Q1. Solution.

$$\left(\cos\alpha + i\sin\alpha\right)^{\frac{1}{5}} = \left(\cos\left(\alpha+2k\pi\right)+i\sin\left(\alpha+2k\pi\right)\right)^{\frac{1}{5}},\ k\in\mathbb{Z}$$

$\therefore$ by De Moivre's theorem,

$$\left(\cos\alpha + i\sin\alpha\right)^{\frac{1}{5}} = \cos\left(\frac{\alpha+2k\pi}{5}\right)+i\sin\left(\frac{\alpha+2k\pi}{5}\right),\ k\in\mathbb{Z}$$

gives the required all possible values of $\left(\cos\alpha + i\sin\alpha\right)^{\frac{1}{5}}$.

Q2. Solution

$$e^{i\frac{\pi}{12}}e^{i\frac{\pi}{4}}e^{i\frac{2\pi}{3}} = e^{i\frac{\pi}{12}+i\frac{\pi}{4}+i\frac{2\pi}{3}} = e^{i\left(\frac{\pi}{12}+\frac{\pi}{4}+\frac{2\pi}{3}\right)} = e^{i\pi}$$

$\therefore$ by Euler's formula; $e^{i\pi} = \cos\left(\pi\right)+i\sin\left(\pi\right) = -1+i\left(0\right) = -1$

$$\Rightarrow\ e^{i\frac{\pi}{12}}e^{i\frac{\pi}{4}}e^{i\frac{2\pi}{3}} = -1$$

Q3. Solution

$$\lim_{n\to\infty}\left(z_1.z_2.\dots.z_n\right)$$
$$= \lim_{n\to\infty}\left(e^{i\frac{\pi}{2^1}}e^{i\frac{\pi}{2^2}}\dots e^{i\frac{\pi}{2^n}}\right) = \lim_{n\to\infty}e^{i\left(\frac{\pi}{2^1}+\frac{\pi}{2^2}+\dots+\frac{\pi}{2^n}\right)} = \lim_{n\to\infty}e^{i\pi\left(\frac{1}{2^1}+\frac{1}{2^2}+\dots+\frac{1}{2^n}\right)}$$

$$\because\begin{cases}\lim_{n\to\infty}\underbrace{\left(\frac{1}{2^1}+\frac{1}{2^2}+\dots+\frac{1}{2^n}\right)}_{\text{a geometric series}} = \frac{a}{1-r},\ \text{where } a=r=\frac{1}{2}\\[2ex] \qquad = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1\end{cases}$$

$$\therefore \lim_{n\to\infty}\left(z_1.z_2.\dots.z_n\right) = e^{i\pi} = \cos\left(\pi\right)+i\sin\left(\pi\right) = -1.$$

Q4. Solution

$\because \forall(x+iy)\in\mathbb{C}$, the polar form is given by; $x+iy = re^{i\theta}$, where

$$r = \left(x^2+y^2\right)^{\frac{1}{2}}\ \text{and}\ \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\therefore (x+iy)^{\frac{s}{t}}+(x-iy)^{\frac{s}{t}}$$

$$=\left(re^{i\theta}\right)^{\frac{s}{t}}-\left(re^{-i\theta}\right)^{\frac{s}{t}}=r^{\frac{s}{t}}\left(e^{i\theta\frac{s}{t}}-e^{-i\theta\frac{s}{t}}\right)$$

$$=r^{\frac{s}{t}}\left(\cos\left(\theta\frac{s}{t}\right)+i\sin\left(\theta\frac{s}{t}\right)+\cos\left(-\theta\frac{p}{q}\right)+i\sin\left(-\theta\frac{s}{t}\right)\right) \quad \because \text{by Euler's formula}$$

$$=r^{\frac{p}{q}}\left(\cos\left(\theta\frac{s}{t}\right)+i\sin\left(\theta\frac{s}{t}\right)+\cos\left(\theta\frac{s}{t}\right)-i\sin\left(\theta\frac{s}{t}\right)\right) \quad \because \{\cos(-\alpha)=\cos\alpha \text{ and } \sin(-\alpha)=-\sin\alpha$$

$$=2r^{\frac{p}{q}}\cos\left(\theta\frac{s}{t}\right)=2\left\{\left(x^2+y^2\right)^{\frac{1}{2}}\right\}^{\frac{s}{t}}\cos\left(\frac{s}{t}\tan^{-1}\left(\frac{y}{x}\right)\right)=2\left(x^2+y^2\right)^{\frac{s}{2t}}\cos\left(\frac{s}{t}\tan^{-1}\left(\frac{y}{x}\right)\right) \quad \square$$

