$$

$$\Rightarrow a^2\sec^2 p\theta d\theta = hdt \qquad \text{(using (4))}$$

$$\Rightarrow a^2\int \sec^2 p\theta d\theta = h\int dt$$

$$\Rightarrow \frac{a^2}{p}\tan p\theta = ht + A \qquad \text{...(5)}$$

Initially when $t = 0$, $\theta = 0$, so from (5), we have $A = 0$. Putting this value of A in (5), we have

$$\frac{a^2}{p}\tan p\theta = ht$$

$$\frac{a^2}{p}\tan p\theta = aVt \qquad \because h = aV$$

$$\tan p\theta = \frac{pVt}{a}$$

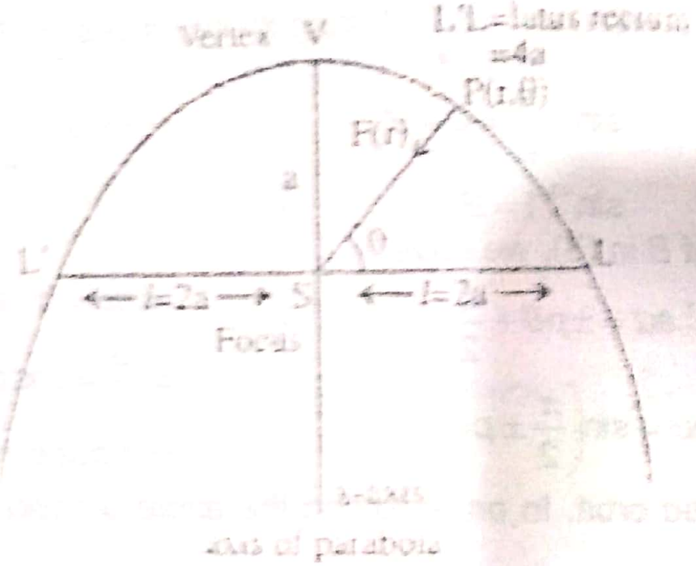
$$\Rightarrow \theta = \frac{1}{p}\tan^{-1}\frac{pVt}{a}$$

Example-14: *Two particles describe, in equal times, the arc of a parabola bounded by the latus rectum, one under an attraction to the focus and the other with constant acceleration g parallel to the axis. Show that the acceleration of the first particle at the vertex of the parabola is* $\frac{16g}{9}$.

Solution: The equation of parabola in polar form is

$$\frac{l}{r}=1+\cos\theta \qquad ...(1)$$

where $l = 2a$ (semi latus rectum). Time from L' to $L = 2$ (time from V to L).



(i) Since the force is an attractive force directed towards the focus S. We have

$$h=r^2\dot{\theta}=\left(\frac{l}{1+\cos\theta}\right)^2\frac{d\theta}{dt}$$

$$\Rightarrow hdt=\frac{l^2d\theta}{(1+\cos\theta)^2}$$

$$\Rightarrow h\int_0^t dt=l^2\int_0^{\frac{\pi}{2}}\frac{d\theta}{(1+\cos\theta)^2}=l^2\int_0^{\frac{\pi}{2}}\frac{d\theta}{\left(2\cos^2\frac{\theta}{2}\right)^2}=\frac{l^2}{2}\int_0^{\frac{\pi}{2}}\sec^4\frac{\theta}{2}d\theta$$

$$\Rightarrow h|t|_0^t=\frac{l^2}{4}\int_0^{\frac{\pi}{2}}\sec^2\frac{\theta}{2}\sec^2\frac{\theta}{2}d\theta=\frac{l^2}{4}\int_0^{\frac{\pi}{2}}\left(1+\tan^2\frac{\theta}{2}\right)\sec^2\frac{\theta}{2}d\theta$$

$$\Rightarrow h(t-0)=\frac{l^2}{4}\int_0^{\frac{\pi}{2}}\left(\sec^2\frac{\theta}{2}+\tan^2\frac{\theta}{2}\sec^2\frac{\theta}{2}\right)d\theta$$

$$\Rightarrow ht=\frac{l^2}{2}\left|\tan\frac{\theta}{2}+3\tan^3\frac{\theta}{2}\right|_0^{\frac{\pi}{2}}=\frac{l^2}{2}\left(\tan\frac{\pi}{4}+\frac{1}{3}\tan^3\frac{\pi}{4}\right)=\frac{l^2}{2}\left(1+\frac{1}{3}\right)=\frac{2l^2}{3}$$

$$\Rightarrow t=\frac{2l^2}{3h}$$

Therefore, the time from V to L is $t=\dfrac{2l^2}{3h}$ and time from L' to L is

$$2t=2\left(\frac{2l^2}{3h}\right)=\frac{4l^2}{3h}=\frac{4(2a)^2}{3h}=\frac{16a^2}{3h} \quad \because l=2a \qquad \text{...(2)}$$

