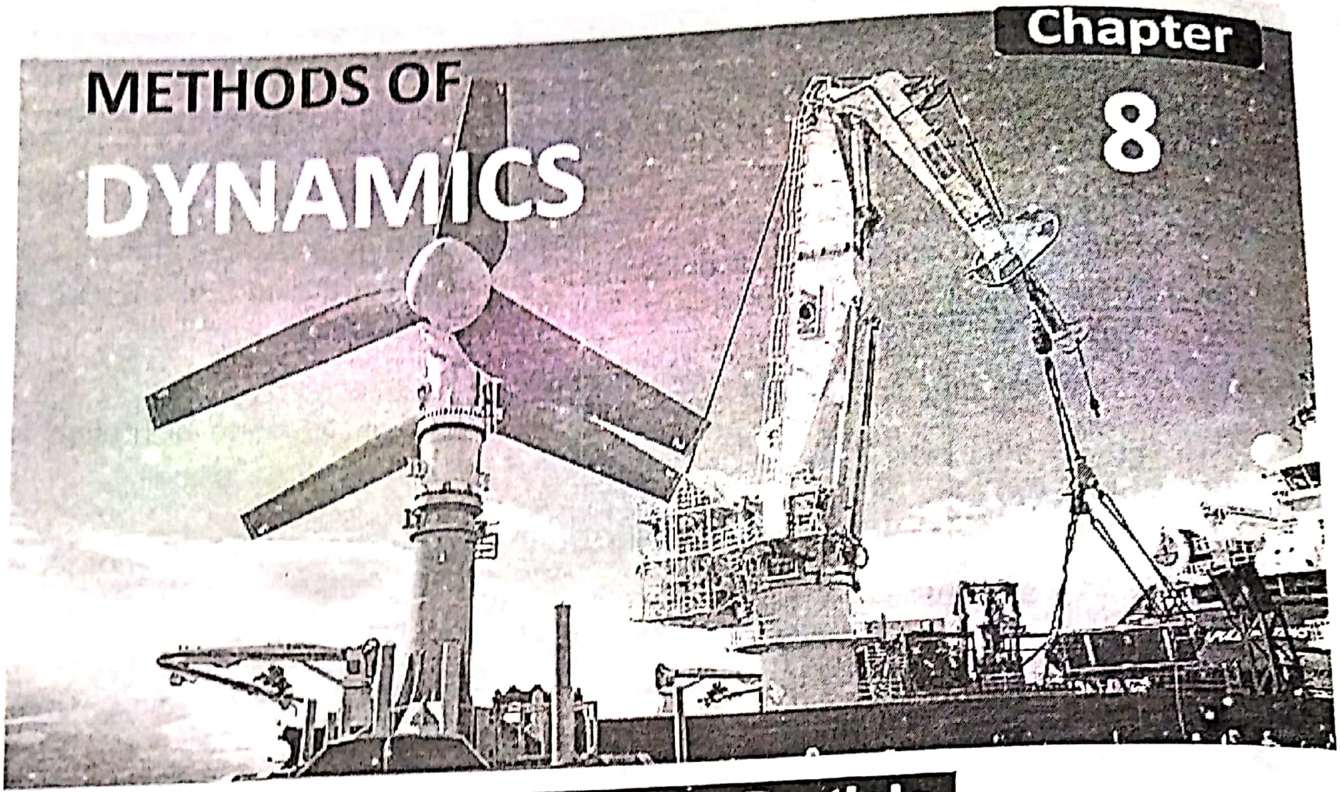
Chapter 8

METHODS OF DYNAMICS



8-1 Equation of Motion of a Particle

If a force $\vec{F}$ acts at time t on a particle of mass m moving with a velocity $\vec{v}$, then the equation of motion (according to Newton's second law) is given by

$$\boxed{\frac{d}{dt}(m\vec{v}) = \vec{F}} \qquad ...(8.1)$$

which can be written as

$$\boxed{m\dot{\vec{v}} = m\frac{d\vec{v}}{dt} = m\vec{a} = \vec{F}} \qquad ...(8.2)$$

where $\dot{\vec{v}} = \frac{d\vec{v}}{dt} = \vec{a}$ is the acceleration at time t. (We shall use one dot and two dots for the operators $\frac{d}{dt}$ and $\frac{d^2}{dt^2}$ respectively). If $\vec{r}$ is the position vector of the particle at time t, with respect to a fixed origin O, then, since $\dot{\vec{r}} = \vec{v}$, equation (8.2) becomes

$$\boxed{m\ddot{\vec{r}} = \vec{F}} \qquad ...(8.3)$$

If the path of the particle is known, then the field of force acting on the particle can be determined by anyone of these three equivalent equations, and conversely. If the force under the influence of which the particle is in motion, is given, then we can obtain the necessary information about the motion of the particle (e.g. position, speed, acceleration).

Central Force: If the force $\vec{F}$ on a particle always passes through a fixed point O, the force is called a *central force* and O is called the *centre of force*. A central force is *repulsive* or *attractive* according as it is directed away from or towards the centre of force.

Principle of Conservation of Momentum: *The momentum of a particle moving under the influence of no force is constant throughout the motion.*

Proof: Since the particle moves under no force, so its equation of motion is

$$\frac{d}{dt}(m\vec{v}) = 0$$

$$\Rightarrow m\vec{v} = \vec{G}$$

where $\vec{G}$ is a constant vector. This completes the proof.

Two-Dimensional Cartesian Form of Equation of Motion:

Suppose that the particle is moving in the *xy*-plane. If $\vec{r} = x\hat{i} + y\hat{j}$ is the position vector and $\vec{F} = X\hat{i} + Y\hat{j}$; the force on the particle at time *t*, then equation of motion (8.3) becomes

$$m\ddot{\vec{r}} = \vec{F}$$

$$m(\ddot{x}\hat{i} + \ddot{y}\hat{j}) = X\hat{i} + Y\hat{j}$$

$$m\ddot{x}\hat{i} + m\ddot{y}\hat{j} = X\hat{i} + Y\hat{j}$$

$$\boxed{m\ddot{x} = X, \qquad m\ddot{y} = Y} \qquad \text{...(8.4)}$$

which are equations of motion in a plane or in two-dimension Cartesian form.

