1-MCQ, Lecture 19, Marks-01

Which of the following would be an annihilator operator for $\pi x \cos\left(\sqrt{2}x\right)$?

D^2+2

$\left(D^2+2\right)^2$ (correct)

$D^2+2\pi$

$\left(D^2+2\pi\right)^2$

2-MCQ, Lecture 20, Marks-01

For the differential equation $y''-y=e^{2x}$, we get $y_1=e^x, y_2=e^{-x}, W(y_1,y_2)=-2, W_1=e^{2x}$ and $W_2=-e^x$.If $u_1'=-\frac{e^{3x}}{2}$, then $u_1=----$.

$-\frac{e^{3x}}{2}$

$\frac{e^{3x}}{2}$

$-\frac{e^{3x}}{6}$ (correct)

$\frac{e^{3x}}{6}$

3-MCQ, Lecture 21, Marks-01

The functions $x, 2x$ and $3x$ are the solutions of a differential equation, then these are linearly$-----$.

independent

dependent (correct)

4-MCQ, Lecture 22, Marks-01

Time period for a body performing Simple Harmonic motion, whose position at any point (x,t) is $x(t)=a\cos t+b\sin t$ is——-.

π

$\frac{\pi}{2}$

$\frac{\pi}{4}$

2π (correct)

5-Descriptive, Lecture 19, Marks-02

Determine the annihilator operator of $px\sin(qx)$,where $p,q\in\mathbb{R}$.

Solution:

$\because D^2(px)=pD^2x=p.0=0$

and $\left(D^2+q^2\right)\sin(qx)=D^2\left(\sin(qx)\right)+q^2\sin(qx)=-q^2\sin(qx)+q^2\sin(qx)=0$

$\therefore$Annihilator of $px\sin(qx)=\left(D^2+q^2\right)^2$ i.e. $\left(D^2+q^2\right)^2\left(px\sin(qx)\right)=0$

6-Descriptive, Lecture 20, Marks-02

For the differential equation:$a_2(x)y'+a_1(x)y+a_0(x)=f(x), a_2(x)\neq 0$, write the formulation for its general solution.

Solution:

$a_2(x)y'+a_1(x)y+a_0(x)=f(x)\Longrightarrow y'+\frac{a_1(x)}{a_2(x)}y+\frac{a_0(x)}{a_2(x)}=\frac{f(x)}{a_2(x)}$

$\Longrightarrow y'+p(x)y+q(x)=g(x)$, where $\frac{a_1(x)}{a_2(x)}=p(x), \frac{a_0(x)}{a_2(x)}=q(x)$ and $\frac{f(x)}{a_2(x)}=g(x)$.

Now its general solution will be given by;

$$\boxed{y = \underbrace{e^{-\int p(x)dx}\int e^{\int p(x)dx} g(x)dx}_{\text{Particular Solution}} + \underbrace{ce^{-\int p(x)dx}}_{\text{Complementary Solution}}}$$

7-Descriptive, Lecture 21, Marks-02

If we have a differential equation: $\frac{d^4y}{dx^4} + ky = x + e^{-x}$, then commemorate the idea of variation of parameters, write down the third column of W_3.

Solution:

$\because$ the given DE is of order 4

$\Longrightarrow$ there will be 4 linearly independent solutions.

$\Longrightarrow$Wronskian would be of order 4×4.

$\Longrightarrow$3rd column: $\begin{pmatrix} 0 \\ 0 \\ 0 \\ x + e^{-x} \end{pmatrix}$

8-Descriptive, Lecture 22, Marks-02

If the position x at any time t, for a body performing simple Harmonic motion is; $x(t) = \frac{\cos t}{2} + \frac{\sqrt{3}\sin t}{2}$, then find its amplitude of the vibration.

