Practice Questions of Lecture 16 to 22_Solution

Q.1: Find the angle between the following pair of lines.

(a) $11x^2+16xy-y^2=0$

(b) $3x^2+7xy+2y^2=0$

Solution:

(a)

If we compare with general equation of pair of straight lines $ax^2+2hxy+by^2=0,$ we get :

$a=11\ ,\ b=-1\ ,\ h=8$

Let θ be the angle between the two lines, then

$$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}$$

$$\tan\theta \quad =\frac{2\sqrt{(8)^2-(11)(-1)}}{11-1}=\frac{2\sqrt{64+11}}{10}=\frac{\sqrt{75}}{5}$$

Hence $\quad \theta=\tan^{-1}\left(\frac{\sqrt{75}}{5}\right)=\frac{\pi}{3}$

(b)

Here $a=3\ ,\ b=2\ ,\ h=\frac{7}{2}$

Let θ be the angle between the two lines, then

$$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}$$

$$\tan\theta \quad =\frac{2\sqrt{\left(\frac{7}{2}\right)^2-(3)(2)}}{3+2}=\frac{2\sqrt{\frac{49-24}{4}}}{5}=\frac{\sqrt{25}}{5}=1$$

Hence $\quad \theta=\tan^{-1}(1)=\frac{\pi}{4}$

Q.2: For what value of λ will the following equation represent a pair of straight lines

$$4x^2-9y^2-2(8+\lambda)x-18y=29+2\lambda$$

Solution:

Compare the given equation with general equation of second degree in x, y which is $ax^2+2hxy+by^2+2gx+2fy+c=0$, we get :

Here $a=4$; $b=-9$; $c=-29-2\lambda$; $h=0$; $g=-(8+\lambda)$; $f=-9$,

Now, the condition for equation of pair of straight lines is:

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$

$$\begin{vmatrix} 4 & 0 & -(8+\lambda) \\ 0 & -9 & -9 \\ -(8+\lambda) & -9 & -29-2\lambda \end{vmatrix}=0$$

Expand the above determinant by 1st row, we get:

$$4(261+18\lambda-81)-(8+\lambda)(0-(72+9\lambda))=0$$
$$1296+216\lambda+9\lambda^2=0$$
$$\lambda^2+24\lambda+144=0$$
$$(\lambda+12)^2=0$$
$$\lambda=-12$$

Q.3: Express the equation $r=1+\sin\theta$ in rectangular coordinates.

Solution:

Given $r=1+\sin\theta$,

Multiply both sides by r, we get

$$r^2=r+r\sin\theta$$

As we know that $x=r\cos\theta, y=r\sin\theta, and\ r=\sqrt{x^2+y^2}$, so above eq. becomes:

$$x^2+y^2=\sqrt{x^2+y^2}+y$$

This is the required equation in rectangular coordinates.

Q.4: Express the equation $r = a\cos\theta, a > 0$ in rectangular coordinate system.

Solution:

Given $r = a\cos\theta$,

Multiply both sides by r, we get

$$r^2 = ar\cos\theta$$
$$x^2 + y^2 = ax$$
$$x^2 + y^2 - ax = 0$$

This is the required equation in Cartesian coordinate system.

Q.5: Find polar coordinates of the point P whose rectangular coordinates are (1, 1).

Solution:

Here

$$x = 1 \textit{ and } y = 1, so$$
$$r = \sqrt{1^2 + 1^2} = \sqrt{2} \textit{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

So, in polar coordinate, the point is $\left(\sqrt{2}, \frac{\pi}{4}\right)$.

Q.6: Find Cartesian coordinates of the point P whose polar coordinates are (16, 30^0).

Solution:

Here

$$r = 16 \textit{ and } \theta = 30^o, so$$
$$x = r\cos\theta = 16\cos 30^o = 14(\text{approx.})$$
$$y = r\sin\theta = 16\sin 30^o = 8$$

Hence required point in Cartesian coordinated is (14,8).

Q.7: Find the eccentricity and length of semi-latus rectum of the conic $\frac{4}{r}=5+4\sin\theta$.

Solution:

$$\textit{Given}\quad \frac{4}{r}=5+4\sin\theta$$

$$r=\frac{4}{5+4\sin\theta}$$

$$r=\frac{5}{5(1+\frac{4}{5}\sin\theta)}=\frac{1}{1+\frac{4}{5}\sin\theta}$$

Compair the equation with

$$r=\frac{l}{1+e\sin\theta}$$

we get

$l=1$(*required length of semi latus rectum*) and

$$\mathrm{e}=\frac{4}{5}$$

Q.8: Identify the conic $\frac{4}{r}=2+\sin\theta$. Find also its eccentricity and the length of latus-rectum.

Solution:

$$\text{Given}\quad \frac{4}{r}=2+\sin\theta$$

$$r=\frac{4}{2+\sin\theta}$$

$$r=\frac{4}{2(1+\frac{1}{2}\sin\theta)}=\frac{4/2}{1+\frac{1}{2}\sin\theta}$$

Compare the equation with

$$r=\frac{l}{1+e\sin\theta}$$

we get

$\mathrm{e}=\frac{1}{2}<1$ (*The conic is an ellipse*)

$l=2(2)=4$(*required length of latus rectum*)

Q.9: Find the angle ψ for the polar curve $r = a(1-\cos\theta)$ at $\theta = \frac{\pi}{2}$.

Solution:

Given $r = a(1-\cos\theta)$,

We know that $\tan\psi = r\frac{d\theta}{dr}$, so first we need to find $\frac{dr}{d\theta}$.

$$\frac{dr}{d\theta} = a(0+\sin\theta) = a\sin\theta$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{1}{a\sin\theta}.$$

Therefore,

$$\tan\psi = a(1-\cos\theta)\frac{1}{a\sin\theta} = \frac{1-\cos\theta}{\sin\theta}$$

at $\theta = \frac{\pi}{2}$,

$$\tan\psi = \frac{1-\cos\left(\pi/2\right)}{\sin\left(\pi/2\right)} = \frac{1-0}{1} = 1$$

$$\Rightarrow \psi = \tan^{-1}(1) = \frac{\pi}{4}$$

Q.10: Find the angle of intersection of the curves $r = 2$ *and* $r = 4\sin\theta$.

Solution:

Here $r_1 = 2$ *and* $r_2 = 4\sin\theta$

Solving both equations to find θ:

$$2 = 4\sin\theta$$

$$\Rightarrow \sin\theta = \frac{1}{2}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

Now, $\frac{dr_1}{d\theta} = 0$ and $\frac{dr_2}{d\theta} = 4\cos\theta$.

As we know that $\tan\psi = r\dfrac{d\theta}{dr}$, *so*

$$\tan\psi_1 = r_1\frac{d\theta}{dr_1} = 1(1/0) = \infty \Rightarrow \psi_1 = \tan^{-1}(\infty) = \frac{\pi}{2}$$

and,

$$\tan\psi_2 = r_2\frac{d\theta}{dr_2} = 4\sin\theta\left(\frac{1}{4\cos\theta}\right) = \tan\theta \Rightarrow \psi_2 = \theta$$

As $\theta = \dfrac{\pi}{6}$, *so* $\psi_2 = \dfrac{\pi}{6}$.

Now angle between two curves $= |\psi_1 - \psi_2| = \left|\dfrac{\pi}{2} - \dfrac{\pi}{6}\right| = \dfrac{\pi}{3}$