Example-1: *A particle moves in such a way that its position vector at time t is $\vec{r} = (a\cos nt)\hat{i} + (b\sin nt)\hat{j}$, where a, b, n are constants and $a > b > 0$. Show that the path of the particle is an ellipse of semi-major and semi-minor axes a, b respectively, and that the field of force is directed towards the centre of the ellipse. Find also the maximum speed.*

Solution:

$$\vec{r} = (a\cos nt)\hat{i} + (b\sin nt)\hat{j} \qquad \text{...(1)}$$

In Cartesian form, we have

$$\vec{r} = x\hat{i} + y\hat{j} \qquad \text{...(2)}$$

Equating (1) and (2), we have

$$x\hat{i} + y\hat{j} = (a\cos nt)\hat{i} + (b\sin nt)\hat{j}$$

$$\Rightarrow x = a\cos nt, \qquad y = b\sin nt$$

$$\Rightarrow \frac{x}{a} = \cos nt, \qquad \frac{y}{b} = \sin nt$$

Squaring and adding, we have

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \cos^2 nt + \sin^2 nt$$

$$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad ...(3$$

This shows that the path of the particle is an ellipse of semi-major and semi-mino axes a, b respectively.

Differentiating (1) w.r.t. t, we get the velocity $\vec{v}$, i.e.

$$\vec{v}=\dot{\vec{r}}=(-na\sin nt)\hat{i}+(nb\cos nt)\hat{j} \qquad ...($$

The magnitude v of velocity $\vec{v}$ is given by

$$v=|\vec{v}|=\sqrt{(-na\sin nt)^2+(bn\cos nt)^2}=\sqrt{n^2a^2\sin^2 nt+n^2b^2\cos^2 nt}$$

$$=|n|\sqrt{a^2\sin^2 nt+b^2(1-\sin^2 nt)}=|n|\sqrt{a^2\sin^2 nt+b^2-b^2\sin^2 nt}$$

$$=|n|\sqrt{(a^2-b^2)\sin^2 nt+b^2}$$

This shows that v is maximum for maximum value of $\sin^2 nt$, i.e. for $\sin^2 nt=1$, s

$$v_{\max}=|n|\sqrt{(a^2-b^2)(1)+b^2}=|n|\sqrt{a^2-b^2+b^2}=|n|\sqrt{a^2}=|na|$$

Now differentiating (4) w.r.t. t, we have

$$\ddot{\vec{r}}=(-n^2a\cos nt)\hat{i}+(-n^2b\sin nt)\hat{j}$$

$$\ddot{\vec{r}}=-n^2(a\cos nt\hat{i}+b\sin nt\hat{j})=-n^2\vec{r}$$

$$\Rightarrow \ddot{\vec{r}}=-n^2\vec{r}$$

$$\Rightarrow m\ddot{\vec{r}}=-mn^2\vec{r},\ \text{where } m \text{ is the mass of the particle}$$

$$\Rightarrow \vec{F}=-mn^2\vec{r}$$

The – ive sign shows that $\vec{F}$ is directed towards origin.

Example-2: *A particle moves in the xy-plane under the influence of a field o which is parallel to the axis of y and varies as the distance from x-axis. Show the force is repulsive, the path of the particle, supposed not straight, is of th* $y=A\cosh x+b\sinh x$*, where A, B, n are constants.*

Solution: If m is the mass of the particle and k is a constant, then the force by

$$\vec{F}=mk^2y\hat{j},\quad \text{where } mk^2 \text{ is proportionality constant}$$

$$\Rightarrow m(\ddot{x}\hat{i}+\ddot{y}\hat{j})=mk^2y\hat{j} \qquad \because \vec{F}=m(\ddot{x}\hat{i}+\ddot{y}\hat{j})$$

$$\Rightarrow \ddot{x}\hat{i}+\ddot{y}\hat{j}=k^2y\hat{j}$$

$$\Rightarrow \ddot{x}=0,\qquad \ddot{y}=k^2y$$

Integrating these equations, we get

$$x=a+bt,\qquad y=ce^{kt}+de^{-kt}$$

where a, b, c, d are constants. Since $b=0$ implies that the path is the st $x=a$, so $b\neq 0$. Eliminating t between these equations, we find

$$y = ce^{-\frac{ka}{b}} e^{\frac{k}{b}x} + de^{\frac{ka}{b}} e^{-\frac{k}{b}x}$$

$$y = ce^{-\frac{ka}{b}}\left(\cosh\frac{k}{b}x + \sinh\frac{k}{b}x\right) + de^{\frac{ka}{b}}\left(\cosh\frac{k}{b}x - \sinh\frac{k}{b}x\right) \quad \because e^{x} = \cosh x + \sinh x$$
$$e^{-x} = \cosh x - \sinh x$$

$$y = (ce^{-\frac{ka}{b}} + de^{\frac{ka}{b}})\cosh\frac{k}{b}x + (ce^{-\frac{ka}{b}} - de^{\frac{ka}{b}})\sinh\frac{k}{b}x$$

$$y = A\cosh nx + B\sinh nx$$

where $A = ce^{-\frac{ka}{b}} + de^{\frac{ka}{b}}$, $B = ce^{-\frac{ka}{b}} - de^{\frac{ka}{b}}$, $n = \frac{k}{b}$

Two-Dimensional Polar Form of Equation of Motion:

Suppose that the motion of a particle is restricted to xy-plane, and (r,θ) are the polar coordinates of the particle, then acceleration in radial and transverse components form is given as

$$\vec{a} = \{\ddot{r} - r(\dot{\theta})^2\}\hat{r} + \{2\dot{r}\dot{\theta} + r\ddot{\theta}\}\hat{s}$$
$$m\vec{a} = m\{\ddot{r} - r(\dot{\theta})^2\}\hat{r} + m\{2\dot{r}\dot{\theta} + r\ddot{\theta}\}\hat{s}$$
$$\vec{F} = F_r\hat{r} + F_\theta\hat{s}$$

where $F_r = m\{\ddot{r} - r(\dot{\theta})^2\}$, $F_\theta = m\{2\dot{r}\dot{\theta} + r\ddot{\theta}\}$ are radial and transverse components of force. Thus, equations of motion in polar coordinates are

$$\boxed{m\{\ddot{r} - r(\dot{\theta})^2\} = F_r, \quad m\{2\dot{r}\dot{\theta} + r\ddot{\theta}\} = F_\theta} \qquad ...(8.5)$$

For a particle moving along a circle of radius a with centre at O, $\dot{r} = 0, \ddot{r} = 0$ so that if a force $\vec{F}$ having radial and transverse components F_r and F_θ acts on it, the equations of motion are

$$\boxed{ma(\dot{\theta})^2 = -F_r, \quad ma\ddot{\theta} = F_\theta} \qquad ...(8.6)$$

