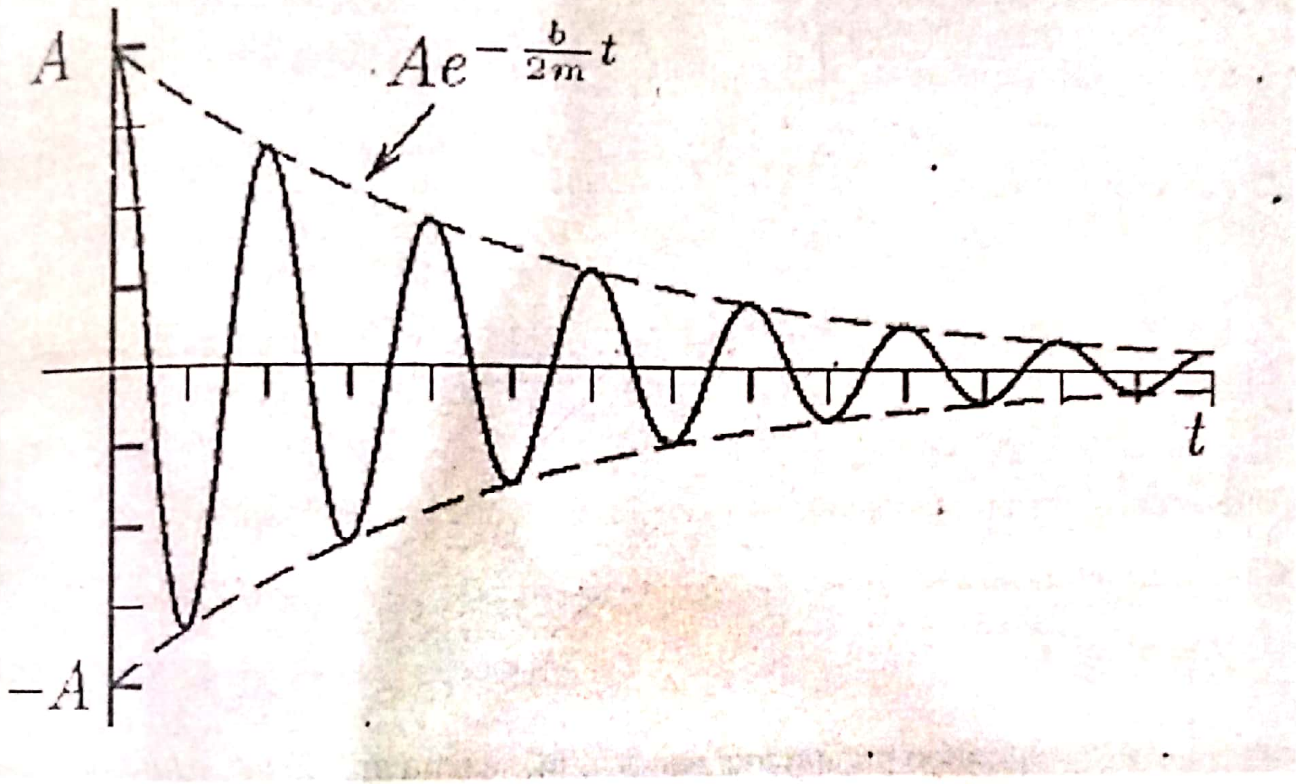
DAMPED AND FORCED OSCILLATOR

RESISTED MOTION

Chapter

11



11-1 Damped Harmonic Oscillator

The simple harmonic oscillator is one of the central problems in physics. It is useful in understanding springs, small amplitude pendulums, electronic circuits, and quantum mechanics. Furthermore, many problems can be considered the sum of a large number, or infinite number, of harmonic oscillators.

Almost everyone has an intuitive understanding of the playground swing, and so it is a good first example. If the person in the swing is neither "pumping" nor being pushed, and if frictional losses are small, one has a simple harmonic oscillator, at least for small amplitudes. If the rider drags his or her feet then there is *damping*.

We consider the case when the damping force is proportional to (the first power of) the velocity.

Suppose a particle of mass m, moving in a straight line is subject to

(i) a restorative force $-m\omega^2 x$, where x is the distance of the particle at time t from a fixed paint O in the line and

(ii) a damping force $-mk\dot{x}$, where k is a positive constant.

Clearly, the equation of motion of the particle is

$$m\ddot{x} = -mk\dot{x} - m\omega^2 x$$

$$\Rightarrow \ddot{x} = -k\dot{x} - \omega^2 x$$

$$\Rightarrow \ddot{x} + k\dot{x} + \omega^2 x = 0 \qquad \ldots(1)$$

or

$$(D^2 + kD + \omega^2)x = 0 \qquad \ldots(2)$$

where $D \equiv \dfrac{d}{dt}$, $D^2 \equiv \dfrac{d^2}{dt^2}$.

The auxiliary equation of the differential equation (2) is

$$p^2 + kp + \omega^2 = 0 \qquad \ldots(3)$$

$$p = \frac{-k \pm \sqrt{k^2 - 4\omega^2}}{2}$$

$$\Rightarrow p = \frac{-k}{2} \pm \sqrt{\frac{k^2}{4} - \omega^2} \qquad \ldots(4)$$

Three cases arise for consideration according as the discriminant is $\lesseqgtr$.

Case-I: Small Damped Case:

When
$$\frac{k^2}{4} - \omega^2 < 0$$

Let $\dfrac{k^2}{4} - \omega^2 = -n^2$, then putting this value in (4); we have

$$p = \frac{-k}{2} \pm \sqrt{-n^2} = \frac{-k}{2} \pm ni$$

Therefore, the general solution of the differential equation (2) is

$$x = ae^{-\frac{kt}{2}} \sin(nt + \alpha) \qquad \ldots(5)$$

where a and α are constants.

In this case the motion is oscillatory and periodic, the period of an oscillation being $\dfrac{2\pi}{n}$.

But the amplitude $ae^{-\frac{kt}{2}}$ is no longer constant. Since k is supposed to be small, the amplitude slowly decreases with time. As $t \to \infty$, the amplitude approaches zero. Therefore, ultimately x becomes and permanently remains zero. We, therefore, say that the motion *decays* with time.

The constant k is called *coefficient of damping*. This case is of *small damping*.

Case-II: Large Damped Case:

When
$$\frac{k^2}{4} - \omega^2 > 0$$

Let $\frac{k^2}{4} - \omega^2 = \lambda^2$, then putting this value in (4), we have

$$p = \frac{-k}{2} \pm \sqrt{\lambda^2} = \frac{-k}{2} \pm \lambda$$

Therefore, the general solution of the differential equation (2) is

$$x = Ae^{\left(-\frac{k}{2}+\lambda\right)t} + Be^{-\left(\frac{k}{2}+\lambda\right)t} \qquad ...(6)$$

where A and B are constants.

The motion is *aperiodic* (non-periodic) and non-oscillatory. It is called a *dead beat.* Both the exponentials decay with time. (In the second exponential the exponent is negative. Therefore, it tends to zero as $t \to \infty$. Also $\lambda - \frac{k}{2} = \sqrt{\frac{k^2}{4} - \omega^2} - \frac{k}{2} =$ a negative number. Hence the first exponential also $\to 0$ as $t \to \infty$).

This is the case of *large damping* as k is supposed to be large.

Case-III: Critically Damped Case:

When
$$\frac{k^2}{4} - \omega^2 = 0$$

The auxiliary equation has equal roots

$$p = -\frac{k}{2}, -\frac{k}{2} = -\omega, -\omega$$

Therefore, the general solution of the differential equation (2) is

$$x = (Ct + D)e^{-\omega t} \qquad ...(7)$$

where C and D are constants.

In this case also the motion is aperiodic and non-oscillatory.
Also $x \to 0$ as $t \to \infty$, i.e. the motion decays with time.

In this case the system is said to be *critically damped.*

11-2 Damped Forced Oscillator

In order to overcome the damping effect of a medium an applied force, called *driving force*, is generally applied. In such a case the system is called *damped forced oscillator.*

The applied force is usually a periodic force. If in addition to the restorative force and damping force described above, the oscillator is subject to an applied force of magnitude $mF\cos pt$ (F and p being constants), acting in the direction of increase of x, then its equation of motion is

$$m\ddot{x} = -mk\dot{x} - m\omega^2 x + mF\cos pt$$
$$\Rightarrow \ddot{x} = -k\dot{x} - \omega^2 x + F\cos pt$$

$$\Rightarrow \ddot{x}+k\dot{x}+\omega^2 x = F\cos pt \qquad \text{...(1)}$$

The solution of this differential equation consists of two parts, the complementary function and the particular integral.