Solution:

Comparing the given: $x(t) = \frac{\cos t}{2} + \frac{\sqrt{3}\sin t}{2} = \frac{1}{2}\cos t + \frac{\sqrt{3}}{2}\sin t$, with the general form of SHM:

$x(t) = c_1 \cos t + c_2 \sin t \Longrightarrow c_1 = \frac{1}{2}$ and $c_2 = \frac{\sqrt{3}}{2}$

$\therefore$Amplitude$= \sqrt{c_1^2 + c_2^2} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

9-Descriptive, Lecture 19, Marks-03

If the complementary solution of $y'' + \pi y = \cos \pi x$ is $y_c = a\cos\sqrt{\pi}x + b\sin\sqrt{\pi}x$, then find the general solution of the higher-order homogeneous differential equation obtained by multiplying the given differential equation with the annihilator operator $D^2 + \pi^2$.

Solution:

Operator of the given DE is:

$\left(D^2 + \pi\right) y = \cos \pi x$

According to given condition;

$\left(D^2 + \pi^2\right)\left(D^2 + \pi\right) y = \left(D^2 + \pi^2\right)\cos \pi x$

$\Longrightarrow \left(D^2 + \pi^2\right)\left(D^2 + \pi\right) y = 0 \qquad \because \begin{cases} \text{Annihilator of } \cos \pi x \text{ is } D^2 + \pi^2 \\ \text{i.e. } \left(D^2 + \pi^2\right)\cos \pi x = 0 \end{cases}$

$\therefore$Associated auxiliary equation to the above Homogeneous DE is;

$\left(m^2 + \pi^2\right)\left(m^2 + \pi\right) = 0 \Longrightarrow m = \pm i\sqrt{\pi}, \pm i\pi$

$\therefore$ the general sol: $\boxed{y = k_1 \cos\sqrt{\pi}x + k_2 \sin\sqrt{\pi}x + k_3 \cos \pi x + k_4 \sin \pi x}$

10-Descriptive, Lecture 20, Marks-03

Find the auxiliary equation associated to the differential equation: $y'' + ay = \sec x$, for $a > 0$ and $a < 0$.

Solution:

For $a > 0$, if $y = e^{mx}$ be its solution, then associated homogeneous DE is;

$m^2e^{mx} + ae^{mx} = 0 \Longrightarrow e^{mx}\left(a + m^2\right) = 0 \Longrightarrow \boxed{m^2 + a = 0} \quad \because e^{mx} \neq 0$

For $a < 0 \exists b > 0$ in $\mathbb{R}$ such that $a = -b$, and if $y = e^{mx}$ be its solution, then associated homogeneous DE is;

$m^2e^{mx} - be^{mx} = 0 \Longrightarrow e^{mx}\left(-b + m^2\right) = 0 \Longrightarrow \boxed{m^2 - b = 0} \quad \because e^{mx} \neq 0$

11-Descriptive, Lecture 21, Marks-03

Determine W by applying the technique of the variation of parameters on the following differential equation:

$\frac{d^3y}{dx^3} + \pi\frac{dy}{dx} = \tan 2x$

Solution:

If $y = e^{mx}$ is the proposed solution of the associated homogeneous DE of the given, then the auxiliary eq: is

$m^3 + \pi m = 0 \Longrightarrow m\left(\pi + m^2\right) = 0 \Longrightarrow m = 0, 0 \pm i\sqrt{\pi}$

$\Longrightarrow y_c = ae^0 + e^0\left(b\cos\sqrt{\pi}x + d\sin\sqrt{\pi}x\right) = a.1 + b\cos\sqrt{\pi}x + d\sin\sqrt{\pi}x$

$\Longrightarrow$fundamental set$= \{1, \cos\sqrt{\pi}x, \sin\sqrt{\pi}x\}$

$\Longrightarrow W\left(1, \cos\sqrt{\pi}x, \sin\sqrt{\pi}x\right) = \begin{vmatrix} 1 & \cos\sqrt{\pi}x & \sin\sqrt{\pi}x \\ 0 & -\sqrt{\pi}\sin\sqrt{\pi}x & \sqrt{\pi}\cos\sqrt{\pi}x \\ 0 & -\pi\cos\sqrt{\pi}x & -\pi\sin\sqrt{\pi}x \end{vmatrix} = \pi\sqrt{\pi}\sin^2\sqrt{\pi}x +$
$\pi\sqrt{\pi}\cos^2\sqrt{\pi}x = \pi\sqrt{\pi}$

12-Descriptive, Lecture 22, Marks-03

Determine the equation of the simple harmonic motion for a mass weighing w lbs, which stretches a spring to a inches from an initial position.

