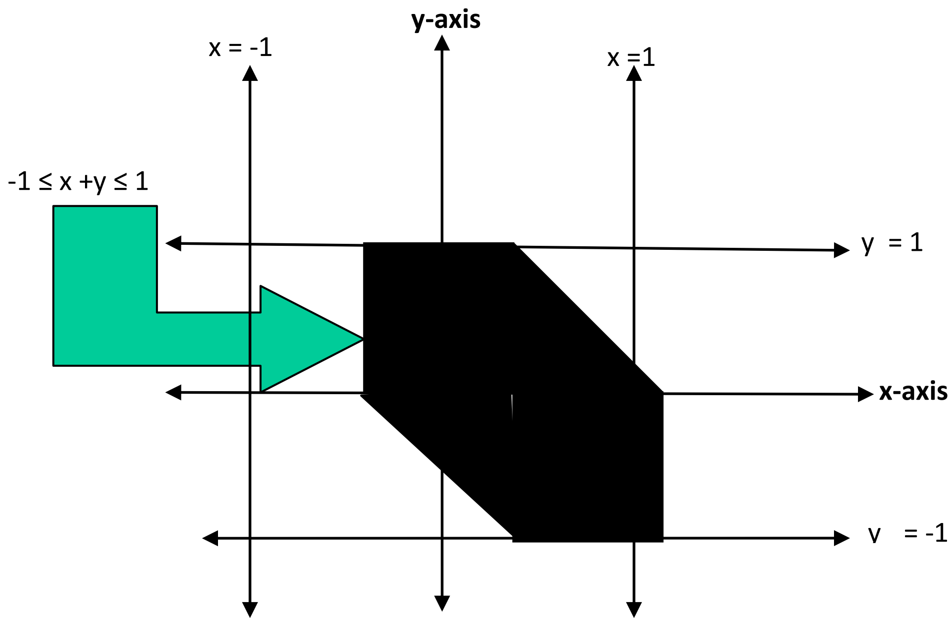
Lecture No-5

Limit of Multivariable Function

Example 1:

$f(x,y) = Sin^{-1}(x+y)$

Domain of f is the region in which $-1 \leq x+y \leq 1$



Domains and Ranges

	Functions	*Domain*	*Range*
1)	$\omega = \sqrt{x^2 + y^2 + z^2}$	Entire space	$[0, \infty)$
2)	$\omega = \dfrac{1}{x^2 + y^2 + z^2}$	Entire space except origin $(x, y, z) \neq (0, 0, 0)$	$(0, \infty)$
3)	$\omega = xy \ln z$	Half space, $z > 0$	$(-\infty, \infty)$

Examples of domain of a function

Example 2:

$$f(x, y) = xy\sqrt{y-1}$$

Domain of f consists of the region in xy-plane where $y \geq 1$.

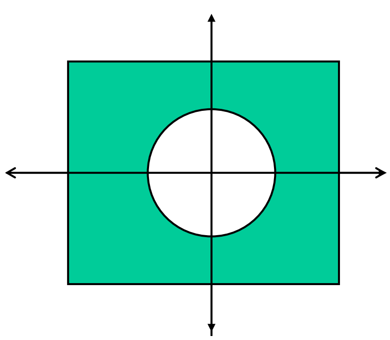
(Here we take $y - 1 \geq 0$ *for real values*.)

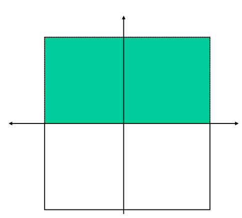
Example 3:

$$f(x, y) = \sqrt{x^2 + y^2 - 4}$$

Domain of f consists of the region in xy-plane where $x^2 + y^2 \geq 4$. It means that the points of the domain lie outside the circle with radius 2.

As shown in the figure





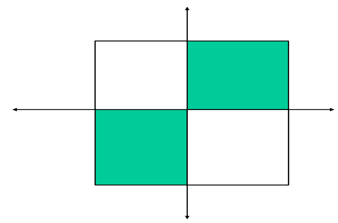
Example 4:

$$f(x, y) = \ln xy$$

For the real values of logarithmic function, $xy > 0$ which is possible:

When $x \prec 0,\quad y \prec 0$ (3rd quadrant) and when $x > 0,\quad y > 0$ (1st quadrant)

Domain of *f* consists of region lying in first and third quadrants in xy-plane as shown below.



