Lecture No -6

Geometry of Continuous Functions

Geometry of continuous functions in one variable or Informal definition of continuity of function of one variable.

A function is continuous if we draw its graph by a pen then the pen is not raised so that there is no gap in the graph of the function

Geometry of continuous functions in two variables or Informal definition of continuity of function of two variables.

The graph of a continuous function of two variables to be constructed from a thin sheet of clay that has been hollowed and pinched into peaks and valleys without creating tears or pinholes.

Continuity of functions of two variables

A function f of two variables is called continuous at the point (x_0, y_0) if

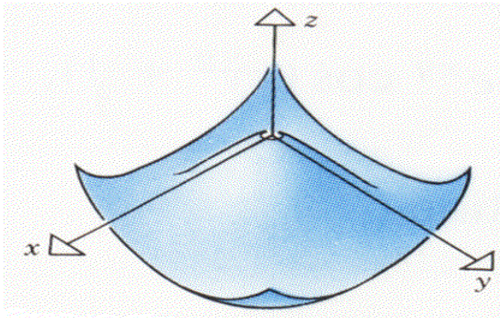
1. $f(x_0,y_0)$ if defined.
2. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists.
3. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.

The requirement that $f(x_0,y_0)$ must be defined at the point (x_0,y_0) eliminates the possibility of a hole in the surface $z = f(x_0,y_0)$ above the point (x_0,y_0).

Justification of three points involving in the definition of continuity.

(1) Consider the function of two variables $x^2 + y^2 \ln(x^2 + y^2)$ now as we know that the Log function is not defined at 0, it means that when x = 0 and y = 0, our function $x^2 + y^2 \ln(x^2 + y^2)$ is not defined.

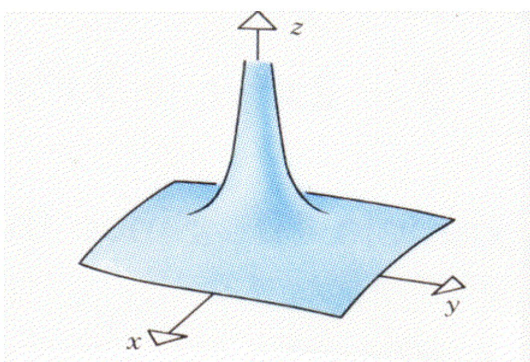
Consequently the surface $z = x^2 + y^2 \ln(x^2 + y^2)$ will have a hole just above the point (0,0) as shown in the graph of $x^2 + y^2 \ln(x^2 + y^2)$



$z = (x^2 + y^2)\ln(x^2 + y^2)$ has a hole at the origin.

(2) The requirement that $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists ensures us that the surface z = f(x,y) of the function f(x, y) doesn't become infinite at (x_0, y_0) or doesn't oscillate widely.

Consider the function of two variables $\frac{1}{\sqrt{x^2+y^2}}$. Now as we know that the Natural domain of the function is whole the plane except origin. Because at origin, we have x = 0 and y =0. In the defining formula of the function, we will have $\frac{1}{0}$ at that point which is infinity. Thus the limit of the function $\frac{1}{\sqrt{x^2+y^2}}$ does not exist at origin. Consequently the surface $z = \frac{1}{\sqrt{x^2+y^2}}$ will approach towards infinity when we approach towards origin as shown in the figure above.



$z = \frac{1}{\sqrt{x^2+y^2}}$ becomes infinite at the origin.

(3) The requirement that

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

ensures us that the surface $z = f(x,y)$ of the function $f(x,y)$ doesn't have a vertical jump or step above the point (x_0,y_0).

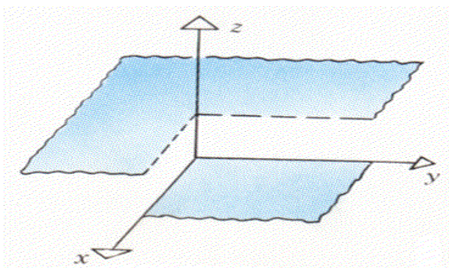
Consider the function of two variables

$$f(x,y) = \begin{cases} 0 & if \;\; x \geq 0 and \; y \geq 0 \\ 1 & otherwise \end{cases}$$

Now as we know that the Natural domain of the function is whole the plane. But you should note that the function has one value "0" for all the points in the plane for which both x and y have nonnegative values. And value "1" for all other points in the plane. Consequently the surface

$$z = f(x,y) = \begin{cases} 0 & if \;\; x \geq 0 and \; y \geq 0 \\ 1 & otherwise \end{cases}$$

has a jump as shown in the figure



$z = \begin{cases} 0 & \text{if } x \geq 0, y \geq 0 \\ 1 & \text{otherwise} \end{cases}$
has a vertical jump
at the origin.