Solution:

$\because$Weight$= F = W = w$lbs$\Longrightarrow mg = w \Longrightarrow m = \frac{w}{g} = \frac{w}{32}$

distance from Initial position$= s = a$ inches$= \frac{a}{12}$ft

$\therefore$By Hook's law, $F = ks \Longrightarrow \frac{w}{32} = k\frac{a}{12} \Longrightarrow k = \frac{3w}{8a}$

Now the eq: of SHM: $m\frac{d^2x}{dt^2} = -kx \Longrightarrow \frac{w}{32}\frac{d^2x}{dt^2} = -\frac{3w}{8a}x \Longrightarrow \boxed{\frac{d^2x}{dt^2} + \frac{12}{a}x = 0}$ is the required equation.

13-Descriptive, Lecture 19, Marks-05

If the complementary solution of $y'' + 3y' + 2y = e^x\sin(2x + 3)$ is $y_c = ae^{-x} + be^{-2x}$, then determine the form of its particular solution by applying **only** the concept of annihilator operator approach.

Solution:

Operator form of the given DE is;

$\left(D^2 + 3D + 2\right)y = e^x\sin(2x + 3)$

And the Annihilator of $e^{1x}\sin(2x+3) = D^2 + 1^2 + 2^2 - 2(1)D \equiv D^2 - 2D + 5$

i.e. $\left(D^2 - 2D + 5\right)e^x\sin(2x + 3) = 0$

$\therefore$Pre-operating by $\left(D^2 - 2D + 5\right)$ on both sides of given D.E

$\Longrightarrow \left(D^2 - 2D + 5\right)\left(D^2 + 3D + 2\right)y = \left(D^2 - 2D + 5\right)e^x\sin(2x + 3)$

$\Longrightarrow \left(D^2 - 2D + 5\right)\left(D^2 + 3D + 2\right)y = 0$

$\Longrightarrow$associated auxiliary eq: of above Homogeneous eq: is;

$\Longrightarrow \left(m^2 - 2m + 5\right)\left(m^2 + 3m + 2\right) = 0$

$\Longrightarrow m = -2, -1$ and $m^2 - 2m + 5 = 0 \Longrightarrow m = 1 + 2i, 1 - 2i$

$\therefore$ the general sol: $\boxed{y = Ae^{-x} + Be^{-2x} + e^{x}(D\cos 2x + E\sin 2x)}$

14-Descriptive, Lecture 20, Marks-05

Find the wronskian for the complementary solution of the differential equation:$y'' + ay = \sec x$,for $a > 0$; using the variation of parameter.

Solution:

For $a > 0$, if $y = e^{mx}$ be its solution, then associated homogeneous DE:

$y'' + ay = 0 \Longrightarrow m^2e^{mx} + ae^{mx} = 0 \Longrightarrow e^{mx}\left(a + m^2\right) = 0$

$\Longrightarrow m^2 + a = 0 \ \because e^{mx} \neq 0$

$\Longrightarrow m = 0 \pm i\sqrt{a}$

$\Longrightarrow$the complementary solution:$y_c = A\cos\left(\sqrt{a}x\right) + B\sin\left(\sqrt{a}x\right)$

$\Longrightarrow$Fundamental Set$= \{y_1, y_2\} = \{\cos\left(\sqrt{a}x\right), \sin\left(\sqrt{a}x\right)\}$

$\Longrightarrow$Wronskian$= W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos\left(\sqrt{a}x\right) & \sin\left(\sqrt{a}x\right) \\ -\sqrt{a}\sin\left(\sqrt{a}x\right) & \sqrt{a}\cos\left(\sqrt{a}x\right) \end{vmatrix} = \sqrt{a}\cos^2\left(\sqrt{a}x\right) + \sqrt{a}\sin^2\left(\sqrt{a}x\right) = \sqrt{a}\left(\cos^2\sqrt{a}x + \sin^2\sqrt{a}x\right) = \sqrt{a}$

15-Descriptive, Lecture 21, Marks-05

Solve the differential equation:$\frac{d^2y}{dx^2} + y = \tan^3 x$ by variation of parameters.Where:

$y_c = ay_1 + by_2 = a\cos x + b\sin x, W\left(y_1, y_2\right) = 1, W_1 = -\sin x\tan^3 x$ and $W_2 = \cos x\tan^3 x$.

