Question-1 (MCQ, Marks-1, Lecture-13)
If the Wronskian of a set of functions is zero, then the set will be——.
Linearly independent *(correct)*
Linearly dependent
Neither dependent nor independent
None of these

Question-2 (MCQ, Marks-1, Lecture-14)
$W(x, x+1) =$———.
1
0
-1(*correct*)
$-x$

Question-3 (MCQ, Marks-1, Lecture-15)
If $y_1 = e^{-x}$ is the 1st solution of the differential equation:$y'' + 2y' + y = 0$, then which of the following will give second?
$xe^{-x} \int \frac{e^{-\int 2dx}}{e^{-2x}} dx$
$e^{-x} \int \frac{e^{-\int 2dx}}{e^{-2x}} dx \quad (correct)$
$e^{-x} \int \frac{e^{-2x}}{e^{-\int 2dx}} dx$
$xe^{-x} \int \frac{e^{-2x}}{e^{-\int 2dx}} dx$

Question-4 (Marks-2, Lecture-13)
Show that the functions $f(x) = \sin x$ and $g(x) = 2\sin x$ are linearly dependent.
Solution:
$W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} \sin x & 2\sin x \\ \cos x & 2\cos x \end{vmatrix} = 2\sin x\cos x - 2\sin x\cos x = 0 \Longrightarrow$ the given functions are linearly dependent.

Question-5 (Marks-2, Lecture-14)
Show that solutions $f(x) = e^x$ and $g(x) = e^{-x}$ of the differential equation: $y'' - y = 0$, will form the fundamental set.
Solution:
$$\because W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^x e^{-x} - e^x e^{-x} = -2 \neq 0$$
$\Longrightarrow \{f, g\}$ is linearly independent and hence will form a fundamental set.

Question-6 (Marks-2, Lecture-15)
Evaluate $e^{-\int p(x)dx}$ by using the $\cos x.y'' - \sin x.y' + e^x y = 0$ by writing it in the standard form: $y'' + p(x)y' + q(x).y = 0$.
Solution:
$\cos x.y'' - \sin x.y' + e^x y = 0 \Longrightarrow y'' - \tan x.y' + \sec x e^x.y = 0$
$\Longrightarrow p(x) = -\tan x \Longrightarrow e^{-\int p(x)dx} = e^{\int \tan x dx} = e^{\int \frac{\sin x}{\cos x} dx} = e^{-\int \frac{-\sin x}{\cos x} dx} = e^{-\ln(\cos x)} = e^{\ln(\cos x)^{-1}} = (\cos x)^{-1} = \sec x$

Question-7(Marks-3, Lecture-13)

Given that $y = Ae^x$ is one parameter family of curves satisfying the differential equation: $\frac{dy}{dx} - y = 0$.Then find a member of family satisfying $y(0) = 1$.

Solution:

Given that $y = Ae^x$ under $y(0) = 1$

Put $x = 0$ and $y = 1$ in the given solution.

$\therefore 1 = Ae^0 \implies A = 1$, put this in the given solution.

$\therefore y = 1e^x = e^x$ is the required member of the given family.

Question-8(Marks-3, Lecture-14)

Determine whether the following functions are linearly dependent or independent;

$f(x) = a, g(x) = ax$ and $h(x) = ax^2$, $a \in \mathbb{R} - \{0\}$.

Solution:

$$\text{Taking } W(f,g,h) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} = \begin{vmatrix} a & ax & ax^2 \\ 0 & a & 2ax \\ 0 & 0 & 2a \end{vmatrix} = 2a^3 \neq 0$$

$\implies$ functions are linearly independent.

Question-9(Marks-3, Lecture-15)

If $y_1(x) = e^{-x}$ be the 1st solution of the differential equation:$y''+2y'+y = 0$, then construct its 2nd solution by using:

$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)dx}}{y_1^2(x)}$, where $p(x)$ is the coefficient of y'.

Solution:

Here $p(x) = 2$

$\implies y_2(x) = y_1(x) \int \frac{e^{-\int p(x)dx}}{y_1^2(x)} = e^{-x} \int \frac{e^{-\int 2dx}}{e^{-2x}} dx = e^{-x} \int \frac{e^{-2x}}{e^{-2x}} dx = e^{-x} \int 1dx = xe^{-x} + c$

Question-10 (Marks-5, Lecture-15)

If $y_1(x) = \sin x$ is the 1st solution of the differential equation: $y'' + 9y = 0$, then find its 2nd solution by using

$y_2(x) = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx$. Also determine their linear dependence or independence.