The first part which is the solution of the equation

$$(D^2+kD+\omega^2)x=0 \qquad \text{...(2)}$$

has already been obtained above in the first article. We have noticed that in each of the three cases that arise, the complementary function *decays* with time. For this reason this part of the solution is called the *transient solution.*

The particular integral, on the other hand, generally does not die out. Therefore, the ultimate motion of the system depends upon that part of the solution which is called the *steady state solution.*

$$P.I. = \frac{1}{D^2+kD+\omega^2}F\cos pt = F\,\text{Re}\frac{1}{D^2+kD+\omega^2}e^{ipt} = F\,\text{Re}\frac{e^{ipt}}{(ip)^2+k(ip)+\omega^2}$$

$$= F\,\text{Re}\frac{e^{ipt}}{-p^2+ikp+\omega^2} = F\,\text{Re}\frac{e^{ipt}}{\omega^2-p^2+ikp}$$

$$= F\,\text{Re}\frac{e^{ipt}(\omega^2-p^2-ikp)}{(\omega^2-p^2+ikp)(\omega^2-p^2-ikp)} = F\,\text{Re}\frac{(\cos pt+i\sin pt)(\omega^2-p^2-ikp)}{(\omega^2-p^2)^2-(ikp)^2}$$

$$= F\frac{(\omega^2-p^2)\cos pt+ikp\sin pt}{(\omega^2-p^2)^2+k^2p^2}$$

$$= \frac{F}{R}\sin(pt+\alpha)$$

where $R=\sqrt{(\omega^2-p^2)^2+k^2p^2}$, $\sin\alpha=\dfrac{\omega^2-p^2}{R}$, $\cos\alpha=\dfrac{kp}{R}$.

Thus the steady state solution is

$$x=\frac{F}{R}\sin(pt+\alpha) \qquad \text{...(2)}$$

where $R=\sqrt{(\omega^2-p^2)^2+k^2p^2}$, $\sin\alpha=\dfrac{\omega^2-p^2}{R}$, $\cos\alpha=\dfrac{kp}{R}$.

The steady state solution which is a periodic function (of time) will not decay with time (as anticipated). The amplitude of the steady state is $\dfrac{F}{R}$ and x will always lie between $-\dfrac{F}{R}$ and $\dfrac{F}{R}$.

Resonance: Resonance is the tendency of a system to oscillate with greater amplitude at some frequencies than at others.

$$R^2=(\omega^2-p^2)^2+k^2p^2=p^4-2p^2\omega^2+\omega^4+k^2p^2$$

$$=p^4-2p^2\omega^2+k^2p^2+\omega^4=p^4-2p^2\left(\omega^2-\frac{k^2}{2}\right)+\omega^4$$

$$R^2 = p^4 - 2p^2\left(\omega^2 - \frac{k^2}{2}\right) + \left(\omega^2 - \frac{k^2}{2}\right)^2 + \omega^4 - \left(\omega^2 - \frac{k^2}{2}\right)^2$$

$$R^2 = \left(p^2 - \left(\omega^2 - \frac{k^2}{2}\right)\right)^2 + \omega^4 - \left(\omega^2 - \frac{k^2}{2}\right)^2$$

If $\omega > \frac{k}{\sqrt{2}}$, R^2 will be minimum, when $p^2 - \left(\omega^2 - \frac{k^2}{2}\right) = 0$

$$\Rightarrow p = \sqrt{\omega^2 - \frac{k^2}{2}}$$

When this condition is satisfied, the amplitude is maximum and the corresponding frequency of the steady state, $\frac{p}{2\pi}$ is called the *resonance frequency.*

Example-1: *A particle of mass m oscillates in a line with natural period $\frac{2\pi}{\omega}$. If an applied force $F\cos pt$ now acts in the line so that the particle is instantaneously at rest at zero time at a distance d from the centre of oscillation, prove that the displacement of the particle from the centre at subsequent time t is* $d\cos \omega t + \frac{F(\cos pt - \cos \omega t)}{(\omega^2 - p^2)m}$.

Solution: Natural period means 'period of a pure SHM.' The equation of motion of the particle is

$$m\ddot{x} = -m\omega^2 x + mF\cos pt$$

(Note that there is no damping force)

$$\Rightarrow m\ddot{x} + m\omega^2 x = F\cos pt$$

$$\Rightarrow \ddot{x} + \omega^2 x = \frac{F}{m}\cos pt$$

$$\Rightarrow (D^2 + \omega^2)x = \frac{F}{m}\cos pt \qquad ...(1)$$

$$C.F. = a\sin(\omega t + \alpha), \text{ a, } \alpha \text{ are constants}$$

and

$$P.I. = \frac{F}{m}\frac{1}{D^2 + \omega^2}\cos pt = \frac{F}{m}\frac{\cos pt}{-p^2 + \omega^2} = \frac{F}{m}\frac{\cos pt}{\omega^2 - p^2}$$

The general solution of (1) is

$$x = a\sin(\omega t + \alpha) + \frac{F}{m}\frac{\cos pt}{\omega^2 - p^2} \qquad ...(2)$$

Differentiating w.r.t. t, we get

$$\dot{x} = \omega a\cos(\omega t + \alpha) - \frac{Fp}{m(\omega^2 - p^2)}\sin pt \qquad ...(3)$$

Initially, when $t = 0$, $x = d$ and $x = 0$, so equations (2) and (3) give

$$d = a\sin\alpha + \frac{F}{m(\omega^2 - p^2)} \qquad \text{...(4)}$$

$$0 = \omega a\cos\alpha$$

And

$$\Rightarrow \alpha = \frac{\pi}{2}$$

Putting this value in (4), we have

$$d = a\sin\frac{\pi}{2} + \frac{F}{m(\omega^2 - p^2)}$$

$$a = d - \frac{F}{m(\omega^2 - p^2)}$$

Putting these values in (2), we have

$$x = \left[d - \frac{F}{m(\omega^2 - p^2)}\right]\sin\left(\omega t + \frac{\pi}{2}\right) + \frac{F}{m}\frac{\cos pt}{\omega^2 - p^2}$$

$$x = \left(d - \frac{F}{m(\omega^2 - p^2)}\right)\cos\omega t + \frac{F}{m(\omega^2 - p^2)}\cos pt$$

which is the required result.

Example-2: *A particle of mass m is moving under the action of the forces* $F_1 = -m\omega^2 x$, $F_2 = mF_0 t$, $F_3 = -2m\mu\dot{x}$. *Assuming that damping is small, set up and solve the equation of motion.* **PU, 2010 (B.A./B.Sc.)**

Solution: The equation of motion is

$$m\ddot{x} = -2m\mu\dot{x} - m\omega^2 x + mF_0 t$$

$$\ddot{x} = -2\mu\dot{x} - \omega^2 x + F_0 t$$

$$\ddot{x} + 2\mu\dot{x} + \omega^2 x = F_0 t$$

$$(D^2 + 2\mu D + \omega^2)x = F_0 t \qquad \text{...(1)}$$

The auxiliary equation for the complementary function is

$$p^2 + 2\mu p + \omega^2 = 0$$

$$\Rightarrow p = \frac{-2\mu \pm \sqrt{4\mu^2 - 4\omega^2}}{2} = -\mu \pm \sqrt{\mu^2 - \omega^2} \qquad \text{...(2)}$$

For small damping, $\mu^2 - \omega^2 < 0$, so let $\mu^2 - \omega^2 = -n^2$, putting this in (2)

$$p = -\mu \pm \sqrt{-n^2} = -\mu \pm ni$$

Therefore,

$$C.F. = ae^{-\mu t}\sin(nt + \alpha) \qquad \text{...(3)}$$

$$P.I. = \frac{1}{D^2 + 2\mu D + \omega^2}F_0 t = \frac{F_0}{\omega^2}\frac{1}{\frac{D^2 + 2\mu D}{\omega^2} + 1}t = \frac{F_0}{\omega^2}\left(1 + \frac{D^2 + 2\mu D}{\omega^2}\right)^{-1}t$$

$$P.I. = \frac{F_0}{\omega^2}\left(1 - \frac{2\mu D}{\omega^2}\right)t \qquad \text{(expanding up to D)}$$

$$P.I. = \frac{F_0}{\omega^2}\left(t - \frac{2\mu}{\omega^2}Dt\right) = \frac{F_0}{\omega^2}\left(t - \frac{2\mu}{\omega^2}\right)$$

The complete solution is

$$x = ae^{-\mu t}\sin(nt+\alpha) + \frac{F_0}{\omega^2}\left(t - \frac{2\mu}{\omega^2}\right)$$

Example-3: *An oscillator moves under the forces:*

restorative force $= -kx$

damping force $= -2\mu\dot{x}$

driving force $= F_0 e^{-at}$

each force being per unit mass. Set up and solve the equation of motion completely.

