Lecture No-3   Elements of three dimensional geometry

**Distance formula in three dimension**

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points such that $PQ$ is not parallel to one of the coordinate axis Then $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ Which is known as Distance fromula between the points P and Q.

**Example of distance formula**

Let us considerthe points A (3, 2, 4), B (6, 10, -1), and C (9, 4, 1)

Then

$|AB| = \sqrt{(6 - 3)^2 + (10 - 2)^2 + (-1 - 4)^2} = \sqrt{98} = 7\sqrt{2}$

$|AC| = \sqrt{(9 - 3)^2 + (4 - 2)^2 + (1 - 4)^2} = \sqrt{49} = 7$

$|BC| = \sqrt{(9 - 6)^2 + (4 - 10)^2 + (1 + 1)^2} = \sqrt{49} = 7$

**Mid point of two points**

If R is the middle point of the line segment PQ, then the co-ordinates of the middle points are

$x = (x_1 + x_2)/2$ ,  
$y = (y_1 + y_2)/2$ ,  
$z = (z_1 + z_2)/2$

Let us consider tow points A(3,2), and B(6,10,-1)

Then the co-ordinates of mid point of AB is

$[(3+6)/2,(2+10)/2,(4-1)/2]$

$= (9/2,6,3/2)$

**Direction Angles**

The direction angles $\alpha, \beta, \gamma$ of a line are defined as

$\alpha$ = Angle between line and the positive x-axis

$\beta$ = Angle between line and the positive y-axis

$\gamma$ = Angle between line and the positive z-axis.

By definition, each of these angles lies between 0 and $\pi$.

**Direction Ratios**

Cosines of direction angles are called direction cosines

Any multiple of direction cosines are called direction numbers or direction ratios of the line $L$.

**Given a point, finding its Direction cosines**

y-axis
Direction angles of a Line

The angles which a line makes with positive x, y and z-axis are known as Direction Angles. In the above figure the blue line has direction angles as $\alpha$, $\beta$, and $\gamma$ which are the angles which blue line makes with x, y and z-axis respectively.

**Direction cosines:**

Now if we take the cosine of the Direction Angles of a line then we get the Direction cosines of that line. So the Direction Cosines of the above line are given by

$$
\cos \alpha = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}
$$

$$
\cos \beta = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}
$$

Similarly,

$$
\cos \gamma = \frac{z}{r} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}
$$

$$
\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.
$$

**Direction cosines and direction ratios of a line joining two points**

- For a line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ the direction ratios are
$x_2 - x_1, y_2 - y_1, z_2 - z_1$ and the directions cosines are \( \frac{x_2 - x_1}{|PQ|}, \frac{y_2 - y_1}{|PQ|}, \frac{z_2 - z_1}{|PQ|} \).

**Example** For a line joining two points $P(1,3,2)$ and $Q(7,-2,3)$ the direction ratios are
\[
7 - 1, -2 – 3 , 3 – 2 \\
6 , -5 , 1
\]
and the directions cosines are
\[
\frac{6}{\sqrt{62}} , \frac{-5}{\sqrt{62}} , \frac{1}{\sqrt{62}}
\]
In two dimensional space the graph of an equation relating the variables $x$ and $y$ is the set of all point $(x, y)$ whose co-ordinates satisfy the equation. Usually, such graphs are curves. In three dimensional space the graph of an equation relating the variables $x$, $y$ and $z$ is the set of all point $(x, y, z)$ whose co-ordinates satisfy the equation. Usually, such graphs are surfaces.

**Intersection of two surfaces**

- Intersection of two surfaces is a curve in three dimensional space.
- It is the reason that a curve in three dimensional space is represented by two equations representing the intersecting surfaces.

**Intersection of Cone and Sphere**

**Intersection of Two Planes**

If the two planes are not parallel, then they intersect and their intersection is a straight line. Thus, two non-parallel planes represent a straight line given by two simultaneous linear equations in $x$, $y$ and $z$ and are known as non-symmetric form of equations of a straight line.
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<th>DESCRIPTION</th>
<th>EQUATION</th>
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<td>xy-plane</td>
<td>Consists of all points of the form (x, y, 0)</td>
<td>z = 0</td>
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<tr>
<td>xz-plane</td>
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**Planes parallel to Co-ordinate Planes**

**General Equation of Plane**

Any equation of the form

\[ ax + by + cz + d = 0 \]

where a, b, c, d are real numbers, represent a plane.

**Sphere**

The standard equation of the sphere of radius \( a \) centered at \((x_0, y_0, z_0)\) is

\[ (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2 \]

The level surfaces of \( f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \) are concentric spheres.
Right Circular Cone

Horizontal Circular Cylinder
Horizontal Elliptic Cylinder

Overview of Lecture # 3

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  Three Dimensional Space
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