Solution:

$\because$we have given the complementary solution and just to find the particular one, which is given by:

$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$,where $u_1 = \int \frac{W}{W_1}dx$ and $u_2 = \int \frac{W_2}{W}dx$.

Now $u_1 = \int \frac{W_1}{W}dx = \int \frac{-\sin x\tan^3 x}{1}dx = -\int \frac{\sin^4 x}{\cos^3 x}dx$

and $u_2 = \int \frac{W_2}{W}dx = \int \frac{\cos x\tan^3 x}{1}dx = \int \frac{\sin^3 x}{\cos^2 x}dx$

Here both the integrals are not of fundamental nature and will remain as such.

$\therefore$ the general sol:$y = y_c + y_p = a\cos x + b\sin x + u_1(x)y_1(x) + u_2(x)y_2(x) = a\cos x + b\sin x + \cos x\left(-\int \frac{\sin^4 x}{\cos^3 x}dx\right) + \sin x\left(\int \frac{\sin^3 x}{\cos^2 x}dx\right)$

$\boxed{y = a\cos x + b\sin x - \cos x\int \frac{\sin^4 x}{\cos^3 x}dx + \sin x\int \frac{\sin^3 x}{\cos^2 x}dx}$

16-Descriptive, Lecture 22, Marks-05

Express the solution:$x(t) = a\cos\omega t - b\sin\omega t$ of Simple Harmonic Motion(SHM) in the form of $x(t) = A\cos(\omega t + \phi)$.

Solution:

Firstly we look for to find A and ϕ

$\because x(t) = A\cos(\omega t - \phi)$ is the required form;

$\therefore$ {by using $\cos\left(\alpha - \beta\right) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\Longrightarrow x(t) = A\left(\cos t\omega\cos\phi + \sin t\omega\sin\phi\right) = \left(A\cos\phi\right)\cos t\omega + \left(A\sin\phi\right)\sin\omega t\left(A\sin\phi\right) = \left(A\cos\phi\right)\cos t\omega + A\sin\phi\sin\omega t$

Comparing this with given:$x(t) = a\cos\omega t - b\sin\omega t$

$\Longrightarrow a = A\cos\phi$ and $b = (-A\sin\phi)$

Squaring and Adding;

$a^2 + b^2 = (A\cos\phi)^2 + (A\sin\phi)^2 = A^2\cos^2\phi + A^2\sin^2\phi = A^2\left(\cos^2\phi + \sin^2\phi\right) = A^2.1 = A^2$

$\Longrightarrow A^2 = a^2 + b^2 \Longrightarrow A = \sqrt{a^2 + b^2}$

Again $a = A\cos\phi \Longrightarrow \frac{a}{A} = \cos\phi$ and $\frac{b}{A} = \sin\phi$

$\Longrightarrow \tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{\frac{b}{A}}{\frac{a}{A}} \Longrightarrow \phi = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{b}{a}\right)$

Now from the given:$x(t) = a\cos\omega t - b\sin\omega t = A\left(\frac{a}{A}\cos\omega t - \frac{b}{A}\sin\omega t\right)$ (*Note this step*)

$\Longrightarrow x(t) = A\left(\frac{a}{A}\cos\omega t - \frac{b}{A}\sin\omega t\right) = A\left(\cos\phi\cos\omega t - \sin\phi\sin\omega t\right) = A\left(\cos\omega t\cos\phi - \sin\omega t\sin\phi\right)$

$\because \{\cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha + \beta)$

$\therefore \boxed{x(t) = A\cos(\omega t + \phi) = \sqrt{a^2 + b^2}\cos\left(\omega t + \tan^{-1}\left(\frac{b}{a}\right)\right)}$ i.e. the required form.