Solution: The equation of motion is

$$\ddot{x} = -2\mu\dot{x} - kx + F_0 e^{-at}$$

$$\ddot{x} + 2\mu\dot{x} + kx = F_0 e^{-at}$$

$$(D^2 + 2\mu D + k)x = F_0 e^{-at} \qquad \text{...(1)}$$

The auxiliary equation for the complementary function is

$$p^2 + 2\mu p + k = 0$$

$$\Rightarrow p = \frac{-2\mu \pm \sqrt{4\mu^2 - 4k}}{2} = -\mu \pm \sqrt{\mu^2 - k} \qquad \text{...(2)}$$

Transient Solution: Case-I: Small Damped Case:

When $\mu^2 - k < 0$, let $\mu^2 - k = -n^2$, then putting this value in (2), we have

$$p = -\mu \pm \sqrt{-n^2} = -\mu \pm ni$$

Therefore, the solution of the differential equation (1) is ...(3)

$$x = ae^{-\mu t}\sin(nt+\alpha)$$

where a and α are constants.

Case-II: Large Damped Case:

When $\mu^2 - k > 0$, let $\mu^2 - k = \lambda^2$, then putting this value in (2), we have

$$p = -\mu \pm \sqrt{\lambda^2} = -\mu \pm \lambda$$

Therefore, the solution of the differential equation (1) is ...(4)

$$x = Ae^{(-\mu+\lambda)i} + Be^{-(-\mu+\lambda)l}$$

where A and B are constants.

Case-III: Critically Damped Case:

When $\mu^2 - k = 0$, the auxiliary equation has equal roots $p = -\mu, -\mu$ ...(5)

Therefore, the solution of the differential equation (1) is

$$x = (Ct + D)e^{-\mu t}, \quad \mu^2 = k$$

where C and D are constants.

Steady State Solution: Steady state solution is given by particular integral, i.e.

$$P.I.=\frac{1}{D^2+2\mu D+k}F_0e^{-at}=\frac{F_0e^{-at}}{(-a)^2+2\mu(-a)+k}=\frac{F_0e^{-at}}{a^2-2\mu a+k}\quad\text{(in each case)}$$

Complete Solution:

Complete solution = Transient solution + Steady state solution

Example-4: *An oscillator moves under the forces:*

restorative force $=-kx$

damping force $=-2\mu\dot{x}$

driving force $=F_0e^{-at}\cos\omega t$

each force being per unit mass. Find the steady state solution.

Solution: The equation of motion is

$$\ddot{x}=-2\mu\dot{x}-kx+F_0e^{-at}\cos\omega t$$

$$\ddot{x}+2\mu\dot{x}+kx=F_0e^{-at}\cos\omega t$$

$$(D^2+2\mu D+k)x=F_0e^{-at}\cos\omega t \qquad \text{...(1)}$$

Steady state solution is

$$x=\frac{1}{D^2+2\mu D+k}F_0e^{-at}\cos\omega t=F_0\,\text{Re}\frac{1}{D^2+2\mu D+k}e^{-at}e^{i\omega t}$$

$$=F_0\,\text{Re}\frac{1}{D^2+2\mu D+k}e^{-(a-i\omega)t}=F_0\,\text{Re}\frac{e^{-(a-i\omega)t}}{(a-i\omega)^2-2\mu(a-i\omega)+k}$$

$$=F_0e^{-at}\,\text{Re}\frac{e^{i\omega t}}{a^2-\omega^2-2ia\omega-2\mu a+2i\mu\omega+k}$$

$$=F_0e^{-at}\,\text{Re}\frac{\cos\omega t+i\sin\omega t}{a^2-\omega^2-2\mu a+k+2i\mu\omega-2ia\omega}$$

$$=F_0e^{-at}\,\text{Re}\frac{\cos\omega t+i\sin\omega t}{(a^2-\omega^2-2\mu a+k)+2i(\mu-a)\omega}$$

$$=F_0e^{-at}\,\text{Re}\frac{\cos\omega t+i\sin\omega t}{(a^2-\omega^2-2\mu a+k)+2i(\mu-a)\omega}\times\frac{(a^2-\omega^2-2\mu a+k)-2i(\mu-a)\omega}{(a^2-\omega^2-2\mu a+k)-2i(\mu-a)\omega}$$

$$=F_0e^{-at}\,\text{Re}\frac{(\cos\omega t+i\sin\omega t)[(a^2-\omega^2-2\mu a+k)-2i(\mu-a)\omega]}{(a^2-\omega^2-2\mu a+k)^2-4i^2(\mu-a)^2\omega^2}$$

$$=F_0e^{-at}\frac{(a^2-\omega^2-2\mu a+k)\cos\omega t+2(\mu-a)\omega\sin\omega t}{(a^2-\omega^2-2\mu a+k)^2+4(\mu-a)^2\omega^2}$$

Example-5 **An oscillator moves under the forces:**

restorative force $=-kx$

damping force $=-2\mu\dot{x}$

$$\text{driving force} = F_0(e^{-at} + e^{-at}\cos\omega t)$$

force being per unit mass. Find the steady state solution.

The equation of motion is

$$\ddot{x} = -2\mu\dot{x} - kx + F_0 e^{-at}\cos\omega t$$

$$\ddot{x} + 2\mu\dot{x} + kx = F_0 e^{-at}\cos\omega t$$

$$(D^2 + 2\mu D + k)x = F_0(e^{-at} + e^{-at}\cos\omega t) \quad \text{...(1)}$$

steady state solution is

$$x = \frac{1}{D^2 + 2\mu D + k} F_0(e^{-at} + e^{-at}\cos\omega t)$$

$$= \frac{1}{D^2 + 2\mu D + k} F_0 e^{-at} + F_0 \operatorname{Re} \frac{1}{D^2 + 2\mu D + k} e^{-at} e^{i\omega t}$$

$$= \frac{F_0 e^{-at}}{(-a)^2 + 2\mu(-a) + k} + F_0 \operatorname{Re} \frac{1}{D^2 + 2\mu D + k} e^{-(a-i\omega)t}$$

$$= \frac{F_0 e^{-at}}{a^2 - 2\mu a + k} + F_0 \operatorname{Re} \frac{e^{-(a-i\omega)t}}{(a-i\omega)^2 - 2\mu(a-i\omega) + k}$$

$$= \frac{F_0 e^{-at}}{a^2 - 2\mu a + k} + F_0 e^{-at} \operatorname{Re} \frac{e^{i\omega t}}{a^2 - \omega^2 - 2ia\omega - 2\mu a + 2i\mu\omega + k}$$

$$= \frac{F_0 e^{-at}}{a^2 - 2\mu a + k} + F_0 e^{-at} \operatorname{Re} \frac{\cos\omega t + i\sin\omega t}{a^2 - \omega^2 - 2\mu a + k + 2i\mu\omega - 2ia\omega}$$

$$= \frac{F_0 e^{-at}}{a^2 - 2\mu a + k} + F_0 e^{-at} \operatorname{Re} \frac{\cos\omega t + i\sin\omega t}{(a^2 - \omega^2 - 2\mu a + k) + 2i(\mu - a)\omega}$$

$$= \frac{F_0 e^{-at}}{a^2 - 2\mu a + k} +$$

$$= F_0 e^{-at} \operatorname{Re} \frac{\cos\omega t + i\sin\omega t}{(a^2 - \omega^2 - 2\mu a + k) + 2i(\mu - a)\omega} \times \frac{(a^2 - \omega^2 - 2\mu a + k) - 2i(\mu - a)\omega}{(a^2 - \omega^2 - 2\mu a + k) - 2i(\mu - a)\omega}$$

$$= \frac{F_0 e^{-at}}{a^2 - 2\mu a + k} + F_0 e^{-at} \operatorname{Re} \frac{(\cos\omega t + i\sin\omega t)[(a^2 - \omega^2 - 2\mu a + k) - 2i(\mu - a)\omega]}{(a^2 - \omega^2 - 2\mu a + k)^2 - 4i^2(\mu - a)^2\omega^2}$$

$$= \frac{F_0 e^{-at}}{a^2 - 2\mu a + k} + F_0 e^{-at} \frac{(a^2 - \omega^2 - 2\mu a + k)\cos\omega t + 2(\mu - a)\omega\sin\omega t}{(a^2 - \omega^2 - 2\mu a + k)^2 + 4(\mu - a)^2\omega^2}$$

mple-6: ***In the absence of external forces, a particle of mass m would execute linear SHM. When a disturbing force $F\cos pt$ acts, the maximum speed attained V. Show that the angular frequency of the free oscillations is $\sqrt{\frac{(F + mpV)p}{mV}}$.***

In the absence of external forces, the equation of motion is

$$m\ddot{x} = -kx$$

$$\Rightarrow \ddot{x} = -\frac{k}{m}x$$

We have SHM. Angular frequency of free oscillation is $\sqrt{\frac{k}{m}}$.

