Lecture 5
Exact Differential Equations

Firstly we express any given first order differential equation say $\frac{dy}{dx} = f(x,y)$ in the form as below;
$M(x,y)dx + N(x,y)dy = 0 - - - - - - (1)$
Now this equation would be exact if its LHS is an exact differential of any function say $F(x,y)$ i.e. $M(x,y)dx + N(x,y)dy = d\left(F(x,y)\right) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$. So the given differential equation becomes;
$d\left(F(x,y)\right) = 0$
On integrating, $\int d\left(F(x,y)\right) = \int 0dx \Longrightarrow \boxed{F(x,y) = C}$, which is the solution of given differential equation.
Working Rule of solution:
i) Integrate M with respect to x treating y as a constant.
ii) In N, choose those terms which do not involve x and integrate these with respect to y.
iii) Sum the above two cases and equate this to a constant.
Symbolically the final solution has the formulation;

$$\boxed{\underbrace{\int M(x,y)dx}_{y-\text{constant}} + \int \left(\text{terms of } N \text{ not containing } x\right)dy = C}$$

Example 1
Solve $\left(3x^2y + 2\right)dx + \left(x^3 + y\right)dy = 0.$
Solution:
Here $M(x,y) = 3x^2y + 2$ and $N(x,y) = x^3 + y$
$\Longrightarrow \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(3x^2y + 2\right) = \frac{\partial}{\partial y}\left(3x^2y\right) + \frac{\partial}{\partial y}2 = 3x^2\frac{\partial}{\partial y}\left(y\right) + 0 = 3x^2$ and
$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(x^3 + y\right) = \frac{\partial}{\partial x}x^3 + \frac{\partial}{\partial x}y = 3x^2 + 0 = 3x^2$
$\therefore \frac{\partial M}{\partial y} = 3x^2 = \frac{\partial N}{\partial x}$
$\Longrightarrow$the given differential equation is exact and its solution is given by;

$$\underbrace{\int M(x,y)dx}_{y-\text{constant}} + \int \left(\text{terms of } N \text{ not containing } x\right)dy = C$$

$$\Longrightarrow \underbrace{\int \left(3x^2y + 2\right)dx}_{y-\text{constant}} + \int ydy = C$$

$$\Longrightarrow \underbrace{\int 3x^2ydx + \int 2dx}_{y-\text{constant}} + \int ydy = C$$

$\Longrightarrow y\int 3x^2dx + 2\int dx + \frac{y^2}{2} = C$
$\Longrightarrow y\left(3\frac{x^3}{3}\right) + 2x + \frac{y^2}{2} = C$
$\Longrightarrow \boxed{x^3y + 2x + \frac{y^2}{2} = C}$ is the required solution.

Example 2
Solve the initial value problem $\left(2y\sin x\cos x + y^2\sin x\right)dx + \left(\sin^2 x - 2y\cos x\right)dy = 0, y(0) = 3.$

Solution:
Here $M(x,y) = 2y\sin x\cos x + y^2\sin x$ and $N(x,y) = \sin^2 x - 2y\cos x$

$$\begin{aligned}\Longrightarrow \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2y\sin x\cos x + y^2\sin x\right) = \frac{\partial}{\partial y}2y\sin x\cos x + \frac{\partial}{\partial y}\left(y^2\sin x\right)\\ &= \sin x\cos x\frac{\partial}{\partial y}\left(2y\right) + \sin x\frac{\partial}{\partial y}\left(y^2\right) = \sin x\cos x\left(2\right) + \sin x\left(2y\right)\\ &= 2\sin x\cos x + 2y\sin x\end{aligned}$$

$$\begin{aligned}\text{and } \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\sin^2 x - 2y\cos x\right) = \frac{\partial}{\partial x}\sin^2 x - \frac{\partial}{\partial x}\left(2y\cos x\right)\\ &= 2\cos x\sin x - 2y\frac{\partial}{\partial x}\left(\cos x\right)\\ &= 2\cos x\sin x - 2y\left(-\sin x\right) = 2\sin x\cos x + 2y\sin x\end{aligned}$$

