Practice Questions of Lecture 10 to 12_Solution

Q #1: Prove that $\cos ec^{-1}z = \frac{1}{i}\log\left(\frac{i+\sqrt{z^2-1}}{z}\right), \quad z \in \mathbb{C}$.

Solution:

$$Let \;\; w = \cos ec^{-1}z$$

$$\Rightarrow \text{cosec}\, w = z$$

$$\Rightarrow \frac{1}{\sin w} = z$$

$$\Rightarrow \sin w = \frac{1}{z} ......(1)$$

As we know that $\sin y = \frac{e^{iy}-e^{-iy}}{2}$, so eq.(1) can be written as:

$$\frac{e^{iw}-e^{-iw}}{2i} = \frac{1}{z}$$

$$\Rightarrow e^{iw}-e^{-iw} = \frac{2i}{z}$$

$$\Rightarrow e^{iw}-\frac{1}{e^{iw}} = \frac{2i}{z}$$

$$\Rightarrow \frac{(\mathrm{e}^{iw})^2-1}{e^{iw}} = \frac{2i}{z}$$

$$\Rightarrow z(\mathrm{e}^{iw})^2 - z = 2i\,\mathrm{e}^{iw}$$

$$\Rightarrow z(\mathrm{e}^{iw})^2 - 2i\,\mathrm{e}^{iw} - z = 0 \qquad .....(2)$$

Which is quadratic in e^{iw}, so apply quadratic formula to eq.(2), we get:

$$\mathrm{e}^{iw} = \frac{-(-2)\pm\sqrt{(-2i)^2-4(\mathrm{z})(-\mathrm{z})}}{2z}$$

$$= \frac{2\pm\sqrt{-4+4z^2}}{2z}$$

$$= \frac{1\pm\sqrt{z^2-1}}{z}$$

If we consider only positive value and ignore negative sign, then

$$\mathrm{e}^{iw} = \frac{1+\sqrt{z^2-1}}{z}, \quad \text{taking log on both sides, we get:}$$

$$iw = \log\left(\frac{1+\sqrt{z^2-1}}{z}\right)$$

$$w = \frac{1}{i}\log\left(\frac{1+\sqrt{z^2-1}}{z}\right)$$

$$\cos ec^{-1} z = \frac{1}{i}\log\left(\frac{1+\sqrt{z^2-1}}{z}\right)$$ Proved.

Q #2: Separate into real and imaginary parts of $\tan^{-1}(x+iy)$.

Solution:

Let $\tan^{-1}(x+iy)=a+ib$ ....(1)

so real part is a and imaginary part is b. We want to find a and b.

First we will show that $\tan^{-1}(x-iy)=a-ib.$

From (1), we have

$$x+iy=\tan(a+ib) \qquad ....(2)$$

As we know that $\begin{cases}\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\\ \tan(i\beta)=i\tanh\beta\end{cases}$

So, from (2), we have

$$x+iy=\frac{\tan a+\tan(ib)}{1-\tan a\tan(ib)}=\frac{\tan a+i\tanh b}{1-i\tan a\tanh b}$$

Taking conjugates of both the sides, we get

$$x-iy=\frac{\overline{\tan a+i\tanh b}}{\overline{1-i\tan a\tanh b}}=\frac{\tan a-i\tanh b}{1+i\tan a\tanh b}$$

$$=\frac{\tan a-\tan(ib)}{1-\tan a\tan(ib)}=\tan(a-ib)$$

whcih can be written as :

$$\tan^{-1}(x-iy)=a-ib \qquad ....(2)$$

Adding (1) and (2), we get

$$\tan^{-1}(x+iy)+\tan^{-1}(x-iy)=a+ib+a-ib=2a$$

$$\tan^{-1}\frac{(x+iy)+(\mathrm{x-iy})}{1-(\mathrm{x+iy})(\mathrm{x-iy})}=2a \qquad \text{because } \tan^{-1}\alpha+\tan^{-1}\beta=\tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$$

