

Practice Qs of Lecture 1 to 3_Solution

Q1. Show that $i^{2n-1} + i^{2n} + i^{2n+1} + i^{2n+2} = 1 - i$, where $n \in E =$ Set of positive even integers.

Solution.

$$\begin{aligned} & i^{2n-1} + i^{2n} + i^{2n+1} + i^{2n+2} \\ &= i^{2n}i^{-1} + i^{2n} + i^{2n}i + i^{2n}i^2 \\ &= (i^2)^n \frac{1}{i} + (i^2)^n + (i^2)^n i + (i^2)^n (-1) \quad \because i^2 = -1 \\ &= (-1)^n \frac{i}{i \cdot i} + (-1)^n + (-1)^n i + (-1)^n (-1) \\ &= (1)(-i) + (1) + (1)i + (1)(-1) \quad \because n \in E \Rightarrow (-1)^n = 1 \\ &= -i + 1 + i - 1 = 0 \quad \square \end{aligned}$$

Q2. Evaluate $\frac{3}{i} - \frac{i}{3}$.

Solution.

$$\begin{aligned} \frac{3}{i} - \frac{i}{3} &= \frac{3}{i} \times \frac{i}{i} - \frac{i}{3} = \frac{3i}{i^2} - \frac{i}{3} = \frac{3i}{(-1)} - \frac{i}{3} = -3i - \frac{i}{3} = \left(-3 - \frac{1}{3}\right)i \\ &= \left(-\frac{10}{3}\right)i \end{aligned}$$

Q3. Discuss how many anti-clock quarter rotations, the following complex numbers will have? $-5i, -6$, and $8i$

Solution.

$\because i$ graphically represents an anti-clock quarter rotation of $\frac{\pi}{2}$ radians.

\Rightarrow

i) $-5i$ represents one clockwise quarter rotation

ii) $-6 = 6(-1) = 6i^2 = 6i \cdot i$, represents two anti-clock rotations

iii) $8i$ represents one anti-clock quarter rotation.

Q4. Simplify $i^{11} + i^{40} + i^{30}$.

Solution. As we know $i^2 = -1$,

$$\begin{aligned}i^{11} + i^{40} + i^{30} &= (i^2)^5 \cdot i + (i^2)^{20} + (i^2)^{15} \\ &= (-1)^5 i + (-1)^{20} + (-1)^{15} \\ &= -i + 1 - 1 = -i\end{aligned}$$

Q5. Discuss why $i \neq 0$

Solution. Let $i = 0$. Multiply both side by i , we get,

$$\Rightarrow i \cdot i = 0 \cdot i$$

$$\Rightarrow i^2 = 0$$

$$\Rightarrow -1 = 0,$$

which is false. Hence $i \neq 0$.

Q6. Find Principle and all other possible arguments of $1+i$.

Solution.

Here given that $z = x + iy = 1 + i$, so $x = 1 > 0$ and $y = 1 > 0$

Therefore, terminal ray of $\text{Arg } z$ lies in 1st quadrant.

$$\therefore \text{Arg}(z) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4},$$

And $\arg(z) = \frac{\pi}{4} + 2n\pi$, $n \in \mathbb{Z}$, that gives the all possible arguments.

Q7. Express $-1 - \sqrt{3}i$ into polar form.

Solution.

$$\text{Let } z = x + iy = -1 - \sqrt{3}i$$

$$\Rightarrow |z| = \sqrt{1+3} = 2$$

$$\text{So, } \theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

Since $-1 - \sqrt{3}i$ lies on 3rd quadrant, so $\theta = \frac{\pi}{3} - \pi = -\frac{2\pi}{3}$

Polar form is: $2\text{cis}\left(-\frac{2\pi}{3}\right)$

Q8. Solve $x = \sqrt{-3 + \sqrt{-3 + \sqrt{-3 + \dots \infty}}}$ and show that difference of its roots is pure imaginary.