Example-3: *A particle of mass m moves on xy-plane under the force $\vec{F} = -\frac{k}{r^4}\vec{r}$, where r is its distance from the origin O. If it starts on the positive x-axis at a distance a from O with speed v_0 in a direction making an angle α with the positive x-direction, prove that at time t, $\ddot{r} = \frac{ma^2v_0^2\sin^2\alpha - k}{mr^3}$.*

Solution: $$\vec{F} = -\frac{k}{r^4}\vec{r} = -\frac{k}{r^4}(r\hat{r}) = -\frac{k}{r^3}\hat{r} \qquad (\because \vec{r} = r\hat{r})$$

$$m\{\ddot{r} - r(\dot{\theta})^2\}\hat{r} + m\{2\dot{r}\dot{\theta} + r\ddot{\theta}\}\hat{s} = -\frac{k}{r^3}\hat{r}$$

Comparing coefficients of $\hat{r}$ and $\hat{s}$, we have

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{k}{r^2} \qquad ...(1)$$

and
$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0$$

$$\Rightarrow 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

$$\Rightarrow \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0 \qquad \because \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 2\dot{r}\dot{\theta} + r\ddot{\theta}$$

$$\Rightarrow \frac{d}{dt}(r^2\dot{\theta}) = 0$$

$\Rightarrow r^2\dot{\theta} = h$, where h is constant of integration

At $t = 0$, the radial component of velocity is along x-axis and the transverse component along y-axis. Hence $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$ at $t = 0$ is

$$\vec{v}_0 = (\dot{r})_{t=0}\hat{i} + (r\dot{\theta})_{t=0}\hat{j} = (v_0\cos\alpha)\hat{i} + (v_0\sin\alpha)\hat{j}$$

Therefore $(\dot{r})_{t=0} = v_0\cos\alpha$

and $(r\dot{\theta})_{t=0} = v_0\sin\alpha$

$$\Rightarrow a(\dot{\theta})_{t=0} = v_0\sin\alpha$$

$$\Rightarrow (\dot{\theta})_{t=0} = \frac{v_0\sin\alpha}{a}$$

Since $r^2\dot{\theta} = h$ is constant, so it must have the same value at $r = a$, i.e.

$$r^2\dot{\theta} = (r^2\dot{\theta})_{t=0} = (r^2)_{t=0}(\dot{\theta})_{t=0} = a^2\left(\frac{v_0\sin\alpha}{a}\right) = av_0\sin\alpha$$

$$\Rightarrow \dot{\theta} = \frac{av_0\sin\alpha}{r^2} \qquad ...(2)$$

Putting the value of $\dot{\theta}$ in (1), we have

$$m\left[\ddot{r} - r\left(\frac{av_0\sin\alpha}{r^2}\right)^2\right] = -\frac{k}{r^2}$$

$$\ddot{r} - r\left(\frac{a^2v_0^2\sin^2\alpha}{r^4}\right) = -\frac{k}{mr^2}$$

$$\ddot{r} - \frac{a^2v_0^2\sin^2\alpha}{r^3} = -\frac{k}{mr^2}$$

$$\ddot{r} = \frac{a^2v_0^2\sin^2\alpha}{r^3} - \frac{k}{mr^2}$$

$$\ddot{r} = \frac{ma^2v_0^2\sin^2\alpha - kr}{mr^3}$$

Example 2.[illegible] A smooth horizontal tube rotates in a horizontal plane about a point of itself with uniform angular velocity ω. At time $t = 0$, a particle inside the tube is at rest relatively to the tube and at a distance a from the point of rotation. Show that at time t the distance of the particle from the point of rotation is $a\cosh\omega t$. Find also the force the tube exerts on the particle.

Solution: Let O be the fixed point of the tube, and the tube OP rotate about Oz where Oz is perpendicular to the plane of rotation (Fig 2.1). Suppose that Ox is fixed axis in this plane and (r,θ), the polar coordinates of the particle at time t. Then $\dot\theta=\omega$, and since the tube is smooth, the force exerted on the particle P by the tube is perpendicular to the radius vector OP, and in the direction of θ increasing.

$$\vec F = F_\theta \hat s$$

$$\Rightarrow m\{\ddot r - r(\dot\theta)^2\}\hat r + m\{2\dot r\dot\theta + r\ddot\theta\}\hat s = F_\theta \hat s$$

$$\Rightarrow F_\theta = m\{2\dot r\dot\theta + r\ddot\theta\} \qquad \ldots(1)$$

and $m\{\ddot r - r(\dot\theta)^2\}=0$

$$\Rightarrow \ddot r - r\omega^2 = 0 \qquad \because \dot\theta=\omega$$

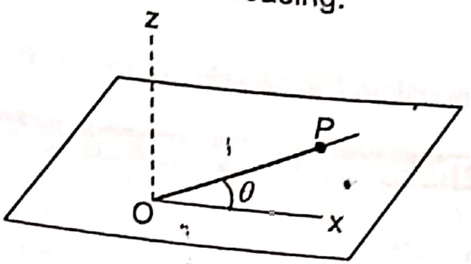


Fig. 9.1

whose solution is

$$r = A\cosh\omega t + B\sinh\omega t$$

$$\dot r = \omega A\sinh\omega t + \omega B\cosh\omega t$$

At $t=0$, the particle is on x-axis, and $r=a, \dot r=0$, so $A=a, B=0$. Hence

$$r = a\cosh\omega t, \quad \dot r = \omega a\sinh\omega t$$

Putting these values in (1), we have

$$F_\theta = m\{2(\omega a\sinh\omega t)(\omega) + r(0)\}$$

$$F_\theta = 2m\omega^2 a\sinh\omega t \qquad \because \dot\theta=\omega \therefore \ddot\theta=0$$

8-2 Work, Power and Energy

Work: If a force $\vec F$, acting on a particle whose position vector at time t is $\vec r$, produces a displacement $d\vec r$, in time dt, then the work done by $\vec F$ on the particle is

$$\boxed{dW = \vec F\cdot d\vec r} \qquad \ldots(9.7)$$

Hence the amount of work done by $\vec F$ in moving the particle from P_1 to P_2 along the path of the particle, is

$$\boxed{W = \int_{P_1}^{P_2}\vec F\cdot d\vec r = \int_{\vec r_1}^{\vec r_2}\vec F\cdot d\vec r} \qquad \ldots(9.8)$$

where $\vec r_1, \vec r_2$ are position vectors of P_1, P_2 respectively.