(ii) To find the time along the arc $L'VL$ under the constant acceleration g vertically downwards.

The time from V to L along the arc is the same as the time from V to S under gravity vertically downwards. Since $VS = a$, so

$$a=ut+\frac{1}{2}gt^2=\frac{1}{2}gt^2 \quad \because u=0$$

$$\Rightarrow t=\sqrt{\frac{2a}{g}}$$

Therefore time from L' to L is

$$2t=2\sqrt{\frac{2a}{g}} \qquad \text{...(3)}$$

Equating (2) and (3), we have

$$2\sqrt{\frac{2a}{g}}=\frac{16a^2}{3h}$$

$$h=\frac{8a^2}{3}\sqrt{\frac{g}{2a}} \qquad \text{...(4)}$$

(iii) If f is the force per unit mass of the first particle towards the focus then differential equation of the orbit is

$$f=h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right) \qquad \text{...(5)}$$

Since $u=\dfrac{1}{r}$, so from (1), we have $lu=1+\cos\theta$

$$\Rightarrow l\frac{du}{d\theta}=-\sin\theta$$

$$\Rightarrow l\frac{d^2u}{d\theta^2}=-\cos\theta$$

$$\Rightarrow l\frac{d^2u}{d\theta^2}=1-lu$$

$$\Rightarrow \frac{d^2u}{d\theta^2}=\frac{1}{l}-u$$

Putting this in (1), we have

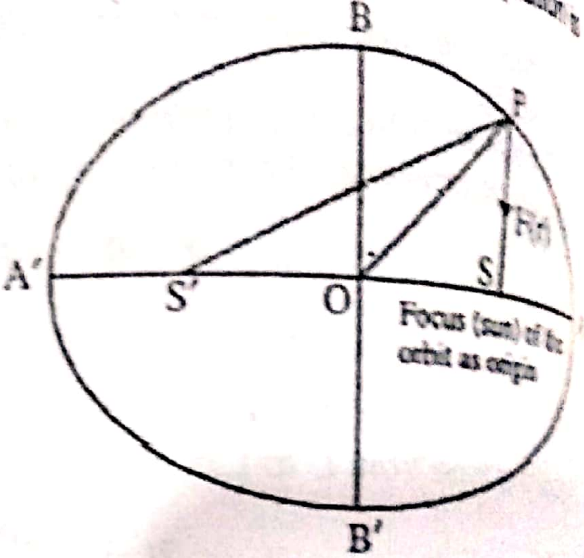
$$f=h^2u^2\left(u+\frac{1}{l}-u\right)=\frac{h^2u^2}{l}=h^2\frac{1}{r^2l}$$

$$f=\left(\frac{8a^2}{3}\sqrt{\frac{g}{2a}}\right)^2\frac{1}{a^2}\frac{1}{2a}$$

$$f=\frac{64a^4}{9}\frac{g^2}{2a}\frac{1}{a^2}\frac{1}{2a}=\frac{16g^2}{9}$$

Example-15: A planet is describing an ellipse about the sun as focus. Show that its velocity away from the sun is greatest when the radius vector to the planet is at right angles to the major axis of the path, and that it then is $\frac{2\pi ac}{T(1-e^2)}$, where 2a the major axis, e the eccentricity, and T is the periodic time.