When the disturbing force $F\cos pt$ acts, the equation of motion is

$$\ddot{x} = -\frac{k}{m}x + \frac{F}{m}\cos pt$$

$$\ddot{x} + \frac{k}{m}x = \frac{F}{m}\cos pt$$

$$\left(D^2 + \frac{k}{m}\right)x = \frac{F}{m}\cos pt$$

Steady solution is

$$x = \frac{1}{D^2 + \frac{k}{m}}\frac{F}{m}\cos pt = \frac{F}{m}\frac{\cos pt}{-p^2 + \frac{k}{m}} = \frac{F\cos pt}{k - mp^2}$$

$$\Rightarrow \dot{x} = \frac{-pF\sin pt}{k - mp^2}$$

$$\Rightarrow V = \frac{pF}{k - mp^2} \qquad (\because \text{maximum value of } \sin pt \text{ is } 1)$$

$$\Rightarrow V(k - mp^2) = pF$$

$$\Rightarrow kV - mp^2V = pF$$

$$\Rightarrow kV = pF + mp^2V$$

$$\Rightarrow k = \frac{(F + mpV)p}{V}$$

Angular frequency of free oscillation is

$$\sqrt{\frac{k}{m}} = \sqrt{\frac{(F + mpV)p}{mV}}$$

Example-7: *An undamped oscillator is subject to the restorative force* $-m\omega_0^2 x$ *an applied force* $mF_0 \sin\omega t$. *Find the motion assuming that initially when* $t = 0, x =$ *and* $\dot{x} = 0$.

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Solution: The equation of motion is

$$m\ddot{x} = -m\omega_0^2 x + mF_0 \sin\omega t$$

$$\ddot{x} + \omega_0^2 x = F_0 \sin\omega t$$

$$(D^2 + \omega_0^2)x = F_0 \sin\omega t$$

Auxiliary equation is

$$D^2+\omega_0^2=0$$

$$\Rightarrow D=\pm i\omega_0$$

Therefore, the complementary function is

$$x_c=A\cos\omega_0 t+B\sin\omega_0 t \qquad ...(2)$$

Particular integral is given by

$$x_p=\frac{1}{D^2+\omega_0^2}F_0\sin\omega t=\frac{F_0\sin\omega t}{-\omega^2+\omega_0^2}=\frac{F_0\sin\omega t}{\omega_0^2-\omega^2} \qquad ...(3)$$

Complete solution is $x=x_c+x_p$

$$\Rightarrow x=A\cos\omega_0 t+B\sin\omega_0 t+\frac{F_0\sin\omega t}{\omega_0^2-\omega^2} \qquad ...(4)$$

Putting $t=0, x=0$ in (4), we have $A=0$. Putting this value of A in (4), we have

$$x=B\sin\omega_0 t+\frac{F_0\sin\omega t}{\omega_0^2-\omega^2} \qquad ...(5)$$

$$\Rightarrow \dot{x}=\omega_0 B\cos\omega_0 t+\frac{\omega F_0\cos\omega t}{\omega_0^2-\omega^2} \qquad ...(6)$$

Putting $t=0, \dot{x}=0$ in (6), we have $0=\omega_0 B+\dfrac{\omega F_0}{\omega_0^2-\omega^2}$

$$B=-\frac{\omega F_0}{\omega_0(\omega_0^2-\omega^2)}$$

Putting this value of B in (5), we have

$$x=-\frac{\omega F_0}{\omega_0(\omega_0^2-\omega^2)}\sin\omega_0 t+\frac{F_0\sin\omega t}{\omega_0^2-\omega^2}$$

$$x=\frac{F_0}{\omega_0(\omega_0^2-\omega^2)}(\omega_0\sin\omega t-\omega\sin\omega_0 t)$$

11-3 Vertical Motion with Air Resistance

Downward Motion: A particle of mass m is released from rest at an initial height above the ground. The resistance of air is assumed to be proportional to the square of the velocity. Let us discuss its motion.

Let the distance fallen through by m in time t be x and air retardation be $-m\lambda v^2$, then the equation of motion is

$$m\ddot{x}=mg-m\lambda v^2 \qquad ...(1)$$

$$\ddot{x}=g-\lambda v^2 \qquad ...(2)$$

$$\ddot{x}=\lambda\left(\frac{g}{\lambda}-v^2\right)$$

For the sake of symmetry we set $\frac{g}{\lambda} = k^2$ so that $\lambda = \frac{g}{k^2}$. Therefore, (2) becomes

$$v\frac{dv}{dx} = \frac{g}{k^2}(k^2 - v^2) \qquad \because \ddot{x} = v\frac{dv}{dx}$$

$$\Rightarrow \frac{vdv}{k^2 - v^2} = \frac{g}{k^2}dx$$

$$\Rightarrow \frac{-2vdv}{k^2 - v^2} = -\frac{2g}{k^2}dx$$

$$\Rightarrow \int_0^v \frac{-2vdv}{k^2 - v^2} = \int_0^x -\frac{2g}{k^2}dx$$

$$\Rightarrow \left[\ln(k^2 - v^2)\right]_0^v = -\frac{2g}{k^2}[x]_0^x$$

$$\Rightarrow \ln(k^2 - v^2) - \ln(k^2 - 0) = -\frac{2g}{k^2}(x - 0)$$

$$\Rightarrow \ln(k^2 - v^2) - \ln k^2 = -\frac{2gx}{k^2}$$

$$\Rightarrow \ln\frac{k^2 - v^2}{k^2} = -\frac{2gx}{k^2}$$

$$\Rightarrow \frac{k^2 - v^2}{k^2} = e^{-\frac{2gx}{k^2}}$$

$$\Rightarrow k^2 - v^2 = k^2 e^{-\frac{2gx}{k^2}}$$

$$\Rightarrow v^2 = k^2\left(1 - e^{-\frac{2gx}{k^2}}\right) \qquad \ldots(3)$$

which gives v in terms of x. Again, equation of motion may be written

$$\frac{dv}{dt} = \frac{g}{k^2}(k^2 - v^2) \qquad \because \ddot{x} = \frac{dv}{dt}$$

$$\Rightarrow \frac{dv}{k^2 - v^2} = \frac{g}{k^2}dt$$

$$\Rightarrow \int_0^v \frac{dv}{k^2 - v^2} = \frac{g}{k^2}\int_0^t dt$$

$$\Rightarrow \frac{1}{2k}\left[\ln\frac{k + v}{k - v}\right]_0^v = \frac{g}{k^2}[t]_0^t$$

$$\Rightarrow \frac{1}{2k}\left(\ln\frac{k + v}{k - v} - \ln\frac{k + 0}{k - 0}\right) = \frac{g}{k^2}(t - 0)$$

$$\Rightarrow \frac{1}{2k}\ln\frac{k + v}{k - v} = \frac{gt}{k^2}$$

$$\Rightarrow \ln\frac{k+v}{k-v}=\frac{2gt}{k}$$

$$\Rightarrow \frac{k+v}{k-v}=e^{\frac{2gt}{k}}$$

$$\Rightarrow \frac{(k+v)-(k-v)}{(k+v)+(k-v)}=\frac{e^{\frac{2gt}{k}}-1}{e^{\frac{2gt}{k}}+1} \quad \text{(By dividendo - componendo)}$$

$$\Rightarrow \frac{k+v-k+v}{k+v+k-v}=\frac{e^{\frac{gt}{k}}-e^{-\frac{gt}{k}}}{e^{\frac{gt}{k}}+e^{-\frac{gt}{k}}}$$

$$\Rightarrow \frac{2v}{2k}=\tanh\frac{gt}{k}$$

$$\Rightarrow v=k\tanh\frac{gt}{k} \qquad ...(4)$$

which gives v in terms of t.