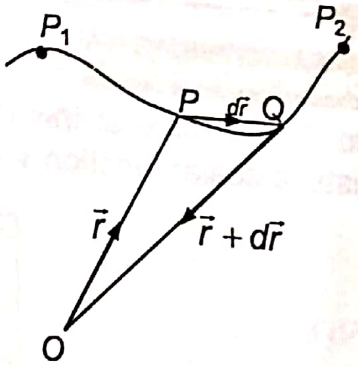


Fig. 9.2

Power: The rate of working on the particle is known as the *power* applied to the particle. Thus, the quantity $P=\dfrac{dW}{dt}$ is called the *power* applied to the particle.

Differentiating (9.7) w.r.t. t, we have

$$\boxed{P = \frac{dW}{dt} = \vec F\cdot\frac{d\vec r}{dt} = \vec F\cdot\vec v} \qquad \ldots(9.10)$$

where $\vec v$ is the velocity of the particle at time t.

Kinetic Energy: If m is the mass of the particle, then the quantity

$$\boxed{T=\frac{1}{2}mv^2} \qquad \text{...(9.11)}$$

is called the *kinetic energy* of the particle, whose speed is v at time t.

Theorem-1 (Principle of Energy): *The total work done on a particle in moving along a curve C from P_1 to P_2 is equal to the increase $T_2 - T_1$ in the kinetic energy where T_1, T_2 are respectively the kinetic energies of the particle at times t_1, corresponding to the positions P_1, P_2.*

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Proof: Let $\vec{F}$ be the force acting on a particle of mass m. The work done by $\vec{F}$, a the particle moves from P_1 to P_2 is given by

$$W=\int_{P_1}^{P_2}\vec{F}\cdot d\vec{r}=\int_{\vec{r}_1}^{\vec{r}_2}\vec{F}\cdot d\vec{r}=\int_{\vec{r}_1}^{\vec{r}_2}m\frac{d\vec{v}}{dt}\cdot d\vec{r}=m\int_{\vec{r}_1}^{\vec{r}_2}d\vec{v}\cdot\frac{d\vec{r}}{dt}=m\int_{\vec{r}_1}^{\vec{r}_2}d\vec{v}\cdot\vec{v}$$

$$=m\int_{\vec{r}_1}^{\vec{r}_2}\vec{v}\cdot d\vec{v}=\frac{1}{2}m\int_{\vec{r}_1}^{\vec{r}_2}d(\vec{v}\cdot\vec{v})=\frac{1}{2}m\left|\vec{v}\cdot\vec{v}\right|_{\vec{r}_1}^{\vec{r}_2}=\frac{1}{2}m\left|v^2\right|_{\vec{r}_1}^{\vec{r}_2}=\frac{1}{2}m[v^2(\vec{r}_2)-v^2(\vec{r}_1)]$$

$$=\frac{1}{2}m[v_2^2-v_1^2],\ \text{where}\ v_1, v_2\ \text{are velocities in the positions}\ P_1, P_2.$$

$$=\frac{1}{2}mv_2^2-\frac{1}{2}mv_1^2=T_2-T_1$$

This completes the proof.

8-3 Principle of Conservation of Energy

Conservative Field: Let $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ be the position vector of a particle at tim and $\vec{F}$, the force at this instant. The field of force $\vec{F}$ is called *conservative*, if th exists, a scalar function V of x, y, z such that

$$\boxed{\vec{F}=-\left(\frac{\partial V}{\partial x}\hat{i}+\frac{\partial V}{\partial y}\hat{j}+\frac{\partial V}{\partial z}\hat{k}\right)=-\nabla V} \qquad \text{...(9.}$$

where
$$\nabla\equiv\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}$$

Potential of Force or Potential Energy: If the field of force $\vec{F}$ is conservative, t $\vec{F}=-\nabla V$. The function $V=V(x,y,z)$ is called the *potential of the force $\vec{F}$* or *potential energy of the particle* at time t or at the position of the particle at time t.

Since curl grad $V=0$, so $\vec{F}$ is *conservative* if and only if curl $\vec{F}=0$.

It may be noticed that an arbitrary constant can always be added to V without altering the field of force. Thus we can always specify the potential of a field of force with respect to a suitable value taken as zero of potential.

Theorem-2: *The work done on the particle in moving it from a position P_1 to P_2 under a conservative field of force is the difference between the potential energies of the particle at P_1 and P_2 respectively.*

Proof: The work done on the particle in moving it from a position P_1 to P_2 is

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} \qquad \ldots(1)$$

Since $\vec{F}$ is conservative field of force, so

$$\vec{F} = -\left(\frac{\partial V}{\partial x} \hat{i} + \frac{\partial V}{\partial y} \hat{j} + \frac{\partial V}{\partial z} \hat{k} \right)$$

Also

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\Rightarrow d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

Putting these values in (1), we have

$$W = \int_{P_1}^{P_2} - \left(\frac{\partial V}{\partial x} \hat{i} + \frac{\partial V}{\partial y} \hat{j} + \frac{\partial V}{\partial z} \hat{k} \right) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= -\int_{P_1}^{P_2} \left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \right)$$

$$= -\int_{P_1}^{P_2} dV \qquad \because dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$

$$= -|V|_{P_1}^{P_2} = -(V_2 - V_1) = V_1 - V_2$$

where V_1 and V_2 are potential energies of the particle at P_1 and P_2 respectively.

Total Energy: If T is the kinetic energy of the particle moving under the influence of a conservative field of force whose potential is V, then

$$E = T + V$$

is called the *total energy* of the particle at the point under consideration.