Solution: The orbit is an ellipse with force centre at S (sun) as origin, its equation is



$$r = \frac{l}{1+e\cos\theta} = (1+e\cos\theta)^{-1}$$

$$\frac{dr}{dt} = -l(1+e\cos\theta)^{-2}(-e\sin\theta)\dot{\theta}$$

$$= \frac{le\sin\theta}{(1+e\cos\theta)^2}\dot{\theta} = \frac{r^2 le\sin\theta}{l^2}\dot{\theta}$$

$$= \frac{he\sin\theta}{l} \qquad \because r^2\dot{\theta} = h$$

$$= \frac{\sqrt{\mu l}\,e\sin\theta}{l} \qquad \because h = \sqrt{\mu l}$$

$$= \frac{e\sqrt{\mu}}{\sqrt{l}}\sin\theta$$

This shows that $\frac{dr}{dt}$ is maximum if $\sin\theta = 1$, i.e. $\theta = \frac{\pi}{2}$. Therefore

$$\frac{dr}{dt}_{max} = \frac{e\sqrt{\mu}}{\sqrt{l}}$$

Using the formula of time period, we have

$$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}}$$

$$\Rightarrow \sqrt{\mu} = \frac{2\pi a^{\frac{3}{2}}}{T}$$

For ellipse,

$$l = \frac{b^2}{a} = \frac{a^2(1-e^2)}{a} = a(1-e^2)$$

Putting these values from (2) and (3) in (1), we have

$$\frac{dr}{dt}_{max} = \frac{2\pi e a^{\frac{3}{2}}}{T\sqrt{a(1-e^2)}}$$

$$= \frac{2\pi ae}{T\sqrt{1-e^2}}$$

... velocity of the planet

EXERCISE 10

Multiple Choice Questions (MCQs)

Four options are given in each of the following questions fill the circle in front of that choice which you think is correct. Cutting or filling two or more circles is not allowed:

Q.1

(i) A force acting on a particle is said to be ________ if its line of action always passes through a fixed point.

(a) applied force (b) friction (c) central force (d) none of these

ⓐ ⓑ ⓒ ⓓ

(ii) The central force is ________ if it is directed towards its centre.

(a) attractive (b) not attractive (c) repulsive (d) none of these

ⓐ ⓑ ⓒ ⓓ

(iii) The central force is ________ if it is directed away from the centre.

(a) attractive (b) repulsive (c) not repulsive (d) none of these

ⓐ ⓑ ⓒ ⓓ

(iv) Orbit of a particle moving under a central force is necessarily ________.

(a) a space curve (b) a plane curve (c) a straight line (d) none of these

ⓐ ⓑ ⓒ ⓓ

(v) The orbit described under a central attractive force varying directly as the distance is ________ having the centre at the centre of the force.

(a) an ellipse (b) a circle (c) a straight line (d) none of these

ⓐ ⓑ ⓒ ⓓ

(vi) A central force $\vec{F}$ is said to be conservative force if $\nabla \times \vec{F} =$

(a) 3 (b) 2 (c) 1 (d) 0

ⓐ ⓑ ⓒ ⓓ

(vii) In a conservative field, the total energy of a particle ________ throughout the motion.

(a) varies rapidly (b) does not remain constant

(c) remains constant (d) none of these

ⓐ ⓑ ⓒ ⓓ

(viii) A central force is conservative if and only if it is a function of

(a) velocity and acceleration (b) distance and velocity

(c) distance alone (d) none of these

ⓐ ⓑ ⓒ ⓓ

(ix) Areal speed of the radius vector $\vec{r}$ joining the particle to the centre of force is

(a) not constant (b) constant

(c) variable (d) none of these

ⓐ ⓑ ⓒ ⓓ

(x) The component of force along the radius vector is

(a) $F_r = m\left[\dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2\right]$ (b) $F_\theta = \dfrac{m}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right)$

(c) $F_r = m\left[\dfrac{dr}{dt} - r\left(\dfrac{d\theta}{dt}\right)\right]$ (d) none of these

ⓐ ⓑ ⓒ ⓓ

(xi) The component of force perpendicular to the radius vector is

(a) $F_r = m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]$ (b) $F_\theta = \frac{m}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$

(c) $F_r = m\left[\frac{dr}{dt} - r\left(\frac{d\theta}{dt}\right)\right]$ (d) none of these

ⓐ ⓑ ⓒ ⓓ

(xii) When the force $\vec{F}$ is an attractive force directed towards origin, then $F_r = -F$ and $F_\theta =$

(a) 3 (b) 2 (c) 1 (d) 0

ⓐ ⓑ ⓒ ⓓ

(xiii) The differential equation of the orbit in polar coordinates is $\frac{d^2u}{d\theta^2} + u =$

(a) $\frac{f}{hu}$ (b) $\frac{f}{h^2u^2}$ (c) $\frac{f}{h^3u^3}$ (d) none of these

ⓐ ⓑ ⓒ ⓓ

(xiv) The length of the radius vector at an apse is known as the apsidal

(a) acceleration (b) velocity (c) distance (d) none of these

ⓐ ⓑ ⓒ ⓓ

(xv) The line joining an apse to the centre of force is ______________ line.