To obtain x in terms of t we eliminate v between equations (3) and (4), so putting the value of v from (4) in (3), we have

$$k^2\tanh^2\frac{gt}{k}=k^2(1-e^{-\frac{2gx}{k^2}})$$

$$\tanh^2\frac{gt}{k}=1-e^{-\frac{2gx}{k^2}}$$

$$e^{-\frac{2gx}{k^2}}=1-\tanh^2\frac{gt}{k}$$

$$e^{-\frac{2gx}{k^2}}=\operatorname{csch}^2\frac{gt}{k}$$

$$e^{\frac{2gx}{k^2}}=\cosh^2\frac{gt}{k}$$

$$e^{\frac{gx}{k^2}}=\cosh\frac{gt}{k}$$

$$\frac{gx}{k^2}=\ln\cosh\frac{gt}{k}$$

$$x=\frac{k^2}{g}\ln\cosh\frac{gt}{k}$$

this gives x in terms of t.

Terminal Velocity or Limiting Velocity: *Terminal velocity* or *limiting velocity* of a particle moving in a resisting medium is the maximum velocity it can attain. v will be maximum when:

$$\frac{dv}{dt}=0$$

Then the above equation of motion (in either form) gives $v=k=\sqrt{\frac{g}{\lambda}}$ which gives the limiting velocity in the case discussed above.

Upward Motion: Suppose a particle is projected vertically upward with an initial speed v_0. Also suppose that the air resistance is, as before, proportional to the square of the velocity.

If x is the height attained by the particle in tim

$$mv\frac{dv}{dx} = -mg - m\lambda v^2$$

$$v\frac{dv}{dx} = -g - \lambda v^2$$

$$v\frac{dv}{dx} = -\lambda\left(\frac{g}{\lambda} + v^2\right)$$

$$v\frac{dv}{dx} = -\frac{g}{k^2}(k^2 + v^2)$$

$$\frac{vdv}{k^2 + v^2} = -\frac{g}{k^2}dx$$

$$\frac{2vdv}{k^2 + v^2} = -\frac{2g}{k^2}dx$$

$$\int_{v_0}^{v}\frac{2vdv}{k^2 + v^2} = -\frac{2g}{k^2}\int_0^x dx$$

$$\left|\ln(k^2 + v^2)\right|_{v_0}^{v} = -\frac{2g}{k^2}|x|_0^x$$

$$\ln(k^2 + v^2) - \ln(k^2 + v_0^2) = -\frac{2g}{k^2}(x - 0)$$

$$\ln(k^2 + v^2) - \ln(k^2 + v_0^2) = -\frac{2gx}{k^2}$$

$$\frac{2gx}{k^2} = \ln(k^2 + v_0^2) - \ln(k^2 + v^2)$$

$$\frac{2gx}{k^2} = \ln\frac{k^2 + v_0^2}{k^2 + v^2}$$

$$x = \frac{k^2}{2g}\ln\frac{k^2 + v_0^2}{k^2 + v^2} \qquad ...(1)$$

Again consider equation of motion

$$\frac{dv}{dt} = -\frac{g}{k^2}(k^2 + v^2)$$

$$\frac{dv}{k^2 + v^2} = -\frac{g}{k^2}dt$$

$$\int_{v_0}^{v}\frac{dv}{k^2 + v^2} = -\frac{g}{k^2}\int_0^t dt$$

$$\frac{1}{k}\left|\tan^{-1}\frac{v}{k}\right|_{v_0}^{v} = -\frac{g}{k^2}|t|_0^t$$

$$\frac{1}{k}\left(\tan^{-1}\frac{v}{k} - \tan^{-1}\frac{v_0}{k}\right) = -\frac{gt}{k^2}$$

$$\frac{gt}{k} = \tan^{-1}\frac{v_0}{k} - \tan^{-1}\frac{v}{k}$$

Example-8: *A small stone of mass m is thrown vertically upward with initial velocity V. If the air resistance at speed v is mkv, where k is a positive constant, show that the stone will return to the starting point with a speed U given by*

$$g - kU = (g + kV)\exp\left\{-\frac{k(U + V)}{g}\right\}$$

Solution: **Upward Motion:** For upward motion, equation of motion of the stone is

$$v\frac{dv}{dx} = -g - kv$$

$$\frac{vdv}{g + kv} = -dx$$

$$-dx = \frac{1}{k}\frac{kvdv}{g + kv}$$

$$-\int dx = \frac{1}{k}\int\frac{(g + kv - g)dv}{g + kv}$$

$$-x = \frac{1}{k}\int\left(\frac{g + kv}{g + kv} - \frac{g}{g + kv}\right)dv$$

$$-x = \frac{1}{k}\int\left(1 - \frac{g}{g + kv}\right)dv$$

$$-x = \frac{1}{k}\left(v - \frac{g}{k}\ln(g + kv)\right) + A \quad ...(1)$$

Applying initial conditions when $x=0, v=V$ in (1), we have

$$0=\frac{1}{k}\left(V-\frac{g}{k}\ln(g+kV)\right)+A$$

$$A=-\frac{V}{k}+\frac{g}{k^2}\ln(g+kV)$$

Putting this value of A in (1), we have

$$-x=\frac{1}{k}\left(v-\frac{g}{k}\ln(g+kv)\right)-\frac{V}{k}+\frac{g}{k^2}\ln(g+kV) \qquad \text{...(2)}$$

If h is maximum height, it is obtained at $v=0$, so putting $x=h, v=0$ in (2), we have

$$-h=\frac{1}{k}\left(0-\frac{g}{k}\ln(g+0)\right)-\frac{V}{k}+\frac{g}{k^2}\ln(g+kV)$$

$$-h=-\frac{g}{k^2}\ln g-\frac{V}{k}+\frac{g}{k^2}\ln(g+kV)$$

$$h=\frac{V}{k}+\frac{g}{k^2}\ln g-\frac{g}{k^2}\ln(g+kV)$$

$$h=\frac{V}{k}+\frac{g}{k^2}[\ln g-\ln(g+kV)$$

$$h=\frac{V}{k}+\frac{g}{k^2}\ln\frac{g}{(g+kV)} \qquad \text{...(3)}$$

Downward Motion: For downward motion, equation of motion of the stone is

$$v\frac{dv}{dx}=g-kv$$

$$\frac{vdv}{g-kv}=dx$$

$$dx=-\frac{1}{k}\frac{-kvdv}{g-kv}$$

$$\int dx=-\frac{1}{k}\int\frac{-kvdv}{g-kv}$$

$$x=-\frac{1}{k}\int\frac{(-g+g-kv)dv}{g-kv}$$

$$x=-\frac{1}{k}\int\left(-\frac{g}{g-kv}+\frac{g-kv}{g-kv}\right)dv$$

$$x=-\frac{1}{k}\int\left(\frac{g}{k}\frac{-k}{g-kv}+1\right)dv$$

$$x=-\frac{1}{k}\left(\frac{g}{k}\ln(g-kv)+v\right)+B \qquad \text{...(4)}$$

When $x=0, v=0$, so putting these in (4)

$$0=-\frac{1}{k}\left(\frac{g}{k}\ln(g-0)+0\right)+B$$

$$B=\frac{g}{k^2}\ln g$$

Putting this in (4), we have

$$x=-\frac{1}{k}\left(\frac{g}{k}\ln(g-kv)+v\right)+\frac{g}{k^2}\ln g \qquad \text{...(5)}$$

Since the stone falls through a distance h given by (3), so putting $x=h, v=U$ in (5), we get

$$h=-\frac{1}{k}\left(\frac{g}{k}\ln(g-kU)+U\right)+\frac{g}{k^2}\ln g \qquad \text{...(6)}$$