Theorem-3 (*Principle of Conservation of Energy*): *In a conservative field of force, the total energy of a particle remains constant throughout the motion.*

Proof: Let $\vec{F}$ be the conservative field of force acting on a particle of mass m. The work done by $\vec{F}$, as the particle moves from P_1 *to* P_2 is given by

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = \int_{\vec{r}_1}^{\vec{r}_2} m \frac{d\vec{v}}{dt} \cdot d\vec{r} = m \int_{\vec{r}_1}^{\vec{r}_2} d\vec{v} \cdot \frac{d\vec{r}}{dt} = m \int_{\vec{r}_1}^{\vec{r}_2} d\vec{v} \cdot \vec{v}$$

$$= m \int_{\vec{r}_1}^{\vec{r}_2} \vec{v} \cdot d\vec{v} = \frac{1}{2} m \int_{\vec{r}_1}^{\vec{r}_2} d(\vec{v} \cdot \vec{v}) = \frac{1}{2} m |\vec{v} \cdot \vec{v}|_{\vec{r}_1}^{\vec{r}_2} = \frac{1}{2} m |v^2|_{\vec{r}_1}^{\vec{r}_2} = \frac{1}{2} m [v^2(\vec{r}_2) - v^2(\vec{r}_1)]$$

$$= \tfrac{1}{2} m [v_2^2 - v_1^2] = \tfrac{1}{2} m v_2^2 - \tfrac{1}{2} m v_1^2, \text{ where } v_1, v_2 \text{ are velocities in the positions } P_1, P_2.$$

$$W = T_2 - T_1 \qquad \ldots(1)$$

where T_1, T_2 are respectively the kinetic energies of the particle at times t_1, t_2 corresponding to the positions P_1, P_2.

∵ Since $\vec{F}$ is conservative field of force, so the work done on the particle in moving it from a position P_1 to P_2 is

$$W = \int_{P_1}^{P_2} \vec{F}\cdot d\vec{r} = \int_{P_1}^{P_2} -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right)\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= -\int_{P_1}^{P_2}\left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right)$$

$$= -\int_{P_1}^{P_2} dV \qquad \because dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

$$= -|V|_{P_1}^{P_2} = -(V_2 - V_1)$$

$$W = V_1 - V_2 \qquad \ldots(2)$$

where V_1 and V_2 are potential energies of the particle at P_1 and P_2 respectively. Equating (1) and (2), we have

$$V_1 - V_2 = T_2 - T_1$$
$$V_1 + T_1 = T_2 + V_2$$

which proves that the total energies at any two points P_1, P_2 are equal.

This shows that in a conservative field of force, the total energy of a particle remains constant throughout the motion.

Example-5: *A particle of mass m moves along x-axis under the influence of a conservative field of force having potential $V(x)$. If the particle is located at positions x_1 and x_2 at respective times t_1 and t_2, prove that if E is the total energy,*

$$t_2 - t_1 = \sqrt{\frac{m}{2}}\int_{x_1}^{x_2}\frac{dx}{\sqrt{E - V(x)}}$$

Solution: Applying principle of conservation of energy, we have

$$E = T + V(x) = \frac{1}{2}mv^2 + V(x)$$

$$\Rightarrow \frac{1}{2}mv^2 = E - V(x)$$

$$\Rightarrow v^2 = \frac{2}{m}[E - V(x)]$$

$$\Rightarrow v = \sqrt{\frac{2}{m}}[E - V(x)]$$

$$\Rightarrow \frac{dx}{dt} = \sqrt{\frac{2}{m}}\sqrt{E - V(x)}$$

$$\Rightarrow dt = \sqrt{\frac{m}{2}}\frac{dx}{\sqrt{E - V(x)}}$$

$$\Rightarrow \int_{t_1}^{t_2} dt = \sqrt{\frac{m}{2}}\int_{x_1}^{x_2}\frac{dx}{\sqrt{E - V(x)}}$$

$$\Rightarrow |t|_{t_1}^{t_2} = \sqrt{\frac{m}{2}}\int_{x_1}^{x_2}\frac{dx}{\sqrt{E - V(x)}}$$

$$\Rightarrow t_2 - t_1 = \sqrt{\frac{m}{2}}\int_{x_1}^{x_2}\frac{dx}{\sqrt{E - V(x)}}$$

Example: Prove that the force field

$$\vec{F}=(y^2-2xyz^3)\hat{i}+(3+2xy-x^2z^3)\hat{j}+(6z^3-3x^2yz^2)\hat{k}$$

is conservative, and determine its potential.

Solution: Consider **PU 2008, 2007 (B.A./B.Sc.) A.J.K. 2012, 2010 (B.A./B.Sc.)**

$$\text{Curl}\,\vec{F}=\nabla\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y^2-2xyz^3 & 3+2xy-x^2z^3 & 6z^3-3x^2yz^2\end{vmatrix}$$

$$=\hat{i}\left[\frac{\partial}{\partial y}(6z^3-3x^2yz^2)-\frac{\partial}{\partial z}(3+2xy-x^2z^3)\right]$$

$$-\hat{j}\left[\frac{\partial}{\partial x}(6z^3-3x^2yz^2)-\frac{\partial}{\partial z}(y^2-2xyz^3)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(3+2xy-x^2z^3)-\frac{\partial}{\partial y}(y^2-2xyz^3)\right]$$

$$=[(-3x^2z^2)-(-3x^2z^2)]\hat{i}-[(-6xyz^2)-(-6xyz^2)]\hat{j}$$

$$+[(2y-2xz^3)-(2y-2xz^3)]\hat{k}$$

$$=(-3x^2z^2+3x^2z^2)\hat{i}-(-6xyz^2+6xyz^2)\hat{j}+(2y-2xz^3-2y+2xz^3)\hat{k}$$

$$=0$$

This shows that $\vec{F}$ is conservative.

Since $\vec{F}$ is conservative, so its potential V satisfies the equation

$$\vec{F}=-\nabla V$$

$$\Rightarrow \nabla V=-\vec{F}$$

$$\Rightarrow \frac{\partial V}{\partial x}\hat{i}+\frac{\partial V}{\partial y}\hat{j}+\frac{\partial V}{\partial z}\hat{k}=-[(y^2-2xyz^3)\hat{i}+(3+2xy-x^2z^3)\hat{j}+(6z^3-3x^2yz^2)\hat{k}]$$

Equating coefficients of $\hat{i},\hat{j},\hat{k}$ on both sides, we have

$$\frac{\partial V}{\partial x}=-(y^2-2xyz^3)=-y^2+2xyz^3 \qquad ...(1)$$

$$\frac{\partial V}{\partial y}=-(3+2xy-x^2z^3)=-3-2xy+x^2z^3 \qquad ...(2)$$

$$\frac{\partial V}{\partial z}=-(6z^3-3x^2yz^2)=-6z^3+3x^2yz^2 \qquad ...(3)$$

Integrating (1) partially w.r.t. x, we have

$$V=-xy^2+x^2yz^3+f(y,z) \qquad ...(4)$$

where $f(y,z)$ is constant of integration (due to partial integration w.r.t. x, it may be function of y and z).