Example 5:

$$f(x, y, z) = e^{xyz}$$

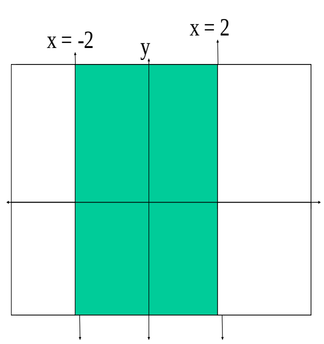
Domain of f consists of the entire region of three dimensional space.

Example 6:

$$f(x, y) = \frac{\sqrt{4 - x^2}}{y^2 + 3}$$

Here we take $4 - x^2 \geq 0$ *for real values of* $f(x, y)$.

Domain of f consists of region in xy - plane where $x^2 \leq 4$ which implies that $-2 \leq x \leq 2$.



Example 7:

$$f(x, y, z) = \sqrt{25 - x^2 - y^2 - z^2}$$

.Here we take $25 - x^2 - y^2 - z^2 \geq 0$ *for real values of* $f(x, y)$. *So,* $x^2 + y^2 + z^2 \leq 5^2$

Domain of f consists of region in three dimensional space occupied by sphere Centre at $(0, 0, 0)$ and radius 5.

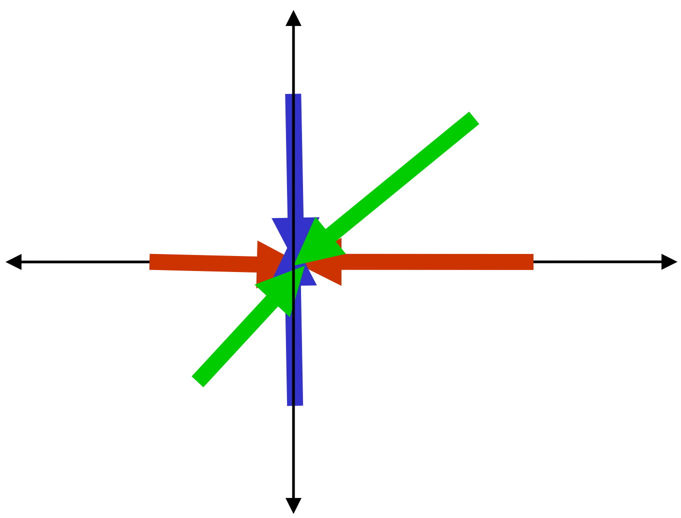
Example 8:

$$f(x, y) = \frac{x^3 + 2x^2 y - xy - 2y^2}{x + 2y}$$

$f(0, 0)$ is not defined but we see that limit exits.

Approaching to (0,0) through x-axis	$f(x, y)$	Approaching to (0,0) through y-axis	$f(x, y)$
$(0.5, 0)$	0.25	(0,0.1)	-0.1
$(0.25, 0)$	0.0625	(0,0.001)	-0.001
(0.1,0)	0.01	(0,0.00001)	0.00001
(-0.25,0)	0.0625	(0,-0.001)	0.001
(-0.1,0)	0.01	(0,-0.00001)	0.00001

Approaching to (0,0) through $y = x$	$f(x, y)$
(0.5,0.5)	-0.25
(0.1,0.1)	-0.09
(0.01,0.01)	-0.0099
(-0.5,-0.5)	0.75
(-0.1,-0.1)	0.11
(-0.01,-0.01)	0.0101



Example 9:

$$f(x,y) = \frac{xy}{x^2+y^2}$$

$f(0, 0)$ is not defined and we see that limit also does not exist.

Approaching to (0,0) through x-axis (y = 0)	f (x,y)	Approaching to (0,0) through y = x	f (x,y)
(0.5,0)	0	(0.5,0.5)	0.5
(0.1,0)	0	(0.25,0.25)	0.5
(0.01,0)	0	(0.1,0.1)	0.5
(0.001,0)	0	(0.05,0.05)	0.5
(0.0001,0)	0	(0.001,0.001)	0.5
(-0.5,0)	0	(-0.5,-0.5)	0.5
(-0.1,0)	0	(-0.25,-0.25)	0.5
(-0.01,0)	0	(-0.1,-0.1)	0.5
(-0.001,0)	0	(-0.05,-0.05)	0.5
(-0.0001,0)	0	(-0.001,-0.001)	0.5