Equating (3) and (6), we have

$$\frac{V}{k}+\frac{g}{k^2}\ln\frac{g}{g+kV}=-\frac{1}{k}\left(\frac{g}{k}\ln(g-kU)+U\right)+\frac{g}{k^2}\ln g$$

$$\frac{V}{k}+\frac{g}{k^2}\ln\frac{g}{g+kV}=-\frac{g}{k^2}\ln(g-kU)-\frac{U}{k}+\frac{g}{k^2}\ln g$$

$$\frac{g}{k^2}\ln\frac{g}{g+kV}+\frac{g}{k^2}\ln(g-kU)-\frac{g}{k^2}\ln g=-\frac{U}{k}-\frac{V}{k}$$

$$\frac{g}{k^2}\ln\frac{g}{g+kV}+\frac{g}{k^2}\ln\frac{g-kU}{g}=-\frac{U+V}{k}$$

$$\frac{g}{k^2}\ln\frac{g}{g+kV}\times\frac{g-kU}{g}=-\frac{U+V}{k}$$

$$\ln\frac{g-kU}{g+kV}=-\frac{k}{g}(U+V)$$

$$\frac{g-kU}{g+kV}=\exp\left\{-\frac{k}{g}(U+V)\right\}$$

$$g-kU=(g+kV)\exp\left\{-\frac{k(U+V)}{g}\right\}$$

Example 9: *A particle of mass m is projected vertically upward with initial speed The air resistance is proportional to the velocity (mkv). Determine the time taken the particle in arriving at the highest point of the path. Find also the energy which dissipated by the viscous medium during this time. (Neglect squares of k).*

Solution: Equation of motion of the particle is

$$m\frac{dv}{dt}=-mg-mkv$$

$$\frac{dv}{dt}=-(g+kv)$$

$$\frac{dv}{g+kv}=-dt$$

$$-\int dt=\frac{1}{k}\int\frac{k\,dv}{g+kv}$$

$$-t=\frac{1}{k}\ln(g+kv)+A \quad ...(1)$$

Initially $t=0, v=V$, so

$$0=\frac{1}{k}\ln(g+kV)+A$$

$$A=-\frac{1}{k}\ln(g+kV)$$

Putting this value of A in (1)

$$-t=\frac{1}{k}\ln(g+kv)-\frac{1}{k}\mathrm{in}(g+kV)$$

$$t=-\frac{1}{k}\ln(g+kv)+\frac{1}{k}\ln(g+kV)$$

$$t=\frac{1}{k}[\ln(g+kV)-\ln(g+kv)]$$

$$t=\frac{1}{k}\ln\frac{g+kV}{g+kv} \quad ...($$

To find the time to reach the heighest point, put $v=0$ in (2), i.e.

$$t=\frac{1}{k}\ln\frac{g+kV}{g}=\frac{1}{k}\ln\left(\frac{g}{g}+\frac{kV}{g}\right)$$

$$t=\frac{1}{k}\ln\left(1+\frac{kV}{g}\right)$$

$$t=\frac{1}{k}\left(\frac{kV}{g}-\frac{k^2V^2}{2g^2}+\ldots\right)$$

$$\Rightarrow t \approx \frac{V}{g} - \frac{kV^2}{2g^2} \qquad \text{(Neglecting squares of } k\text{)}$$

Initially the particle possesses kinetic energy only and at the highest point it possesses potential energy only. For finding the latter energy we have to find the maximum height attained by the particle. The equation of motion can also be written

$$mv\frac{dv}{dx} = -mg - mkv$$

$$v\frac{dv}{dx} = -(g + kv)$$

$$\frac{vdv}{g + kv} = -dx$$

$$-\int dx = \frac{1}{k}\int \frac{kvdv}{g + kv}$$

$$-x = \frac{1}{k}\int \frac{(g + kv - g)dv}{g + kv}$$

$$-x = \frac{1}{k}\int \left(\frac{g + kv}{g + kv} - \frac{g}{g + kv}\right)dv$$

$$-x = \frac{1}{k}\int \left(1 - \frac{g}{g + kv}\right)dv$$

$$-x = \frac{v}{k} - \frac{g}{k^2}\ln(g + kv) + B \qquad \text{...(3)}$$

When $x = 0, v = V$

$$0 = \frac{V}{k} - \frac{g}{k^2}\ln(g + kV) + B$$

$$B = \frac{g}{k^2}\ln(g + kV) - \frac{V}{k}$$

Putting this value of B in (3), we have

$$-x = \frac{v}{k} - \frac{g}{k^2}\ln(g + kv) + \frac{g}{k^2}\ln(g + kV) - \frac{V}{k}$$

$$x = -\frac{v}{k} + \frac{g}{k^2}\ln(g + kv) - \frac{g}{k^2}\ln(g + kV) + \frac{V}{k}$$

When $x = h, v = 0$, so

$$h = \frac{g}{k^2}\ln g - \frac{g}{k^2}\ln(g + kV) + \frac{V}{k}$$

$$h = \frac{V}{k} + \frac{g}{k^2}[\ln g - \ln(g + kV)]$$

$$h = \frac{V}{k} + \frac{g}{k^2}\ln\frac{g}{g + kV}$$

Therefore, energy dissipated during the whole path is

$$\frac{1}{2}mV^2 - mgh = \frac{1}{2}mV^2 - mg\left[\frac{V}{k} + \frac{g}{k^2}\ln\frac{g}{g + kV}\right] = \frac{1}{2}mV^2 - mg\left[\frac{V}{k} - \frac{g}{k^2}\ln\frac{g + kV}{g}\right]$$

$$= \frac{1}{2}mV^2 - mg\left[\frac{V}{k} - \frac{g}{k^2}\ln\left(1 + \frac{kV}{g}\right)\right]$$

$$= \frac{1}{2}mV^2 - mg\left[\frac{V}{k} - \frac{g}{k^2}\left(\frac{kV}{g} - \frac{k^2V^2}{2g^2} + \frac{k^3V^3}{3g^3} - \dots\right)\right]$$

$$= \frac{1}{2}mV^2 - mg\left[\frac{V}{k} - \frac{V}{k} + \frac{V^2}{2g} - \frac{kV^3}{3g^2} + \dots\right]$$

$$= \frac{1}{2}mV^2 - \frac{1}{2}mV^2 + m\frac{kV^3}{3g} - \dots$$

$$= \frac{mkV^3}{3g} \qquad \text{(neglecting squares of } k \text{ and higher order terms)}$$

Example-10: *A particle falls under gravity in a medium which opposes the mo with a force proportional to the velocity of the body. The limiting velocity acquired the particle is such that it would be attained in time T if the body were falling vacuum. Show that the medium being present the body acquires half the limi velocity in a time 0.693 T.*

Solution: Equation of motion of the particle is

$$m\frac{dv}{dt} = mg - mkv$$

$$\frac{dv}{dt} = g - kv$$

For limiting velocity, put $\frac{dv}{dt} = 0$, i.e. $v_{\text{lim}} = \frac{g}{k}$. Using the equation $v = gt$, the tim required for attaining this velocity in vacuum is

$$T = \frac{v_{\text{lim}}}{g} = \frac{g/k}{g} = \frac{1}{k}$$

$$\Rightarrow k = \frac{1}{T}$$

Putting this value of k in (1), we have

$$\frac{dv}{dt} = g - \frac{v}{T}$$

$$dt = \frac{dv}{g - \frac{v}{T}}$$

$$\int dt = \int \frac{dv}{g - \frac{v}{T}}$$

$$t = -T\ln\left(g - \frac{v}{T}\right) + A \quad ...(2)$$

When $t = 0$, $v = 0$, so from (2), we have

$$0 = -T\ln g + A$$

$$A = T\ln g$$

Putting this value of A in (2), we have

$$t = -T\ln\left(g - \frac{v}{T}\right) + T\ln g$$

$$t = T\ln\frac{g}{g - v/T} \quad ...(3)$$

When $v = \frac{g}{2k} = \frac{1}{2}gT$

Putting this in (3), we have

$$t = T\ln\frac{g}{g - g/2} = T\ln 2 = 0.693T$$

Example-11: *A small body of mass m falls with an initial velocity v_0 from a cer height above the ground. The air resistance is mk^2v^2. Find the velocity at an instant. Find also the limiting velocity.*

Solution: Equation of motion of the particle is

$$m\frac{dv}{dt} = mg - mk^2v^2$$

$$\frac{dv}{dt} = g - k^2v^2 \quad ...(1)$$

$$\frac{dv}{g - k^2v^2} = dt$$

$$\int dt = \int \frac{dv}{g - k^2v^2}$$

$$t = \frac{1}{k^2}\frac{1}{2\sqrt{g}}\ln\left(\frac{\sqrt{g} + kv}{\sqrt{g} - kv}\right) + A \quad ...(2)$$

Put $t = 0, v = v_0$ in (1)

$$0=\frac{1}{k^2}\frac{1}{2\sqrt{g}}\ln\left(\frac{\sqrt{g}+kv_0}{\sqrt{g}-kv_0}\right)+A$$

$$A=-\frac{1}{k^2}\frac{1}{2\sqrt{g}}\ln\left(\frac{\sqrt{g}+kv_0}{\sqrt{g}-kv_0}\right)$$

