Practice Questions of Lecture 7 to 9_Solution

Practice Qs of Lecture 7:

Q #1: Solve the equation: $e^{2ix} = 0 + i0$, where $x \in \mathbb{R}$.

Solution:

Here we will use the fact that $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{2ix} = 0 + i0 \Rightarrow \cos 2x + i\sin 2x = 0 + i0 \Rightarrow \cos 2x = 0 \;\; and \;\; \sin 2x = 0$$

$$\Rightarrow 2x = (2n+1)\frac{\pi}{2} \;\; and \;\; 2x = n\pi, \; n \in \mathbb{Z}$$

$$\Rightarrow x = (2n+1)\frac{\pi}{4} \; and \;\; x = \frac{n\pi}{2}, \; n \in \mathbb{Z}.$$

In particular $x = \frac{\pi}{4}$ and $x = 0, \frac{\pi}{2}.$

$, \Rightarrow \nexists \; x \in \mathbb{R}$ which simultaneously can satisfy the vanishing of both real and imaginary parts of given function.

Q #2: Show that the period of e^{2ix} is $2\pi i$, where $x \in \mathbb{R}$.

Solution:

For any complex valued function, $p \in \mathbb{C}$ is period $\Leftrightarrow f(z) = f(z+p)$

$$\therefore e^{i2x+2\pi i} = e^{i(2\pi+2x)} = \cos(2\pi+2x) + i\sin(2\pi+2x) \;\; \because \text{by Euler's formula}$$

$$= \cos 2x + i\sin 2x = e^{2ix} \quad \because \begin{cases} \cos(2\pi+2x) = \cos 2x \\ \sin(2\pi+2x) = \sin 2x \end{cases}$$

$\Rightarrow 2\pi i$ is a period of e^{2ix}.

Q #3:

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2 \in \mathbb{C}$, then show that $e^{z_1}e^{z_2} = e^{x_1+x_2}\left[\cos(y_1+y_2) + i\sin(y_1+y_2)\right]$.

Solution:

$$e^{z_1}e^{z_2} = e^{x_1+iy_1}e^{x_2+iy_2} = e^{x_1}e^{x_2}e^{iy_1}e^{iy_2} = e^{x_1+x_2}\left(\cos y_1 + i\sin y_1\right)\left(\cos y_2 + i\sin y_2\right)$$
$$= e^{x_1+x_2}\left(\cos y_1 \cos y_2 + i\cos y_1 \sin y_2 + i\sin y_1 \cos y_2 + i^2 \sin y_1 \sin y_2\right)$$
$$= e^{x+u}\left(\cos y_1 \cos y_2 + i\cos y_1 \sin y_2 + i\sin y_1 \cos y_2 - \sin y_1 \sin y_2\right)$$
$$= e^{x+u}\left(\left\{\cos y_1 \cos y_2 - -\sin y_1 \sin y_2\right\} + i\left\{\cos y_1 \sin y_2 + \sin y_1 \cos y_2\right\}\right)$$
$$= e^{x+u}\left(\cos\left(y_1+y_2\right) + i\sin\left(y_1+y_2\right)\right) \quad \square$$

Q #4: Show that $\left|e^{iz}\right| = e^{-y}$, for $z = \left(x+iy\right) \in \mathbb{C}$.

Solution:

$$\left|e^{iz}\right| = \left|e^{i(x+iy)}\right| = \left|e^{ix-y}\right| \qquad \because i^2 = -1$$

$$\left|e^{-y}e^{ix}\right| = \left|e^{-y}\right|\left|e^{ix}\right| \qquad As\left|z_1 z_z\right| = \left|z_1\right|\left|z_2\right|$$
$$\Rightarrow e^{-y}\left|\cos x + i\sin x\right| = e^{-y}(\sqrt{\cos^2 x + \sin^2 x}) = e^{-y}$$

Practice Qs of Lecture 8:

Q #5: If $e^{2ix} = \cos 2x + i\sin 2x$, then show that $\sin 2x = \dfrac{e^{2ix} - e^{-2ix}}{2i}$.

Solution:

$$e^{2ix} = \cos 2x + i\sin 2x \cdots\cdots(1)$$
$$\Rightarrow e^{-2ix} = \cos(-2x) - i\sin(-2x) = \cos 2x - i\sin 2x \cdots\cdots(2)$$

subtract eq. (1) and eq. (2),

$$e^{2ix} - e^{-2ix} = 2i\sin 2x \Rightarrow \boxed{\sin 2x = \frac{e^{2ix} - e^{-2ix}}{2i}} \quad \square$$

Q #6: If $e^{2ix} = \cos 2x + i\sin 2x$, then show that $\cos 2x = \dfrac{e^{2ix} + e^{-2ix}}{2}$.