Example
Check whether the limit exists or not for the function

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{x^2}{x^2+y^2}$$

Solution:

First we will calculate the Limit of the function along x-axis and we get

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{x^2}{x^2+0} = 1 \qquad \textbf{(Along x-axis, y = 0)}$$

Now we will find out the limit of the function along y-axis and we note that the limit is

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{y^2}{y^2+0} = 1 \qquad (\textbf{Along y-axis, x = 0}).$$

Now we will find out the limit of the function along the line y = x and we note that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{x^2}{x^2+x^2} = \frac{1}{2} \qquad \textbf{(Along y = x)}$$

It means that limit of the function at (0, 0) doesn't exist because it has different values along different paths. Thus the function cannot be continuous at (0, 0). And also note that the function is not defined at (0, 0) and hence it doesn't satisfy two conditions of the continuity.

Example
Check the continuity of the function at (0,0)

$$f(x,y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{x^2+y^2} & \textit{if } (x,y) \neq (0,0) \\ 1 & \textit{if } (x,y) = (0,0) \end{cases}$$

Solution:
First we will note that the function is defined on the point where we have to check the Continuity that is the function has value at (0, 0). Next we will find out the Limit of the function at (0, 0) and in evaluating this limit, we use the result $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and note that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\mathrm{Sin}(x^2+y^2)}{x^2+y^2}$$

$$= 1 = f(0,0)$$

This shows that f is continuous at (0,0)

CONTINUITY OF FUNCTION OF THREE VARIABLES

A function f of three variables is called *continuous at a point* (x_0, y_0, z_0) if

1. $f(x_0, y_0, z_0)$ if defined.
2. $\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z)$ exists.
3 $\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = f(x_0, y_0, z_0)$.

EXAMPLE

Check the continuity of the function

$$f(x,y,z)=\frac{y+1}{x^2+y^2-1}$$

Solution:

First of all, note that the given function is not defined on the cylinder $x^2+y^2-1=0$.
Thus the function is not continuous on the cylinder $x^2+y^2-1=0$
However, $f(x,y,z)$ is continuous at **all other points** of its domain.

RULES FOR CONTINOUS FUNCTIONS

1) If g and h are continuous functions of one variable, then $f(x,y)=g(x)h(y)$ is a continuous function of x and y.

2) If g is a continuous function of one variable and h is a continuous function of two variables, then their composition $f(x,y)=g(h(x,\ y))$ is a continuous function of x and y.

3) A composition of continuous functions is continuous.

4) A sum, difference, or product of continuous functions is continuous.

5) A quotient of continuous function is continuous, expect where the denominator is zero.

EXAMPLE OF PRODUCT OF FUNCTIONS TO BE CONTINUED

In general, any function of the form $f(x,y)=Ax^m x^n$ (m and n non-negative integers) is continuous everywhere in the domain because it is the product of continuous functions Ax^m and x^n.

The function of the form $f(x,y)=3x^2x^5$ is continuous every where in the domain because it is the product of continuous functions $g(x)=3x^2$ and $h(y)=y^5$.

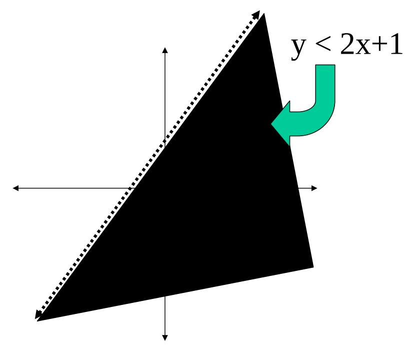
CONTINUOUS EVERYWHERE

A function *f* that is continuous at each point of a region R in 2-dimensional space or 3-dimensional space is said to be *continuous on R.* A function that is continuous at every point in 2-dimensional space or 3-dimensional space is called continuous *everywhere* or simply *continuous.*

EXAMPLES

(1) $f\ (x, y) = \ln(2x - y + 1)$

The function f is continuous in the whole region where $2x > y - 1,\ y < 2x + 1$. And its region is shown in figure below.



(2) $f(x, y) = e^{1-xy}$

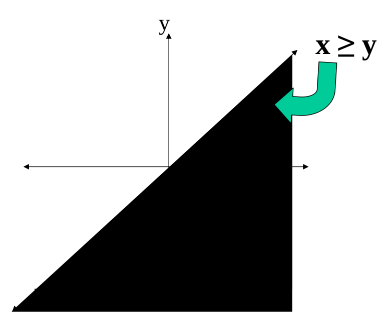
The function f is continuous in the whole region of xy-plane.

(3) $f(x, y) = \tan^{-1}(y - x)$

The function f is continuous in the whole region of xy- plane.