Putting this value of A in (1), we have

$$t=\frac{1}{k^2}\frac{1}{2\sqrt{g}}\ln\left(\frac{\sqrt{g}+kv}{\sqrt{g}-kv}\right)-\frac{1}{k^2}\frac{1}{2\sqrt{g}}\ln\left(\frac{\sqrt{g}+kv_0}{\sqrt{g}-kv_0}\right)$$

$$t=\frac{1}{2k^2\sqrt{g}}\left[\ln\left(\frac{\sqrt{g}+kv}{\sqrt{g}-kv}\right)-\ln\left(\frac{\sqrt{g}+kv_0}{\sqrt{g}-kv_0}\right)\right]$$

$$t=\frac{1}{2k^2\sqrt{g}}\ln\left(\frac{(\sqrt{g}+kv)(\sqrt{g}-kv_0)}{(\sqrt{g}-kv)(\sqrt{g}+kv_0)}\right)$$

For limiting velocity put $\frac{dv}{dt}=0$ in (1), i.e. $g-k^2v^2=0$

$$\Rightarrow v_{\text{lim}}=\frac{\sqrt{g}}{k}$$

Example-12: *A particle is fired upward with a velocity u in a medium whose resistance is proportional to the square of the velocity. Show that the particle returns to the point of projection with a speed $\frac{uV}{\sqrt{u^2+V^2}}$, where V is the limiting velocity of the particle in the medium.*

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Solution: **Upward Motion**: For upward motion, equation of motion is

$$v\frac{dv}{dx}=-g-kv^2$$

$$\frac{vdv}{g+kv^2}=-dx$$

$$-dx=\frac{vdv}{g+kv^2}$$

$$-2k\int dx=\int\frac{2kvdv}{g+kv^2}$$

$$-2kx=\ln(g+kv^2)+A \qquad ...(1)$$

Applying initial conditions when

$x=0, v=u$ in (1), we have

$$0=\ln(g+ku^2)+A$$

$$A=-\ln(g+ku^2)$$

Putting this value of A in (1)

$$-2kx=\ln(g+kv^2)-\ln(g+ku^2)$$

$$2kx=\ln\frac{g+ku^2}{g+kv^2} \qquad ...(2)$$

At maximum height $x=h, v=0$

$$2kh=\ln\frac{g+ku^2}{g} \qquad ...(3)$$

Downward Motion: For downward motion, equation of motion is

$$v\frac{dv}{dx}=g-kv^2 \qquad \text{...(a)}$$

$$dx=\frac{vdv}{g-kv^2}$$

$$\int dx=\int\frac{vdv}{g-kv^2}$$

$$-2k\int dx=\int\frac{-2kvdv}{g-kv^2}$$

$$-2kx=\ln(g-kv^2)+B \qquad \text{...(4)}$$

Applying initial conditions when $x=0, v=0$ in (4), we have

$$0=\ln g+B$$

$$B=-\log g$$

Putting this value of B in (4), we have

$$-2kx=\ln(g-kv^2)-\ln g$$

$$2kx=\ln g-\ln(g-kv^2)$$

$$2kx=\ln\frac{g}{g-kv^2} \qquad \text{...(5)}$$

Let v' be the speed of the particle with which it returns to point of projection from the height h, so putting $v=v'$, $x=h$ in (5), we have

$$2kh=\ln\frac{g}{g-kv'^2} \qquad \text{...(6)}$$

Equating (3) and (6), we have

$$\ln\frac{g}{g-kv'^2}=\ln\frac{g+ku^2}{g}$$

$$\frac{g}{g-kv'^2}=\frac{g+ku^2}{g}$$

$$\frac{g^2}{g+ku^2}=g-kv'^2$$

$$kv'^2=g-\frac{g^2}{g+ku^2}$$

$$kv'^2=\frac{g^2+gku^2-g^2}{g+ku^2}$$

$$v'^2=\frac{gu^2}{g+ku^2} \qquad \text{...(7)}$$

Since V is the limiting velocity and for limiting velocity, $\frac{dv}{dt}=0$, so putting these values in (a), we have $g-kV^2=0$

$$\Rightarrow g=kV^2$$

Putting this value of g in (7), we have

$$v'^2=\frac{kV^2u^2}{kV^2+ku^2}=\frac{u^2V^2}{u^2+V^2}$$

$$v'=\frac{uV}{\sqrt{u^2+V^2}}$$

Example-13: *A particle is projected vertically upward with velocity u in a medium offering a resistance kv^2 per unit mass. Prove that it returns to the starting point after a time $\frac{1}{\sqrt{kg}}\{\alpha+\ln(\sec\alpha+\tan\alpha)\}$, where $\tan\alpha=u\sqrt{\frac{k}{g}}$.*

Solution: **Upward Motion**: For upward motion, equation of motion is

$$\frac{dv}{dt}=-g-kv^2$$

$$\frac{dv}{dt}=-k\left(\frac{g}{k}+v^2\right)$$

$$\frac{dv}{\frac{g}{k}+v^2}=-kdt$$

$$\int\frac{dv}{\frac{g}{k}+v^2}=-k\int dt$$

$$\frac{1}{\sqrt{\frac{g}{k}}}\tan^{-1}\frac{v}{\sqrt{\frac{g}{k}}}=-kt+A$$

$$\sqrt{\frac{k}{g}}\tan^{-1}\sqrt{\frac{k}{g}}v=-kt+A \qquad ...(1)$$

Putting $t=0, v=u$ in (1), we have

$$\sqrt{\frac{k}{g}}\tan^{-1}\sqrt{\frac{k}{g}}u=0+A$$

$$A=\sqrt{\frac{k}{g}}\tan^{-1}\sqrt{\frac{k}{g}}u$$

Putting this value of A in (1), we have

$$\sqrt{\frac{k}{g}}\tan^{-1}\sqrt{\frac{k}{g}}v=-kt+\sqrt{\frac{k}{g}}\tan^{-1}\sqrt{\frac{k}{g}}u$$

Time t_1 required to reach maximum height is given by putting $v=0$

$$0=-kt_1+\sqrt{\frac{k}{g}}\tan^{-1}\sqrt{\frac{k}{g}}u$$

$$t_1=\frac{1}{\sqrt{kg}}\tan^{-1}\sqrt{\frac{k}{g}}u \qquad ...(2)$$

$$t_1=\frac{1}{\sqrt{kg}}\alpha \qquad ...(3)$$

where $\tan\alpha=u\sqrt{\frac{k}{g}}$ ...(4)

To find maximum height, we consider the equation

$$v\frac{dv}{dx}=-g-kv^2$$

$$\frac{vdv}{g+kv^2}=-dx$$

$$-dx=\frac{vdv}{g+kv^2}$$

$$-2k\int dx=\int\frac{2kvdv}{g+kv^2}$$

$$-2kx=\ln(g+kv^2)+B \qquad ...(5)$$

Applying initial conditions when $x=0, v=u$ in (1), we have

$$0=\ln(g+ku^2)+B$$

$$B=-\ln(g+ku^2)$$

Putting this value of B in (5)

$$-2kx=\ln(g+kv^2)-\ln(g+ku^2)$$

$$x=\frac{1}{2k}\ln\frac{g+ku^2}{g+kv^2} \qquad ...(6)$$

At maximum height $x=h, v=0$

$$h=\frac{1}{2k}\ln\frac{g+ku^2}{g} \qquad ..(7)$$

Downward Motion: For downward motion, equation of motion is

$$v\frac{dv}{dx}=g-kv^2$$

$$dx=\frac{vdv}{g-kv^2}$$

$$\int dx=\int\frac{vdv}{g-kv^2}$$

$$-2k\int dx=\int\frac{-2kvdv}{g-kv^2}$$

$$-2kx=\ln(g-kv^2)+C \qquad ...(8)$$

Applying initial conditions when $x=0, v=0$ in (8), we have

$$0=\ln g+C$$

$$C = -\log g$$

Putting this value of C in (8), we have

$$-2kx = \ln(g - kv^2) - \ln g$$

$$2kx = \ln g - \ln(g - kv^2)$$

$$2kx = \ln\frac{g}{g - kv^2}$$

$$e^{2kx} = \frac{g}{g - kv^2}$$

$$e^{-2kx} = \frac{g - kv^2}{g}$$

$$ge^{-2kx} = g - kv^2$$

$$kv^2 = g - ge^{-2kx}$$

$$v^2 = \frac{g}{k}(1 - e^{-2kx})$$

$$v^2 = \frac{g}{k}e^{-2kx}(e^{2kx} - 1)$$

$$v = \sqrt{\frac{g}{k}}e^{-kx}\sqrt{e^{2kx} - 1}$$

$$\frac{dx}{dt} = \sqrt{\frac{g}{k}}e^{-kx}\sqrt{e^{2kx} - 1}$$

$$\frac{e^{kx}dx}{\sqrt{e^{2kx} - 1}} = \sqrt{\frac{g}{k}}dt$$

$$\int\frac{e^{kx}dx}{\sqrt{e^{2kx} - 1}} = \sqrt{\frac{g}{k}}\int dt$$

Put $z = e^{kx} \Rightarrow dz = ke^{kx}dx$

$$\frac{1}{k}\int\frac{dz}{\sqrt{z^2 - 1}} = \sqrt{\frac{g}{k}}\int dt$$

$$\frac{1}{k}\ln(z + \sqrt{z^2 - 1}) = \sqrt{\frac{g}{k}}t + D$$

$$\frac{1}{k}\ln(e^{kx} + \sqrt{e^{2kx} - 1}) = \sqrt{\frac{g}{k}}t + D \quad \text{...(9)}$$

