Solution To Practice Exercise 1

Q1. Show that $i^{2n-1}+i^{2n}+i^{2n+1}+i^{2n+2}=1-i$, where $n\in E=$ Set of positive even integers.

Solution.

$$i^{2n-1}+i^{2n}+i^{2n+1}+i^{2n+2}$$
$$=i^{2n}i^{-1}+i^{2n}+i^{2n}i+i^{2n}i^2$$
$$=\left(i^2\right)^n\frac{1}{i}+\left(i^2\right)^n+\left(i^2\right)^n i+\left(i^2\right)^n(-1) \qquad \because i^2=-1$$
$$=\left(-1\right)^n\frac{i}{i.i}+\left(-1\right)^n+\left(-1\right)^n i+\left(-1\right)^n(-1)$$
$$=(1)(-i)+(1)+(1)i+(1)(-1) \qquad \because n\in E\Rightarrow\left(-1\right)^n=1$$
$$=-i+1+i-1=0 \quad \square$$

Q2. Evaluate $\frac{3}{i}-\frac{i}{3}$.

Solution.

$$\frac{3}{i}-\frac{i}{3}=\frac{3}{i}\times\frac{i}{i}-\frac{i}{3}=\frac{3i}{i^2}-\frac{i}{3}=\frac{3i}{(-1)}-\frac{i}{3}=-3i-\frac{i}{3}=\left(-3-\frac{1}{3}\right)i$$
$$=\left(-\frac{10}{3}\right)i$$

Q3. Discuss how many anti-clock quarter rotations, the following complex numbers will have? $-5i,-6,$ *and* $8i$

Solution.

$\because i$ graphically represents an anti-clock quarter rotation of $\frac{\pi}{2}$ radians.

$\Rightarrow$

i) $-5i$ represents one clockwise quarter rotation

ii) $-6=6(-1)=6i^2=6i\cdot i$, represents two anti-clock rotations

iii) $8i$ represents one anti-clock quarter rotation.

Q4. Simplify $i^{11}+i^{40}+i^{30}$.

Solution. As we know $i^2=-1$,

$$\begin{aligned}i^{11}+i^{40}+i^{30} &= (i^2)^5.i+(i^2)^{20}+(i^2)^{15}\\ &= (-1)^5 i+(-1)^{20}+(-1)^{15}\\ &= -i+1-1=-i\end{aligned}$$

Q5. Discuss why $i\neq 0$

Solution. Let $i=0$. Multiply both side by i, we get,

$\Rightarrow i.i=0.i$

$\Rightarrow i^2=0$

$\Rightarrow -1=0,$

which is false. Hence $i\neq 0$.

Q6. Find Principle and all other possible arguments of $1+i$.

Solution.

Here given that $z=x+iy=1+i,\ so\ \ x=1>0\ and\ y=1>0$

Therefore, terminal ray of $\text{Arg}\,z$ lies in 1st quadrant.

$$\therefore Arg(z)=\tan^{-1}\left(\frac{y}{x}\right)=\tan^{-1}\left(\frac{1}{1}\right)=\tan^{-1}(1)=\frac{\pi}{4},$$

And $\arg(z)=\frac{\pi}{4}+2n\pi,\ n\in Z$, that gives the all possible arguments.

Q7. Express $-1-\sqrt{3}i$ into polar form.

Solution.

Let $z=x+iy=-1-\sqrt{3}i$

$\Rightarrow |z|=\sqrt{1+3}=2$

So, $\theta=\tan^{-1}(\sqrt{3})=\frac{\pi}{3}$

Since $-1-\sqrt{3}i$ lies on 3rd quadrant, so $\theta=\frac{\pi}{3}-\pi=-\frac{2\pi}{3}$

Polar form is: $2cis(-\frac{2\pi}{3})$

Q8. Solve $x=\sqrt{-3+\sqrt{-3+\sqrt{-3+\cdots\infty}}}$ and show that difference of its roots is pure imaginary.