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0 \text{ (along } y = 0)$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0.5 \text{ (along } y = x)$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \text{ does not exist.}$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

Let (x, y) approach $(0, 0)$ along the line $y = x$.

$$f(x,y)=\frac{xy}{x^2+y^2}$$

$$f(x,x)=\frac{xx}{x^2+x^2}=\frac{x^2}{2x^2}=\frac{1}{2} \qquad x\neq 0$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \frac{1}{2} \qquad \textit{Along the line } y=x$$

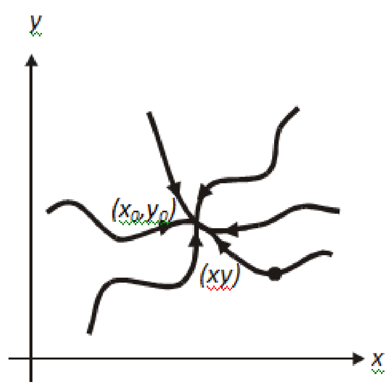
Now let (x, y) approach $(0, 0)$ along x-axis. On x-axis, $y=0$.

$$f(x,0)=\frac{x\times 0}{x^2+0^2}=\frac{0}{x^2}=0 \qquad x\neq 0$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} =0 \qquad \textit{Along the line } x-axis.$$

|Therefore $f(x,y)$ assumes two different values, as (x,y) approaches $(0,0)$ along two different paths. So $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

We can approach a point in space through infinite paths some of them are shown in the figure below:



Rule for Non-Existence of a Limit

If in $\lim\limits_{(x,y)\to(a,b)} f(x,y)$, we get two or more different values, as (x,y) approaches (a,b) along two different paths, then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist.

The paths along which (a,b) is approached may be straight *lines or plane curves through* (a,b)

Example

$$\lim_{(x,y)\to(2,1)} \frac{x^3+2x^2y-x-2y^2}{x+2y}$$
$$=\frac{\lim\limits_{(x,y)\to(2,1)}\left(x^3+2x^2y-x-2y^2\right)}{\lim\limits_{(x,y)\to(2,1)}\left(x+2y\right)}$$

Example

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$$

We set $x=r\cos\theta,\ \ y=r\sin\theta$, then

$$\frac{xy}{\sqrt{x^2+y^2}}=\frac{(r\cos\theta)(r\sin\theta)}{\sqrt{(r\cos\theta)^2+(r\sin\theta)^2}}$$
$$=\frac{\left(r^2\cos\theta\sin\theta\right)}{r\sqrt{\cos^2\theta+\sin^2\theta}}=\frac{\left(r\cos\theta\sin\theta\right)}{\sqrt{1}}$$
$$=r\cos\theta\sin\theta, \qquad\qquad r>0$$

$$Since\ r=\sqrt{x^2+y^2},\quad so\, as\ (x,y)\to(0,0)$$
$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}=\lim_{r\to0}\ r\cos\theta\sin\theta <=0\times\cos\theta\sin\theta=0$$

$$Note\ that\ \left|\cos\theta\sin\theta\right|<1 \quad for\ all\ values\ of\ \theta.$$

RULES FOR LIMIT

If $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L_1$ *and* $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = L_2$, then

(a) $\lim_{(x,y)\to(x_0,y_0)} cf(x,y) = cL_1$ (if *c* is constant)

(b) $\lim_{(x,y)\to(x_0,y_0)} \{f(x,y) + g(x,y)\} = L_1 + L_2$

(c) $\lim_{(x,y)\to(x_0,y_0)} \{f(x,y) - g(x,y)\} = L_1 - L_2$

(d) $\lim_{(x,y)\to(x_0,y_0)} \{f(x,y)g(x,y)\} = L_1 L_2$

(e) $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L_1}{L_2}$ (if $L_2 = 0$)

$\lim_{(x,y)\to(x_0,y_0)} c = c$ (*c* is a constant), $\lim_{(x,y)\to(x_0,y_0)} x_0 = x_0$, $\lim_{(x,y)\to(x_0,y_0)} y_0 = y_0$

Similarly for the function of three variables.

Overview of lecture# 5

In this lecture we recall you all the limit concept which are prerequisite for this course and you can find all these concepts in the chapter # 16 (topic # 16.2) of your recommended book "*Calculus By Howard Anton*".