When $t = 0, x = 0$

$$\frac{1}{k}\ln(1 + \sqrt{1 - 1}) = 0 + D$$

$$\frac{1}{k}\ln 1 = D$$

$$D = 0$$

Putting this value in (9), we have

$$\frac{1}{k}\ln(e^{kx} + \sqrt{e^{2kx} - 1}) = \sqrt{\frac{g}{k}}t$$

If t_2 is time to reach point of projection from height h, then

$$\frac{1}{k}\ln(e^{kh} + \sqrt{e^{2kh} - 1}) = \sqrt{\frac{g}{k}}t_2$$

$$t_2 = \frac{1}{\sqrt{gk}}\ln(e^{kh} + \sqrt{e^{2kh} - 1}) \quad \text{...(10)}$$

From (7), we have

$$2kh = \ln\left(1 + \frac{ku^2}{g}\right) = \ln(1 + \tan^2\alpha)$$

$$2kh = \ln(\sec^2\alpha)$$

$$2kh = 2\ln\sec\alpha$$

$$kh = \ln\sec\alpha$$

$$e^{kh} = \sec\alpha$$

$$e^{2kh} = \sec^2\alpha$$

$$e^{2kh} - 1 = \sec^2\alpha - 1$$

$$e^{2kh} - 1 = \tan^2\alpha$$

$$\sqrt{e^{2kh} - 1} = \tan\alpha$$

Putting these values in (10), we have

$$t_2 = \frac{1}{\sqrt{gk}}\ln(\sec\alpha + \tan\alpha)$$

Required time is

$$t_1 + t_2 = \frac{1}{\sqrt{kg}}\alpha + \frac{1}{\sqrt{gk}}\ln(\sec\alpha + \tan\alpha)$$

$$= \frac{1}{\sqrt{kg}}\{\alpha + \ln(\sec\alpha + \tan\alpha)\}$$

Example [illegible] Suppose that the motion of a falling spherical drop of liquid is opposed by a force proportional to the surface area of the drop to the density ρ of the medium through which it is falling and to nth power of its velocity. The density of the droplet is ρ_1 and it is assumed that the shape remains spherical. Show that the limiting velocity of the drop varies as the nth root of the radius of the drop. Gravity g is acting vertically downward.

Solution: Let r and v be the radius and velocity of the drop respectively. Then $4\pi r^2$ is the surface area of the drop. Now force of resistance $\propto 4\pi r^2 \rho v^n$, so equation of motion is

$$m\ddot{x} = mg - 4k\pi r^2 \rho v^n, \qquad \text{where } k \text{ is proportinality constant}$$

$$\ddot{x} = g - \frac{4k\pi r^2 \rho v^n}{m} \quad ...(1)$$

Since limiting velocity is attained at $\ddot{x}=0$, so from (1), we have

$$g - \frac{4k\pi r^2 \rho v^n_{lim}}{m} = 0$$

$$\Rightarrow 4k\pi r^2 \rho v^n_{lim} = gm$$

$$\Rightarrow v^n_{lim} = \frac{gm}{4k\pi r^2 \rho}$$

$$\Rightarrow v_{lim} = \left(\frac{gm}{4k\pi r^2 \rho}\right)^{\frac{1}{n}} \quad ...(2)$$

$$\text{Density} = \frac{\text{mass}}{\text{volume}}$$

$$\rho_1 = \frac{m}{\frac{4}{3}\pi r^3} = \frac{3m}{4\pi r^3}$$

$$m = \frac{4}{3}\pi r^3 \rho_1$$

Putting this value in (2), we have

$$v_{lim} = \left(\frac{4g\pi r^3 \rho_1}{3 \times 4k\pi r^2 \rho}\right)^{\frac{1}{n}} = \left(\frac{gr\rho_1}{3k\rho}\right)^{\frac{1}{n}}$$

$$v_{lim} = \left(\frac{g\rho_1}{3k\rho}\right)^{\frac{1}{n}} r^{\frac{1}{n}} \propto r^{\frac{1}{n}}$$

EXERCISE 11

Multiple Choice Questions (MCQs)

Four options are given in each of the following questions fill the circle in front of that choice which you think is correct. Cutting or filling two or more circles is not allowed:

11.1

(i) If a particle of mass m is released from rest at an initial height above the ground assuming the resistance of air proportional to the square of the velocity, then its equation of motion is $m\ddot{x} =$

(a) $mg - m\lambda v^2$ (b) $mg - m\lambda v$ (c) $mg - m\lambda v^3$ (d) none of these ⓐ ⓑ ⓒ ⓓ

(ii) If a particle of mass m is released from rest at an initial height above the ground assuming the resistance of air proportional to the velocity, then its equation of motion is $m\ddot{x} =$

(a) $mg - m\lambda v^2$ (b) $mg - m\lambda v$ (c) $mg - m\lambda v^3$ (d) none of these ⓐ ⓑ ⓒ ⓓ

(iii) If a particle of mass m is released from rest at an initial height above the ground assuming the resistance of air proportional to the cube of the velocity, then its equation of motion is $m\ddot{x} =$.

(a) $mg - m\lambda v^2$ (b) $mg - m\lambda v$ (c) $mg - m\lambda v^3$ (d) none of these ⓐ ⓑ ⓒ ⓓ

(iv) The maximum velocity of a particle moving in a resisting medium is called ______________ velocity.

(a) initial (b) escape (c) terminal (d) final ⓐ ⓑ ⓒ ⓓ

(v) For limiting velocity, the acceleration of the particle is

(a) negative (b) zero (c) positive (d) none of these ⓐ ⓑ ⓒ ⓓ

(vi) If a particle of mass m is projected vertically upward with some initial speed assuming the resistance of air proportional to the square of the velocity, then its equation of motion is $m\ddot{x} =$

(a) $-mg - m\lambda v^2$ (b) $mg + m\lambda v^2$ (c) $mg + m\lambda v$ (d) none of these ⓐ ⓑ ⓒ ⓓ

(vii) The small damped case for $\ddot{x} + k\dot{x} + \omega^2 x = 0$ is that

(a) $k^2 - 4\omega^2 < 0$ (b) $k^2 - 4\omega^2 = 0$

(c) $k^2 - 4\omega^2 > 0$ (d) none of these ⓐ ⓑ ⓒ ⓓ

(viii) The over damped case for $\ddot{x} + k\dot{x} + \omega^2 x = 0$ is that

(a) $k^2 - 4\omega^2 < 0$ (b) $k^2 - 4\omega^2 = 0$

(c) $k^2 - 4\omega^2 > 0$ (d) none of these ⓐ ⓑ ⓒ ⓓ

(ix) The critically damped case for $\ddot{x} + k\dot{x} + \omega^2 x = 0$ is that

(a) $k^2 - 4\omega^2 < 0$ (b) $k^2 - 4\omega^2 = 0$

(c) $k^2 - 4\omega^2 > 0$ (d) none of these ⓐ ⓑ ⓒ ⓓ

SUMMARY

- Equation of motion of harmonic oscillator is $\ddot{x} + k\dot{x} + \omega^2 x = 0$.
- Equation of motion of damped forced oscillator is $\ddot{x} + k\dot{x} + \omega^2 x = F(t)$.
- Terminal velocity or limiting velocity of a particle moving in a resisting medium is the maximum velocity it can attain.

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