$\therefore \frac{\partial M}{\partial y} = 2\sin x\cos x + 2y\sin x = \frac{\partial N}{\partial x}$
$\Longrightarrow$ the given differential equation is exact and its solution is given by;

$\underbrace{\int M(x,y)dx}_{y-\text{constant}} + \int (\text{terms of } N \text{ not containing } x)\, dy = C$

$\Longrightarrow \underbrace{\int \left(2y\sin x\cos x + y^2\sin x\right) dx}_{y-\text{constant}} + \int (0)\, dy = C$

$\Longrightarrow \int 2y\sin x\cos x dx + \int y^2\sin x dx + k = C$

$\Longrightarrow 2y\int \cos x\sin x dx + y^2\int \sin x dx = C - k$

$\because \left\{ \begin{array}{c} \int f'(x)f(x)dx = \frac{f(x)^2}{2} \text{ and here } f(x) = \sin x \text{ and } f'(x) = \cos x \\ \text{and } \int \sin x dx = -\cos x \end{array} \right.$

$\Longrightarrow 2y\frac{\sin^2 x}{2} + y^2(-\cos x) = C - k = K$,where $C - k = K$ say!

$\Longrightarrow y\sin^2 x - y^2\cos x = K - - - -(1)$

To find the value of K, we use the initial value condition in (1) i.e. put $x = 0$ and $y = 3$.

$\therefore (1) \Longrightarrow 3\sin^2(0) - 3^2\cos(0) = K$

$\Longrightarrow 0 - 9(1)^2 = K \Longrightarrow \boxed{K = -9}$, put this in (1)

$\therefore (1) \Longrightarrow y\sin^2 x - y^2\cos x = -9 \Longrightarrow \boxed{y^2\cos x - y\sin^2 x = 9}$, is the required solution of given initial value problem.

Example 3

Solve the DE $\left(e^{2y} - y\cos xy\right) dx + \left(2xe^{2y} - x\cos xy + 2y\right) dy = 0$.

Solution:

Here $M(x,y) = e^{2y} - y\cos xy$ and $N = 2xe^{2y} - x\cos xy + 2y$

$\Longrightarrow \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(e^{2y} - y\cos xy\right) = \frac{\partial}{\partial y}e^{2y} - \frac{\partial}{\partial y}y\cos xy = 2e^{2y} - \left(y\frac{\partial}{\partial y}\cos xy + \cos xy\frac{\partial}{\partial y}y\right)$

$= 2e^{2y} - (y(-x\sin xy) + \cos xy(1))$

$= 2e^{2y} - \cos xy + xy\sin xy$

and $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(2xe^{2y} - x\cos xy + 2y\right) = \frac{\partial}{\partial x}2xe^{2y} - \frac{\partial}{\partial x}x\cos xy + \frac{\partial}{\partial x}2y$

$= 2e^{2y}\frac{\partial}{\partial x}x - \left(x\frac{\partial}{\partial x}\cos xy + \cos xy\frac{\partial}{\partial x}x\right) + \frac{\partial}{\partial x}2y$

$= 2e^{2y}.1 - (x(-y\sin xy) + \cos xy.1) + 0$

$= 2e^{2y} - -xy\sin xy + \cos xy$

$= 2e^{2y} - \cos xy + xy\sin xy$

$\therefore \frac{\partial M}{\partial y} = 2e^{2y} - \cos xy + xy\sin xy = \frac{\partial N}{\partial x} \Longrightarrow$ the given differential equation is exact and its solution is given by;

$\underbrace{\int M(x,y)dx}_{y-\text{constant}} + \int (\text{terms of } N \text{ not containing } x)\, dy = C$

$\Longrightarrow \underbrace{\int \left(e^{2y} - y\cos xy\right) dx}_{y-\text{constant}} + \int 2ydy = C$

$\Longrightarrow \int e^{2y}dx - \int y\cos xydx + 2\int ydy = C$

$\Longrightarrow e^{2y}\int dx - \int (y\cos xy)\, dx + 2\left(\frac{1}{2}y^2\right) = C$

$\Longrightarrow e^{2y}x - \sin xy + y^2 = C \qquad \because \left\{\int f'(x)\cos(f(x))\, dx = \sin(f(x))\right.$

$\Longrightarrow \boxed{xe^{2y} - \sin xy + y^2 = C}$, is the required solution of given differential equation.