Integrating (2) partially w.r.t. y, we have

$$V=-3y-xy^2+x^2yz^3+g(x,z) \quad \ldots(5)$$

Integrating (3) partially w.r.t. z, we have

$$V=-\frac{3}{2}z^4+x^2yz^3+h(x,y) \quad \ldots(6)$$

Combining (4), (5) and (6), we have

$$V=x^2yz^3-xy^2-3y-\frac{3}{2}z^4$$

Example-7: *Determine whether*

$$\vec{F}=(x^2y-z^3)\hat{i}+(3xyz+xz^2)\hat{j}+(2x^2yz+yz^4)\hat{k}$$

PU, 2008 (B.A/B.Sc)

is conservative.

Solution: Consider

$$\text{Curl } \vec{F}=\nabla\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2y-z^3 & 3xyz+xz^2 & 2x^2yz+yz^4\end{vmatrix}$$

$$=\hat{i}\left[\frac{\partial}{\partial y}(2x^2yz+yz^4)-\frac{\partial}{\partial z}(3xyz+xz^2)\right]-\hat{j}\left[\frac{\partial}{\partial x}(2x^2yz+yz^4)-\frac{\partial}{\partial z}(x^2y-z^3)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(3xyz+xz^2)-\frac{\partial}{\partial y}(x^2y-z^3)\right]$$

$$=[(2x^2z+z^4)-(3xy+2xz)]\hat{i}-[(4xyz)-(-3z^2)]\hat{j}$$

$$+[(3yz+z^2)-(x^2)]\hat{k}$$

$$=(2x^2z+z^4-3xy-2xz)\hat{i}-(4xyz+3z^2)\hat{j}+(3yz+z^2-x^2)\hat{k}$$

$$\neq 0$$

This shows that $\vec{F}$ is not conservative.

Example-8: *(a) Show that $\vec{F}=-kr^3\vec{r}$ is conservative.*

(b) *Find the potential energy of a particle moving in the field of force in (a).*

(c) *If a particle of mass m moves with velocity $\vec{v}=\dfrac{d\vec{r}}{dt}$ in this field, show that, if E is constant total energy, then $\dfrac{1}{2}mv^2+\dfrac{1}{5}kr^5=E$. What important physical principle does this illustrate?*

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Solution: (a) Consider

$$\text{curl}\vec{F}=\nabla\times\vec{F}=\nabla\times(-kr^3\vec{r})=-k\nabla\times(r^3\vec{r})$$

$$=-k[r^3(\nabla\times\vec{r})+(\nabla r^3)\times\vec{r}] \qquad \because \nabla\times\phi\vec{A}=\phi\nabla\times\vec{A}+\nabla\phi\times\vec{A}$$

$$=-k[r^3(0)+(3r^2\nabla r)\times\vec{r}] \qquad \because \nabla\times\vec{r}=0,\ \nabla r^3=3r^2\nabla r$$

$$\operatorname{curl}\vec{F} = -k\left[0 + \left(3r^2\frac{\vec{r}}{r}\right)\times\vec{r}\right] \qquad \because \nabla r = \frac{\vec{r}}{r}$$

$$= -k(3r\vec{r}\times\vec{r}) = -3kr(\vec{r}\times\vec{r}) = 0 \qquad \because \vec{r}\times\vec{r} = 0$$

This shows that $\vec{F} = -kr^3\vec{r}$ is conservative.

(b) If V is the potential energy of a particle moving in the field of force $\vec{F} = -kr^3\vec{r}$, then

$$\vec{F} = -\nabla V$$

$$\Rightarrow \nabla V = -\vec{F}$$

$$\Rightarrow \nabla V = -(-kr^3\vec{r}) = kr^3\vec{r}$$

$$\Rightarrow \nabla V = \nabla\left(\frac{k}{5}r^5\right) \qquad \because \nabla r^5 = 5r^4\frac{\vec{r}}{r} = 5r^3\vec{r}$$

$$\Rightarrow V = \frac{k}{5}r^5$$

(c) Kinetic energy $= T = \frac{1}{2}mv^2$

Potential energy $= V = \frac{k}{5}r^5$

Kinetic energy + Potential energy = Total energy

Hence

$$\frac{1}{2}mv^2 + \frac{1}{5}kr^5 = E$$

Earth's Gravitational Field: Earth pulls all materials with a force known as the gravitational force. The vertical acceleration $\vec{g}$ produced by this force in freely falling bodies is known as the *gravitational acceleration.*

If m is the mass of the particle, then the gravitational force $\vec{W}$ is given by

$$m\vec{g} = \vec{W}$$

Taking y-axis in the vertical direction, we have

$$\vec{g} = -g\hat{j}$$

so that

$$\vec{W} = -mg\hat{j}$$

$\vec{W}$ is called the *gravitational field of force*. For particles whose heights from the surface of the earth are small as compared to the radius of the earth, g varies very slowly with height and can be considered constant. In the following we shall treat it as a constant.

If $V = mgy$, where y is the height of the particle at time t, then

$$\frac{\partial V}{\partial y} = mg$$

[illegible]

Principle of Angular Momentum

[illegible]

$$\Rightarrow \frac{d^2\vec{r}}{dt^2} = \frac{1}{m}(a\cos\omega t\,\hat{i}) + \frac{1}{m}(b\sin\omega t\,\hat{j})$$

Integrating both sides w.r.t. t, we have

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{1}{m\omega}(a\sin\omega t\,\hat{i}) + \frac{1}{m\omega}(-b\cos\omega t\,\hat{j}) + \vec{C} \qquad \text{...(1)}$$

Applying initial conditions in (1) when $t = 0, \vec{v} = 0$, we have

$$0 = \frac{1}{m\omega}(a\sin 0\,\hat{i}) + \frac{1}{m\omega}(-b\cos 0\,\hat{j}) + \vec{C}$$

$$0 = 0 - \frac{b}{m\omega}\hat{j} + \vec{C}$$

$$\vec{C} = \frac{b}{m\omega}\hat{j}$$

Putting this value of $\vec{C}$ in (1), we have

$$\vec{v} = \frac{1}{m\omega}(a\sin\omega t\,\hat{i}) + \frac{1}{m\omega}(-b\cos\omega t\,\hat{j}) + \frac{b}{m\omega}\hat{j}$$

$$\vec{v} = \frac{a}{m\omega}(\sin\omega t\,\hat{i}) + \frac{b}{m\omega}(1 - \cos\omega t\,\hat{j})$$