(4) $f(x, y) = \sqrt{y - x}$

The function is continuous where x ≥ y



Partial Derivative

Let f a function of x and y. If we hold y constant, say $y = y_0$ and view x as a variable, then $f(x,\ y_0)$ is a function of x alone. If this function is differentiable at $x = x_0$, then the value of this derivative is denoted by $f_x(x_0,\ y_0)$ and is called the Partial derivative of f with respect of x at the point $(x_0,\ y_0)$.

Similarly, if we hold x constant, say $x = x_0$ and view y as a variable, then $f(x_0, y)$ is a function of y alone. If this function is differentiable at $y = y_0$, then the value of this derivative is denoted by $f_y(x_0, y_0)$ and is called the Partial derivative of f with respect of y at the point (x_0, y_0).

Example

Let $f(x, y) = 2x^3y^2 + 2y + 4x$ be a surface. Find the partial derivative of f with respect to x and y at point (1, 2).

Solution:

Treating y as a constant and differentiating with respect to x, we obtain

$$f_x(x, y) = 6x^2y^2 + 4$$

Treating x as a constant and differentiating with respect to y, we obtain

$$f_y(x, y) = 4x^3y + 2$$

Substituting x = 1 and y = 2 in these partial-derivative formulas yields.

$$f_x(1, 2) = 6(1)^2(2)^2 + 4 = 28$$

$$f_y(1, 2) = 4(1)^3(2) + 2 = 10$$

Example

Let $z = 4x^2 - 2y + 7x^4y^5$ be a surface. Find the partial derivative of z with respect to x and y.

Solution:

$$Z = 4x^2 - 2y + 7x^4y^5$$

$$\frac{\partial z}{\partial x} = 8x + 28x^3y^5$$

$$\frac{\partial z}{\partial y} = -2 + 35x^4y^4$$

Example

Let $z = f(x, y) = x^2 sin^2 y$ be a surface. Find the partial derivative of z with respect to x and y.

Solution:

$$z = f(x, y) = x^2 sin^2 y$$

Then to find the derivative of f with respect to x, *we* treat y as a constant.

Therefore, $\frac{\partial z}{\partial x} = f_x = 2x\sin^2 y$

Then to find the derivative of f with respect to y, *we* treat x as a constant.

Therefore,

$$\frac{\partial z}{\partial y} = f_y = x^2 2\sin y \cos y$$
$$= x^2 \sin 2y$$

Example

Let $z = \ln\left(\frac{x^2+y^2}{x+y}\right)$ be a surface. Find the partial derivative of z with respect to x and y.

Solution:

By using the properties of the ln we can write it as

$$z = \ln(x^2 + y^2) - \ln(x + y)$$
$$\frac{\partial z}{\partial x} = \frac{1}{x^2+y^2} \,.\, 2x - \frac{1}{x+y}$$
$$= \frac{2x^2 + 2xy - x^2 - y^2}{(x^2+y^2)(x+y)}$$
$$= \frac{x^2 + 2xy - y^2}{(x^2+y^2)(x+y)}$$

Similarly, (or by symmetry)

$$\frac{\partial z}{\partial y} = \frac{y^2 + 2xy - x^2}{(x^2+y^2)(x+y)}$$

Example: Find the partial derivative of z with respect to x and y.

$$z = x^4 \sin(xy^3)$$
$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[x^4 \sin(xy^3)\right]$$
$$= x^4 \frac{\partial}{\partial x}\left[\sin(xy^3)\right] + \sin(xy^3)\frac{\partial}{\partial x}(x^4)$$
$$= x^4 \cos(xy^3) y^3 + \sin(xy^3) 4x^3$$
$$\frac{\partial z}{\partial x} = x^4 y^3 \cos(xy^3) + \sin(xy^3)$$
$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left[x^4 \sin(xy^3)\right]$$
$$= x^4 \frac{\partial}{\partial y}\left[\sin(xy^3)\right] + \sin(xy^3)\frac{\partial}{\partial y}(x^4)$$
$$= x^4 \cos(xy^3).3xy^2 + \sin(xy^3).0$$
$$= 3x^5 y^2 \cos(xy^3)$$

Example: Find the partial derivative of z with respect to x and y.

$$z = \cos(x^5y^4)$$

$$\frac{\partial z}{\partial x} = -\sin(x^5y^4)\frac{\partial}{\partial x}(x^5y^4)$$
$$= -5x^4y^4\sin(x^5y^4)$$
$$\frac{\partial z}{\partial y} = -\sin(x^5y^4)\frac{\partial}{\partial y}(x^5y^4)$$
$$= -4x^5y^3\sin(x^5y^4)$$

Example: Find the partial derivative of w with respect to x, y and z.

$$w = x^2 + 3y^2 + 4z^2 - xyz$$

$$\frac{\partial w}{\partial x} = 2x - yz$$

$$\frac{\partial w}{dy} = 6y - xz$$

$$\frac{dw}{dz} = 8z - xy$$