Example 4

Solve $2xydx + \left(x^2 - 1\right) dy = 0$.

Solution

Here $M(x,y) = 2xy$ and $N(x,y) = x^2 - 1$

$\Longrightarrow \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2xy) = 2x$ and $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(x^2 - 1\right) = 2x$

$\therefore \frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x} \Longrightarrow$ the given differential equation is exact and its solution is given by;

$$\underbrace{\int M(x,y)dx}_{y-\text{constant}} + \int (\text{terms of } N \text{ not containing } x)\, dy = C$$

$$\Longrightarrow \underbrace{\int 2xydx}_{y-\text{constant}} + \int (-1)\, dy = C$$

$\Longrightarrow 2y \int xdx + (-1) \int dy = C$

$\Longrightarrow 2y \left(\frac{1}{2}x^2\right) + (-1)\, y = C \Longrightarrow y\left(x^2-1\right) = C \Longrightarrow \boxed{y = \frac{C}{(x^2-1)}}$, is the required solution of given differential equation.

Example 5

Solve the initial value problem $\left(\cos x \sin x - xy^2\right) dx + y\left(1-x^2\right) dy = 0, \;\; y(0) = 2.$

Solution:

Here $M(x,y) = \cos x \sin x - xy^2$ and $N(x,y) = y\left(1-x^2\right) = y - x^2y$

$\Longrightarrow \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\cos x \sin x - xy^2\right)$

$= \frac{\partial}{\partial y}\left(\cos x \sin x\right) - \frac{\partial}{\partial y} xy^2$

$= 0 - x\frac{\partial}{\partial y} y^2 = -2xy$

and $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} y\left(1-x^2\right) = y\frac{\partial}{\partial x}\left(1-x^2\right)$

$= y\left(\frac{\partial}{\partial x}1 - \frac{\partial}{\partial x}x^2\right) = y\left(0-2x\right) = -2xy$

$\therefore \frac{\partial M}{\partial y} = -2xy = \frac{\partial N}{\partial x} \Longrightarrow$ the given differential equation is exact and its solution is now given by;

$$\underbrace{\int M(x,y)dx}_{y-\text{constant}} + \int (\text{terms of } N \text{ not containing } x)\, dy = C$$

$$\Longrightarrow \underbrace{\int \left(\cos x \sin x - xy^2\right) dx}_{y-\text{constant}} + \int ydy = C$$

$\Longrightarrow \int \cos x \sin x dx - \int xy^2 dx + \frac{1}{2}y^2 = C$

$\because \left\{\int f'(x)f(x)dx = \frac{f(x)^2}{2}\right.$ and here $f(x) = \sin x$ and $f'(x) = \cos x$

$\Longrightarrow \frac{\sin^2 x}{2} - y^2 \int xdx + \frac{1}{2}y^2 = C$

$\Longrightarrow \frac{\sin^2 x}{2} - y^2\left(\frac{1}{2}x^2\right) + \frac{1}{2}y^2 = C - - - -(1)$

Now put the initial condition $y(0) = 2$ (put $x = 0$ and $y = 2$ in (1))

$\therefore (1) \Longrightarrow \frac{\sin^2 0}{2} - 2^2\left(\frac{1}{2}0^2\right) + \frac{1}{2}2^2 = C$

$\Longrightarrow 0 - 0 + 2 = C \Longrightarrow C = 2$, put this in (1)

$\therefore (1) \Longrightarrow \frac{\sin^2 x}{2} - y^2\left(\frac{1}{2}x^2\right) + \frac{1}{2}y^2 = 2$

$\Longrightarrow \frac{\sin^2 x - x^2y^2 + y^2}{2} = 2 \Longrightarrow \sin^2 x - x^2y^2 + y^2 = 4$

$\Longrightarrow 1 - \cos^2 x - x^2y^2 + y^2 = 4$

$\Longrightarrow -x^2y^2 + y^2 - \cos^2 x = 4 - 1$

$\Longrightarrow \boxed{y^2(1-x^2) - \cos^2 x = 3}$, is the required solution of the given differential equation.