Solution:

Given that $y'' + 9y = 0 \implies p(x) = 0$

$\therefore y_2(x) = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx = \sin x \int \frac{e^{-\int (0)dx}}{\sin^2 x} dx = \sin x \int \frac{e^c}{\sin^2 x} dx = \sin x \int A \csc^2 x dx$, where $A = e^c \neq 0$

$\implies y_2(x) = A \sin x \int \csc^2 x dx = -A \sin x \cot x = -A \cos x$ i.e. the required 2nd solution.

Now for linear independence:

$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin x & -A \cos x \\ \cos x & A \sin x \end{vmatrix} = A \sin^2 x + A \cos^2 x = A = e^c \neq 0$

$\Longrightarrow \{y_1, y_2\}$ is linearly independent.

Question-11 (Marks-5, Lecture-14)

Solve the differential equation:$y'' + ay = 0, a > 0$ subject to the boundary conditions: $y(0) = 1$ and $y\left(\frac{\pi}{2\sqrt{a}}\right) = -1$.

Solution:

Let $y = e^{mx}$ is the solution of the given differential equation.

$\Longrightarrow \ y' = me^{mx} \Longrightarrow y'' = m^2e^{mx}$

$\therefore$the given differential equation becomes;

$m^2e^{mx} + ae^{mx} = 0 \Longrightarrow e^{mx}\left(m^2 + a\right) = 0$

$\Longrightarrow m^2 + a = 0 \quad \because e^{mx} \neq 0$

$\Longrightarrow m = \pm i\sqrt{a}$, where the roots are imaginary.

$\therefore$ the solution is of the form: $y = \alpha \cos \sqrt{a}x + \beta \sin \sqrt{a}x$

Put $y(0) = 1$ and $y\left(\frac{\pi}{2\sqrt{a}}\right) = -1$ in the above;

$1 = \alpha \cos 0 + \beta \sin 0 \Longrightarrow \boxed{\alpha = 1}$

For $y\left(\frac{\pi}{2\sqrt{a}}\right) = -1; \quad -1 = \alpha \cos\left(\sqrt{a}\frac{\pi}{2\sqrt{a}}\right) + \beta \sin\left(\sqrt{a}\frac{\pi}{2\sqrt{a}}\right) \Longrightarrow -1 = \alpha \cos\left(\frac{1}{2}\pi\right) + \beta \sin\left(\frac{1}{2}\pi\right)$

$\Longrightarrow -1 = \alpha(0) + \beta.1 \Longrightarrow \boxed{\beta = -1}$

$\therefore$ the required particular solution:$\boxed{y = \cos \sqrt{a}x - \sin \sqrt{a}x}$

Question-12 (Marks-5, Lecture-13)

The two solutions of the differential equation: $y'' + 2y' + y = 0$ are e^{-x} and xe^{-x},then prove or disprove that $y = Ae^{-x} + Bxe^{-x}$ is its general solution.

Solution:

$y = Ae^{-x} + Bxe^{-x} \Longrightarrow y' = \frac{d}{dx}\left(Ae^{-x} + Bxe^{-x}\right) = -e^{-x}\left(A - B + Bx\right)$

$y'' = \frac{d}{dx}\left(-e^{-x}\left(A - B + Bx\right)\right) = e^{-x}\left(A - 2B + Bx\right)$

$\therefore y'' + 2y' + y$

$= e^{-x}\left(A - 2B + Bx\right) + 2\left(-e^{-x}\left(A - B + Bx\right)\right) + Ae^{-x} + Bxe^{-x}$

$= e^{-x}\left(A - 2B + Bx - 2\left(A - B + Bx\right) + A + Bx\right) = e^{-x}\left(A - 2B + Bx + 2B - 2A - 2Bx + A + Bx\right)$

$= e^{-x}(0) = 0$

$\Longrightarrow y = Ae^{-x} + Bxe^{-x}$ is the general solution of the given differential equation.