After simplification, we get

$$\tan^{-1}\frac{2x}{1-x^2-y^2}=2a$$

Hence,

$$a = \frac{1}{2}\tan^{-1}\frac{2x}{1-x^2-y^2} \qquad \text{.............}(3)$$

To find b, we subtract (1) and (2), we have

$$\tan^{-1}(x+iy)\tan^{-1}(x-iy) = a+ib-(a-ib) = 2ib$$

$$\tan^{-1}\frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)} = 2ib \qquad \text{because } \tan^{-1}\alpha - \tan^{-1}\beta = \tan^{-1}\left(\frac{\alpha-\beta}{1+\alpha\beta}\right)$$

$$\tan^{-1}\frac{2yi}{1+x^2+y^2} = 2ib$$

$$or \quad \frac{2yi}{1+x^2+y^2} = \tan(2ib) = i\tanh 2b$$

$$Hence, \quad 2b = \tanh^{-1}\frac{2y}{1+x^2+y^2}$$

$$or \qquad b = \frac{1}{2}\tanh^{-1}\frac{2y}{1+x^2+y^2} \qquad \text{.........}(4)$$

Q #3: For any complex number z, prove that $\sinh^{-1} z = \log(\text{z}+\sqrt{z^2+1})$.

Solution:

Let $w = \sinh^{-1} z$. Then $z = \sinh w$. So

$$= \frac{e^{w} + e^{-w}}{2}$$

$$= \frac{e^{2w} - 1}{2e^{w}}$$

Hence $2e^{w} z = e^{2w} - 1$

$$\Rightarrow e^{2w} - 2e^{w} z - 1 = 0$$

which is a quadratic equation in e^{w}. Therefore

$$e^{w} = \frac{2z \pm \sqrt{4z^{2} + 4}}{2}$$

$$= z \pm \sqrt{z^{2} + 1}$$

Now, taking logarithm of both the sides,

$$w = \log(z + \sqrt{z^{2} + 1})$$

Which is the required result.

Q #4: Find $\operatorname{Log} z$ if

(i) $z = 2i$ (ii) $z = -i$

(iii) $z = x,\ x > 0$ (iv) $z = 1 + \sqrt{3}i$

Solution:

(i)

here $z = 2i = 0 + 2i$,

Thus $|z| = \sqrt{0^{2} + 2^{2}} = 2$

and $Arg(\text{z}) = \tan^{-1}(\frac{y}{x}) = \tan^{-1}(\frac{2}{0}) = \tan^{-1}(\infty) = \frac{\pi}{2}$,

therefore $Log(2i) = \ln|z| + iArg(\text{z}) = \ln 2 + \frac{\pi}{2} i$.

(ii)

here $z=-i=0+(-1)i,$

Thus $|z|=\sqrt{0^2+(-1)^2}=1$

and $Arg(\mathrm{z})=\tan^{-1}(\frac{y}{x})=\tan^{-1}(\frac{-1}{0})=\tan^{-1}(\infty)=\frac{\pi}{2}.$

Since $z=-i$ lies below the $x-axis,$ so $Arg(\mathrm{z})=\frac{\pi}{2}-\pi=-\frac{\pi}{2}.$

Therefore $Log(-i)=\ln|z|+iArg(\mathrm{z})=\ln 1-\frac{\pi}{2}i=-\frac{\pi}{2}i.$

(iii)

here $z=x=x+0i,$

Thus $|z|=\sqrt{x^2+0^2}=x$

and $Arg(\mathrm{z})=\tan^{-1}(\frac{y}{x})=\tan^{-1}(\frac{0}{x})=\tan^{-1}(0)=0.$

Therefore $Log(\mathrm{x})=\ln|z|+iArg(\mathrm{z})=\ln x.$

(iv)

here $z=1+\sqrt{3}i,$

Thus $|z|=\sqrt{1^2+(\sqrt{3})^2}=2$

and $Arg(\mathrm{z})=\tan^{-1}(\frac{y}{x})=\tan^{-1}(\frac{\sqrt{3}}{1})=\tan^{-1}(\sqrt{3})=\frac{\pi}{3},$

Therefore $Log(1+\sqrt{3}i)=\ln|z|+iArg(\mathrm{z})=\ln 2+\frac{\pi}{3}i.$