Q5. Solution

$\because$ *we have given*;

$$\left(\cos x+i\sin x\right)^2=\cos 2x+i\sin 2x$$

$$\Rightarrow \left(\cos x\right)^2+\left(i\sin x\right)^2+2\cos x\left(i\sin x\right)=\cos 2x+i\sin 2x$$

$$\Rightarrow \left(\cos^2 x-\sin^2 x\right)+i\left(2\sin x\cos x\right)=\cos 2x+i\sin 2x$$

Comparing real parts;

$$\boxed{\cos 2x=\cos^2 x-\sin^2 x}$$

Q6. Solution

$\because$ we have given;

$$x-\frac{1}{x}=2i\cos y$$

$$\Rightarrow \frac{x^2-1}{x}=2i\cos y$$

$$\Rightarrow x^2-1=\left(2i\cos y\right)x$$

$$\Rightarrow x^2-\left(2i\cos y\right)x-1=0$$

which is quadratic in *x;*

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{(2i\,\text{cosy})\pm\sqrt{(2i\,\text{cosy})^2-4(1)(-1)}}{2}=\frac{(2i\,\text{cosy})\pm\sqrt{-4\cos^2 y+4}}{2}$$

$$=\frac{(2i\,\text{cosy})\pm 2\sqrt{-\cos^2 y+1}}{2}=i\,\text{cosy}\pm\sqrt{-\cos^2 y+1}=i\,\text{cosy}\pm\sqrt{\sin^2 y} \quad \because \cos^2 y=-\sin^2 y+1$$

$$\boxed{x=i\,\text{cosy}\pm\sin y}$$

Q7. solution

$1+i=1+i(1)\Rightarrow x=y=1>0\Rightarrow$ Terminal ray of $\text{Arg}(z)$ lies in 1st quadrant.

$$\Rightarrow r=\sqrt{x^2+y^2}=\sqrt{1^2+1^2}=\sqrt{2} \text{ and } \text{Arg}(z)=\tan^{-1}\left(\frac{1}{1}\right)=\tan^{-1}(1)=\frac{\pi}{4}$$

$$\therefore \text{polar form}: 1+i=r\,\text{cis}\,\theta=\sqrt{2}cis\left(\frac{\pi}{4}\right)$$

$$\Rightarrow (1+i)^{200}$$

$$=\left(\sqrt{2}cis\left(\frac{\pi}{4}\right)\right)^{200}=\left(\sqrt{2}\right)^{200}\left(cis\left(\frac{\pi}{4}\right)\right)^{200}$$

$$=\left(2^{\frac{1}{2}}\right)^{200} cis\left(200\frac{\pi}{4}\right) \quad \because \text{By De Moivre's theorem}$$

$$=2^{100}cis(50\pi)=2^{100}cis(49\pi+\pi)=2^{50}cis(\pi) \quad \because \theta'=\theta+2k\pi, \text{where } k\in\mathbb{Z}$$

$$=2^{100}(\cos\pi+i\sin\pi)=2^{100}(-1+i(0))=-2^{100}$$

Q8. solution

$$x^4-4=0$$

$$\Rightarrow (x^2)^2-(2)^2=0$$

$$\Rightarrow (x^2-2)(x^2+2)=0$$

$$\Rightarrow (x^2-2)=0 \ \ and\ \ (x^2+2)=0$$

$$\Rightarrow x=\pm\sqrt{2},\pm i\sqrt{2}$$

$$\therefore \text{Sum of roots} = \left(\sqrt{2}\right)+\left(-\sqrt{2}\right)+\left(i\sqrt{2}\right)+\left(-i\sqrt{2}\right)=0$$

Q9. Solution

Let

$$x^4=i=0+i(1)=\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)=cis\left(\frac{\pi}{2}\right)=cis\left(\frac{\pi}{2}+2k\pi\right)=\text{cis}\left((4k+1)\frac{\pi}{2}\right),\ k\in\mathbb{Z}$$

$$\Rightarrow x_k = \left\{\text{cis}\left((4k+1)\frac{\pi}{2}\right)\right\}^{\frac{1}{4}} = \text{cis}\left((4k+1)\frac{\pi}{8}\right) \quad \because \text{by De Moivre's theorem}$$

Put $k = 0,1,2,3$

$\Rightarrow$

$$x_0 = \text{cis}\left((4(0)+1)\frac{\pi}{8}\right) = cis\left(\frac{\pi}{8}\right)$$

$$x_1 = \text{cis}\left((4(1)+1)\frac{\pi}{8}\right) = cis\left(\frac{5\pi}{8}\right)$$

$$x_2 = \text{cis}\left((4(2)+1)\frac{\pi}{8}\right) = cis\left(\frac{9\pi}{8}\right)$$

$$x_3 = \text{cis}\left((4(3)+1)\frac{\pi}{8}\right) = cis\left(\frac{13\pi}{8}\right)$$