Integrating both sides w.r.t. t, we have

$$\vec{r} = \frac{a}{m\omega^2}(-\cos\omega t\,\hat{i}) + \frac{b}{m\omega^2}(\omega t - \sin\omega t\,\hat{j}) + \vec{D} \qquad \text{...(2)}$$

Applying initial conditions in (2) when $t = 0, \vec{r} = 0$, we have

$$0 = \frac{a}{m\omega^2}(-\cos 0\,\hat{i}) + \frac{b}{m\omega^2}(0 - \sin 0\,\hat{j}) + \vec{D}$$

$$0 = -\frac{a}{m\omega^2}\hat{i} + \vec{D} \quad \text{or} \quad \vec{D} = \frac{a}{m\omega^2}\hat{i}$$

Putting this value of $\vec{D}$ in (2), we have

$$\vec{r} = \frac{a}{m\omega^2}(-\cos\omega t\,\hat{i}) + \frac{b}{m\omega^2}(\omega t - \sin\omega t\,\hat{j}) + \frac{a}{m\omega^2}\hat{i}$$

$$\vec{r} = \frac{a}{m\omega^2}(1 - \cos\omega t\,\hat{i}) + \frac{b}{m\omega^2}(\omega t - \sin\omega t\,\hat{j})$$

Example-12: A particle of mass m moves under the influence of the force $\vec{F} = a(\sin\omega t\,\hat{i} + \cos\omega t\,\hat{j})$. If the particle is initially at rest at the origin, prove that work done up to time t is given by $\frac{a^2}{m\omega^2}(1 - \cos\omega t)$, and that the instantaneous power applied is $\frac{a^2}{m\omega}\sin\omega t$.

Solution: It is given that

$$\vec{F}\cdot d\vec{r} = a(\sin\omega t\hat{i} + \cos\omega t\hat{j})\cdot\frac{a}{m\omega}[(1-\cos\omega t)\hat{i} + \sin\omega t\hat{j}]dt$$

$$= \frac{a^2}{m\omega}(\sin\omega t\hat{i} + \cos\omega t\hat{j})\cdot[(1-\cos\omega t)\hat{i} + \sin\omega t\hat{j}]dt$$

$$= \frac{a^2}{m\omega}[\sin\omega t(1-\cos\omega t + \cos\omega t\sin\omega t]dt$$

$$= \frac{a^2}{m\omega}[\sin\omega t - \cos\omega t\sin\omega t + \cos\omega t\sin\omega t]dt$$

$$= \frac{a^2}{m\omega}\sin\omega t dt$$

Required work done is

$$W = \int_0^t \vec{F}\cdot d\vec{r} = \int_0^t \frac{a^2}{m\omega}\sin\omega t dt = \frac{a^2}{m\omega}\int_0^t \sin\omega t dt$$

$$= -\frac{a^2}{m\omega^2}|\cos\omega t|_0^t = -\frac{a^2}{m\omega^2}(\cos\omega t - \cos 0) = -\frac{a^2}{m\omega^2}(\cos\omega t - 1)$$

$$= \frac{a^2}{m\omega^2}(1-\cos\omega t)$$

The instantaneous power applied is

$$P = \vec{F}\cdot\vec{v} = \vec{F}\cdot\frac{d\vec{r}}{dt}$$

$$= a(\sin\omega t\hat{i} + \cos\omega t\hat{j})\cdot\frac{a}{m\omega}[(1-\cos\omega t)\hat{i} + \sin\omega t\hat{j}]$$

$$= \frac{a^2}{m\omega}(\sin\omega t\hat{i} + \cos\omega t\hat{j})\cdot[(1-\cos\omega t)\hat{i} + \sin\omega t\hat{j}]$$

$$= \frac{a^2}{m\omega}[\sin\omega t(1-\cos\omega t) + \cos\omega t\sin\omega t]$$

$$= \frac{a^2}{m\omega}(\sin\omega t - \cos\omega t\sin\omega t + \cos\omega t\sin\omega t)$$

$$= \frac{a^2}{m\omega}\sin\omega t$$

Example-13: *A particle is moved by a force* $\vec{F} = 20\hat{i} - 30\hat{j} + 15\hat{k}$ *along a straight line from the point A to the point B with position vectors* $2\hat{i} + 7\hat{j} - 3\hat{k}$ *and* $5\hat{i} - 3\hat{j} - 6\hat{k}$ *respectively. Find the work done.*

Solution: Here

$$\vec{r} = (5\hat{i} - 3\hat{j} - 6\hat{k}) - (2\hat{i} + 7\hat{j} - 3\hat{k}) = 3\hat{i} - 10\hat{j} - 3\hat{k}$$

Required work done is

$$W = \vec{F}\cdot\vec{r} = (20\hat{i} - 30\hat{j} - 15\hat{k})\cdot(3\hat{i} - 10\hat{j} - 3\hat{k})$$

$$= (20)(3) + (-30)(-10) + (15)(-3)$$

$$= 60 + 300 - 45$$

$$= 315$$

Example [illegible] A particle of mass m moves along the curve defined by $\vec{r} = (a\cos\omega t)\hat{i} + (b\sin\omega t)\hat{j}$. Find the torque and the angular momentum about the origin.

Solution: Here

$$\vec{r} = (a\cos\omega t)\hat{i} + (b\sin\omega t)\hat{j}$$

$$\dot{\vec{r}} = (-a\omega\sin\omega t)\hat{i} + (b\omega\cos\omega t)\hat{j}$$

$$\ddot{\vec{r}} = (-a\omega^2\cos\omega t)\hat{i} + (-b\omega^2\sin\omega t)\hat{j}$$

$$= -\omega^2(a\cos\omega t\,\hat{i} + b\sin\omega t\,\hat{j})$$

$$= -\omega^2\vec{r}$$

$$\vec{F} = m\ddot{\vec{r}} = -m\omega^2\vec{r}$$

Torque is given by

$$\vec{\tau} = \vec{r}\times\vec{F} = \vec{r}\times(-m\omega^2\vec{r})$$

$$= -m\omega^2(\vec{r}\times\vec{r})$$

$$= 0 \qquad \because \vec{r}\times\vec{r} = 0$$

Angular momentum is given by

$$\vec{J} = \vec{r}\times m\dot{\vec{r}} = (a\cos\omega t\,\hat{i} + b\sin\omega t\,\hat{j})\times m(-a\omega\sin\omega t\,\hat{i} + b\omega\cos\omega t\,\hat{j})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\omega t & b\sin\omega t & 0 \\ -am\omega\sin\omega t & bm\omega\cos\omega t & 0 \end{vmatrix} = m\omega\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\omega t & b\sin\omega t & 0 \\ -a\sin\omega t & b\cos\omega t & 0 \end{vmatrix}$$

$$= m\omega(ab\cos^2\omega t + ab\sin^2\omega t)\hat{k}$$

$$= mab\omega(\cos^2\omega t + \sin^2\omega t)\hat{k}$$

$$= mab\omega\hat{k}$$

Example [illegible] A shell of mass M is moving with speed V. An internal explosion generates an amount of energy E and breaks the shell into two portions whose masses are in the ratio $m_1 : m_2$. The fragments continue to move in the original line of motion of the shell. Show that their speeds are $V + \sqrt{\frac{2m_2E}{m_1M}}$ and $V - \sqrt{\frac{2m_1E}{m_2M}}$.