Solution:

$$e^{2ix} = \cos 2x + i\sin 2x \cdots\cdots(1)$$
$$\Rightarrow e^{-2ix} = \cos(-2x) - i\sin(-2x) = \cos 2x - i\sin 2x \cdots\cdots(2)$$

Add eq. (1) and eq. (2),

$$e^{2ix}+e^{-2ix}=2\cos 2x \Rightarrow \boxed{\cos 2x=\frac{e^{2ix}+e^{-2ix}}{2}} \quad \square$$

Q #7: Show that $\left\{\frac{n\pi}{2}\right\}_{n\in\mathbb{Z}}$ is the solution set of equation $\cos z = 0.$

Solution:

We know that $\cos z \frac{e^{iz}+e^{-iz}}{2}$. Now consider,

$$\begin{aligned}
&\cos z = 0\\
&\Rightarrow \frac{e^{iz}+e^{-iz}}{2}=0\\
&\Rightarrow e^{iz}+e^{-iz}=0\\
&\Rightarrow e^{iz}+\frac{1}{e^{iz}}=0\\
&\Rightarrow \frac{e^{2iz}+1}{e^{iz}}=0\\
&\Rightarrow e^{2iz}+1=0\\
&\Rightarrow e^{2iz}=-1.................(1)
\end{aligned}$$

Since $\cos\pi + i\sin\pi = -1$, so we can write eq. (1) as:

$$\begin{aligned}
e^{2iz} &= \cos\pi + i\sin\pi\\
&= \cos(\pi+2\,\mathrm{k}\,\pi)+i\sin(\pi+2\,\mathrm{k}\,\pi) \qquad \text{for } \mathrm{k}\in\mathbb{Z}\\
&= e^{(\pi+2\mathrm{k}\pi)\mathrm{i}}.................(2)
\end{aligned}$$

Now we know that $e^{z_1}=e^{z_2}$ *if and only if* $z_1 = z_2 + 2im\pi,\ m\in\mathbb{Z}$,

So, from eq. (2), we have,

$$\begin{aligned}
2iz &= (\pi+2k\pi)\mathrm{i}+2im\pi\\
&= (1+2k+2m)\pi i\\
&= n\pi i, \quad \textit{where } n = 1+2k+2m \text{ is an integer}\\
&\Rightarrow 2z = n\pi\\
&\Rightarrow z = \frac{n\pi}{2}
\end{aligned}$$

Q #8: Prove that

$$1+\tan^2 z = \sec^2 z \text{ for all } z\in\mathbb{C}.$$

Solution:

We know that for any complex number z,

$$\tan z=\frac{\sin z}{\cos z}=\frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}\ \ and\ \ \sec z=\frac{1}{\cos z}=\frac{2}{e^{iz}+e^{-iz}},$$

$$So, \text{L.H.S}=1+\tan^2 z=1+\left(\frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}\right)$$

$$=1+\frac{e^{2iz}-2+e^{-2iz}}{i^2(e^{2iz}+2+e^{-2iz})}$$

$$=1-\frac{e^{2iz}-2+e^{-2iz}}{e^{2iz}+2+e^{-2iz}} \qquad \because i^2=1$$

$$=\frac{e^{2iz}+2+e^{-2iz}-(e^{2iz}-2+e^{-2iz})}{e^{2iz}+2+e^{-2iz}}$$

$$=\frac{4}{e^{2iz}+2+e^{-2iz}}=\left(\frac{2}{e^{iz}+e^{-iz}}\right)^2=\sec^2 z=\text{R.H.S}$$

Practice Qs of Lecture 9:

Q #9: Prove that $\sin iy = i\sinh y$.

(Hint: Use $\sin y=\frac{e^{iy}-e^{-iy}}{2i}$)

Solution:

$$\text{we have}\quad \sin iy=\frac{e^{i(\text{iy})}-e^{-i(\text{iy})}}{2i}=\frac{e^{-y}-e^{y}}{2i}=\frac{e^{-y}-e^{y}}{2i}\times\frac{i}{i}$$

$$=\frac{e^{-y}-e^{y}}{2i^2}\times i=i\frac{e^{y}-e^{-y}}{2}=i\sinh y$$

Q #10: Prove that $\cosh iy=\cos y$.

(Hint. Use $\cosh y=\frac{e^{y}+e^{-y}}{2}$)

Solution:

$$\cosh iy=\frac{e^{iy}+e^{-iy}}{2}=\cos y \qquad \text{,by definition}$$

Q #11: Prove that $\cosh^2 x-\sinh^2 x=1.$

Solution:

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \\ &= \frac{e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})}{4} \\ &= \frac{4}{4} \\ &= 1\end{aligned}$$

Q #12: Show that $\coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$.

Solution:

$$\begin{aligned}\coth x &= \frac{\cosh x}{\sinh x} \\ &= \frac{\frac{e^x + e^{-x}}{2}}{\frac{e^x - e^{-x}}{2}}, \\ &= \frac{e^x + e^{-x}}{2} \times \frac{2}{e^x - e^{-x}}, \\ &= \frac{e^x + e^{-x}}{e^x - e^{-x}}.\end{aligned}$$

Q #13:

Show that $\operatorname{csch} x = \dfrac{2}{e^x - e^{-x}}$.

Solution:

$$\begin{aligned}\operatorname{csch} x &= \frac{1}{\sinh x} \\ &= \frac{1}{\frac{e^x - e^{-x}}{2}}, \\ &= 1 \times \frac{2}{e^x - e^{-x}}, \\ &= \frac{2}{e^x - e^{-x}}.\end{aligned}$$