Solution.

$$x = \sqrt{-3 + \sqrt{-3 + \sqrt{-3 + \dots \infty}}}$$

Squaring;

$$x^2 = -3 + \underbrace{\sqrt{-3 + \sqrt{-3 + \dots \infty}}}_x$$

$$\Rightarrow x^2 = -3 + x \Rightarrow x^2 - x + 3 = 0$$

$$\Rightarrow x = \frac{-(-1) \pm \sqrt{1 - 4(1)(3)}}{2} = \frac{1 \pm \sqrt{-12}}{2} = \frac{1 \pm i\sqrt{11}}{2} \text{ are the roots of given equation.}$$

$$\text{Hence roots are: } x_1 = \frac{1 + i\sqrt{11}}{2} \text{ and } x_2 = \frac{1 - i\sqrt{11}}{2}$$

$$\text{Now difference} = x_1 - x_2 = \left(\frac{1 + i\sqrt{11}}{2} \right) - \left(\frac{1 - i\sqrt{11}}{2} \right) = \frac{1 + i\sqrt{11} - 1 + i\sqrt{11}}{2} = \frac{2i\sqrt{11}}{2} = i\sqrt{11} \text{ i.e.}$$

pure imaginary.

Q9. For a complex number $z \in \mathbb{C}$ if, $|z| = 3$ and $\text{Arg}(z) = \frac{\pi}{3}$, then find $\frac{1}{z}$.

Solution.

$$\frac{1}{z} = \frac{z}{zz} = \frac{z}{|z|^2} \quad \because |z|^2 = \bar{z}z$$

$$= \frac{1}{(3)^2} z = \frac{1}{9} (|z| \text{cis} \theta) = \frac{1}{9} \left(3 \left(\cos \left(\frac{\pi}{3} \right) + i \sin \left(\frac{\pi}{3} \right) \right) \right)$$

$$= \frac{1}{3} \left(\frac{1}{2} + i \frac{\sqrt{3}}{2} \right) = \frac{1}{6} (1 + \sqrt{3}i)$$

Q10. Let $|z_1| = 2$ and $z_2 = -1 + \sqrt{3}i$. Then by using Triangular Inequality, find the extreme values of $\left| \frac{z_1 + z_2}{2} \right|$.

Solution.

$$\because \text{given that } |z_1| = 2 \text{ and } z_2 = -1 + \sqrt{3}i. \Rightarrow |z_2| = \sqrt{1 + 3} = 2$$

Now by using Triangular Inequality,

$$\left| |z_1| - |z_2| \right| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

$$|1 - 2| \leq |z_1 + z_2| \leq 1 + 2$$

$$\Rightarrow 1 \leq |z_1 + z_2| \leq 3$$

$$\Rightarrow \frac{1}{2} \leq \left| \frac{z_1 + z_2}{2} \right| \leq \frac{3}{2}$$

$$\Rightarrow \max \left\{ \left| \frac{z_1 + z_2}{2} \right| \right\} = \frac{3}{2}, \min \left\{ \left| \frac{z_1 + z_2}{2} \right| \right\} = \frac{1}{2}$$

Q11. Show that the locus of point $P(z)$ satisfying; $\left| \frac{1 + i\bar{z}}{z + 1} \right| = 2$ is

$$3x^2 + 3y^2 + 8x - 2y + 3 = 0.$$

Solution.

Let $z = x + iy$

$$\Rightarrow \left| \frac{1 + i\bar{z}}{z + 1} \right| = 1$$

$$\Rightarrow \left| \frac{1 + i(\overline{x + iy})}{(x + iy) + 1} \right| = 1$$

$$\Rightarrow \left| \frac{1 + i(\overline{x + iy})}{(x + iy) + 1} \right| = 2 \quad \because \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$$

$$\Rightarrow |1 + i(x - iy)| = 2|(x - iy) + 1|$$

$$\Rightarrow |(1 + y) + ix| = 2|(x + 1) - iy|$$

$$\Rightarrow \sqrt{(1 + y)^2 + x^2} = 2\sqrt{(x + 1)^2 + (-y)^2}$$

$$\Rightarrow (1 + y)^2 + x^2 = 4((x + 1)^2 + (-y)^2)$$

$$\Rightarrow 1 + 2y + y^2 + x^2 = 4(x^2 + 2x + 1 + y^2)$$

$$\Rightarrow 1 + 2y + y^2 + x^2 = 4x^2 + 8x + 4 + 4y^2$$

$$\Rightarrow 4x^2 + 8x + 4 + 4y^2 - 1 - 2y - y^2 - x^2 = 0$$

$$\Rightarrow 3x^2 + 3y^2 + 8x - 2y + 3 = 0$$