[D.U. 2001 B.A.B.Sc]

Solution: Let v_1, v_2 be the velocities of the masses m_1, m_2 after collision in the

Putting the value of u from (1) in (2), we have

$$m\left(\frac{Mv}{m}\right)^2 + Mv^2 = 2mgh$$

$$m\frac{M^2v^2}{m^2} + Mv^2 = 2mgh$$

$$\frac{M^2v^2}{m} + Mv^2 = 2mgh$$

$$M^2v^2 + Mmv^2 = 2m^2gh$$

$$M(M+m)v^2 = 2m^2gh$$

$$v^2 = \frac{2m^2gh}{M(M+m)}$$

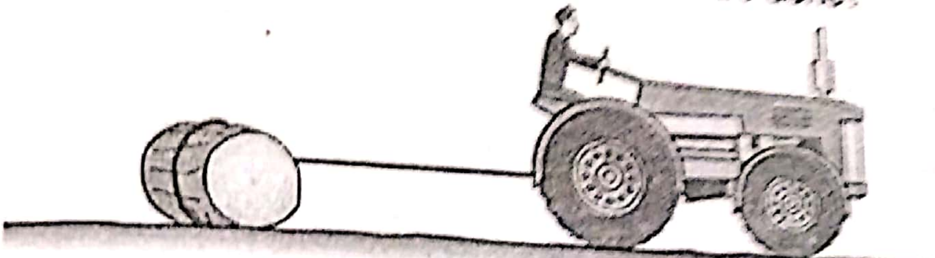
$$v = \sqrt{\frac{2m^2gh}{M(M+m)}}$$

EXERCISE 8

Short Questions

Q.1 Solve / write answers of the following short questions:

(i) A tractor pulls a felled tree along the ground for a distance of 200 m. If the tractor exerts a force of 500 N, how much work will be done?



(ii) Find the kinetic energy of a football player of mass 90 kg running at 6 m/s.

(iii) Find the kinetic energy of an elephant of mass 6 tonnes charging at 10 m/s.

(iv) Find the kinetic energy of a racing car of mass 1.5 tonnes travelling at 300 km per hour.

(v) Find the kinetic energy of a bullet of mass 20 grams moving at 400 m s^{-1}.

(vi) Find the kinetic energy of a meteorite of mass 20 kg as it enters the earth's atmosphere at 8 km s^{-1}.

(vii) A gardener moves a wheel barrow 30 metres along a level, straight path. The work done by the gardener is 120 J, and the barrow is initially and finally at rest. Calculate the average force resisting the motion.

(viii) A ball of mass 1.2 kg moving with initial speed 20 m s^{-1} comes to rest after travelling 30 metres across a horizontal surface. Find the work done against resisting forces, and hence calculate the mean resisting force.

(ix) A small block is pulled along a rough horizontal surface at a constant speed of 2 m s^{-1} by a constant force. This force has magnitude 25 N and acts at an angle of 30° to horizontal. Calculate work done by the force in 10 seconds.

(x) A particle of mass 2 kg falls freely from rest. Calculate the kinetic energy of the particle after it has descended 20 metres.

(xi) A stone of mass 0.8 kg is thrown vertically upwards with speed 10 m/s. Calculate initial kinetic energy of the stone, and the height to which it will rise.

Long Questions

Q.2 An aircraft of mass 1.8 tonnes landing on an aircraft carrier at 144 kilometres per hour is brought to rest by a parachute brake and an arrester cable. If 30% of the work is done by the parachute, calculate the work done by the cable.

Whether the force field $\vec{F} = y^2\hat{i} + x^2z^3\hat{j} + z^3\hat{k}$ is conservative?

Q.3 Show that the force field $\vec{F} = x^2\hat{i} + 2yz\hat{j} + y^2\hat{k}$ is conservative.

Q.4 Show that the force $\vec{F} = 3x^2y^2\hat{i} + (2x^3y + \cos z)\hat{j} - y\sin z\hat{k}$ is conservative.

Q.5

SUMMARY

- If the force $\vec{F}$ on a particle always passes through a fixed point O, the force is called a central force and O is called the centre of force. A central force is repulsive or attractive according as it is directed away from or towards the centre of force.
- The momentum of a particle moving under the influence of no force is constant throughout the motion.
- The work done by $\vec{F}$ in moving the particle from P_1 to P_2 along the path of the particle is $W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$.
- Rate of working on the particle is known as the power applied to particle. Thus, the quantity $P = \dfrac{dW}{dt}$ is called the power applied to the particle.
- If m is the mass of the particle, then the quantity $T = \frac{1}{2}mv^2$ is called the kinetic energy of the particle, whose speed is v at time t.
- The total work done on a particle in moving it along a curve C from P_1 to P_2 is equal to the increase $T_2 - T_1$ in the kinetic energy, where T_1, T_2 are respectively the kinetic energies of the particle at times t_1, t_2 corresponding to the positions P_1, P_2.
- Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of a particle at time t and $\vec{F}$, the force at this instant. The field of force $\vec{F}$ is called conservative, if there exists, a scalar function V of x, y, z such that $\vec{F} = -\nabla V$.
- If the field of force $\vec{F}$ is conservative, then $\vec{F} = -\nabla V$. The function $V = V(x, y, z)$ is called the potential of the force $\vec{F}$ or the potential energy of the particle at time t or at the position of the particle at time t.
- The work done on the particle in moving it from a position P_1 to P_2 under a conservative field of force is the difference between the potential energies of the particle at P_1 and P_2 respectively.
- In a conservative field of force, the total energy of a particle remains constant throughout the motion.

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