Solution.

$$x=\sqrt{-3+\sqrt{-3+\sqrt{-3+\cdots\infty}}}$$

Squaring;

$$x^2=-3+\underbrace{\sqrt{-3+\sqrt{-3+\cdots\infty}}}_{x}$$

$$\Rightarrow x^2=-3+x\Rightarrow x^2-x+3=0$$

$$\Rightarrow x=\frac{-(-1)\pm\sqrt{1-4(1)(3)}}{2}=\frac{1\pm\sqrt{-12}}{2}=\frac{1\pm i\sqrt{11}}{2}$$ are the roots of given equation.

Hence roots are: $x_1=\frac{1+i\sqrt{11}}{2}$ *and* $x_2=\frac{1-i\sqrt{11}}{2}$

Now difference $=x_1-x_2=\left(\frac{1+i\sqrt{11}}{2}\right)-\left(\frac{1-i\sqrt{11}}{2}\right)=\frac{1+i\sqrt{11}-1+i\sqrt{11}}{2}=\frac{2i\sqrt{11}}{2}=i\sqrt{11}$ i.e pure imaginary.

Q9. For a complex number $z\in\mathbb{C}$ if, $|z|=3$ and $Arg(z)=\frac{\pi}{3}$, then find $\frac{1}{\bar{z}}$.

Solution.

$$\frac{1}{\bar{z}}=\frac{z}{\bar{z}z}=\frac{z}{|z|^2}\quad\because |z|^2=\bar{z}z$$

$$=\frac{1}{(3)^2}z=\frac{1}{9}\left(|z|cis\theta\right)=\frac{1}{9}\left(3\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right)\right)$$

$$=\frac{1}{3}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=\frac{1}{6}\left(1+\sqrt{3}i\right)$$

Q10. Let $|z_1|=2$ and $z_2=-1+\sqrt{3}i$. Then by using Triangular Inequality, find the extreme values of $\left|\frac{z_1+z_2}{2}\right|$.

Solution.

$\because$ given that $|z_1|=2$ and $z_2=-1+\sqrt{3}i$. $\Rightarrow |z_2|=\sqrt{1+3}=2$

Now by using Triangular Inequality,

$$||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$$

$$|1-2| \le |z_1 + z_2| \le 1+2$$

$$\Rightarrow 1 \le |z_1 + z_2| \le 3$$

$$\Rightarrow \frac{1}{2} \le \left|\frac{z_1 + z_2}{2}\right| \le \frac{3}{2}$$

$$\Rightarrow \max\left\{\left|\frac{z_1 + z_2}{2}\right|\right\} = \frac{3}{2},\ \min\left\{\left|\frac{z_1 + z_2}{2}\right|\right\} = \frac{1}{2}$$

Q11. Show that the locus of point $P(z)$ satisfying; $\left|\frac{1+i\bar{z}}{\bar{z}+1}\right| = 2$ is

$3x^2 + 3y^2 + 8x - 2y + 3 = 0$.

Solution.

Let $z = x + iy$

$$\Rightarrow \left|\frac{1+i\bar{z}}{\bar{z}+1}\right| = 1$$

$$\Rightarrow \left|\frac{1+i\left(\overline{x+iy}\right)}{\overline{(x+iy)}+1}\right| = 1$$

$$\Rightarrow \frac{\left|1+i\left(\overline{x+iy}\right)\right|}{\left|\overline{(x+iy)}+1\right|} = 2 \qquad \because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$\Rightarrow |1+i(x-iy)| = 2|(x-iy)+1|$$

$$\Rightarrow |(1+y)+ix| = 2|(x+1)-iy|$$

$$\Rightarrow \sqrt{(1+y)^2 + x^2} = 2\sqrt{(x+1)^2 + (-y)^2}$$

$$\Rightarrow (1+y)^2 + x^2 = 4\left((x+1)^2 + (-y)^2\right)$$

$$\Rightarrow 1 + 2y + y^2 + x^2 = 4(x^2 + 2x + 1 + y^2)$$

$$\Rightarrow 1 + 2y + y^2 + x^2 = 4x^2 + 8x + 4 + 4y^2$$

$$\Rightarrow 4x^2 + 8x + 4 + 4y^2 - 1 - 2y - y^2 - x^2 = 0$$

$$\Rightarrow 3x^2 + 3y^2 + 8x - 2y + 3 = 0$$