(a) a focal line (b) an apse (c) an initial (d) none of these

ⓐ ⓑ ⓒ ⓓ

(xvi) The angle between two consecutive apse lines is an

(a) angle of friction (b) angle of projection

(c) apsidal angle (d) angle of depression

ⓐ ⓑ ⓒ ⓓ

(xvii) The analytic condition for a point with radius vector r to be an apse is $\frac{dr}{d\theta} =$

(a) 1 (b) 0 (c) -1 (d) -2

ⓐ ⓑ ⓒ ⓓ

(xviii) If ϕ is the angle between the radius vector and the tangent at some point, then $\tan\phi =$

(a) $r\frac{d\theta}{dr}$ (b) $\frac{1}{r}\frac{dr}{d\theta}$ (c) $\frac{dr}{d\theta}$ (d) none of these

ⓐ ⓑ ⓒ ⓓ

(xix) If ϕ is the angle between the radius vector and the tangent at some point, then $\cot\phi =$

(a) $r\frac{d\theta}{dr}$ (b) $\frac{1}{r}\frac{dr}{d\theta}$ (c) $\frac{dr}{d\theta}$ (d) none of these

ⓐ ⓑ ⓒ ⓓ

(xx) Kepler's ______________ Law states that each planet describes an ellipse with the sun at one focus.

(a) Third (b) Second (c) First (d) none of these

ⓐ ⓑ ⓒ ⓓ

(xxi) Kepler's ________ Law states that the radius vector drawn from the sun to the planet sweeps out equal areas in equal times.
(a) Third (b) Second (c) First (d) none of these
ⓐ ⓑ ⓒ ⓓ

(xxii) Kepler's ________ Law states that the squares of the periods of the planets are proportional to the cubes of the semi-major axes of their orbits.
(a) Third (b) Second (c) First (d) none of these
ⓐ ⓑ ⓒ ⓓ

Short Questions

Q.2 Solve / write answers of the following short questions:

(i) State Kepler's First Law. **PU, 2013; 2011 (BS Math)**

(ii) State Kepler's Second Law. **PU, 2013; 2011 (BS Math)**

(iii) State Kepler's Third Law. **PU, 2013; 2011 (BS Math)**

(iv) What is an apsidal angle?

(v) Define a central force.

(vi) When the central force is attractive; when it is repulsive?

(vii) What is an apse?

SUMMARY

- If the force $\vec{F}$ on a particle always passes through a fixed point O, the force is called a central force and O is called the centre of force.
- A central force is repulsive or attractive according as it is directed away from or towards the centre of force.
- The orbit of a particle moving under a central force is necessarily a plane curve.
- The orbit described under a central attractive force varying directly as the distance, is an ellipse having the centre at the centre of the force.
- When a particle moves under a central force, the areal velocity is constant.
- Differential equation of the orbit is $\frac{d^2u}{d\theta^2}+u=\frac{f}{h^2u^2}$.
- Differential equation of the orbit in pedal form is $\frac{h^2}{p^3}\frac{dp}{dr}=f$.
- An apse is a point on an orbit at which the radius vector drawn from the centre of force is a maximum or minimum.

- The length of the radius vector at such a point is known as the ap[illegible] distance.
- The line joining an apse to the centre of force is an apse line.
- The angle between two consecutive apse lines is an apsidal angle.
- The radius vector through an apse and the tangent at it to the orb[illegible] perpendicular.
- The planet will describe an ellipse, a parabola or a hyperbola acc[illegible] as total energy per unit mass, E, is negative, zero or positive.
- Kepler's First Law: Each planet describes an ellipse with the sun a[illegible] focus.
- Kepler's Second Law: The radius vector drawn from the sun to the [illegible] sweeps out equal areas in equal times.
- Kepler's Third Law: The squares of the periods of the planets [illegible] proportional to the cubes of the semi-major axes of their orbits.