

# Calculus II

MTH301



Virtual University of Pakistan

Knowledge beyond the boundaries

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## LECTURE No. 1

### INTRODUCTION

- Calculus is the mathematical tool used to analyze changes in physical quantities.
- Calculus is also Mathematics of Motion and Change.
- Where there is motion or growth, where variable forces are at work producing acceleration, Calculus is right mathematics to apply.

**Differential Calculus** deals with the Problem of Finding

- (1) Rate of change
- (2) Slope of curve

Velocities and acceleration of moving bodies. Firing angles that give cannons their maximum range. The times when planets would be closest together or farthest apart.

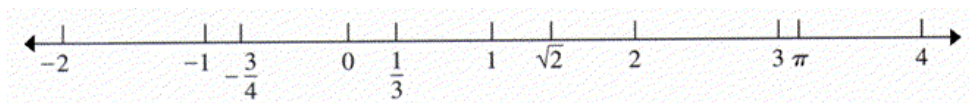
**Integral Calculus** deals with the Problem of determining a *function* from information about its *rates of Change*.

Integral Calculus enables us

- (1) To calculate lengths of curves.
- (2) To find areas of irregular regions in plane.
- (3) To find the volumes and masses of arbitrary solids
- (4) To calculate the future location of a body from its present position and knowledge of the forces acting on it.

### Reference Axis System

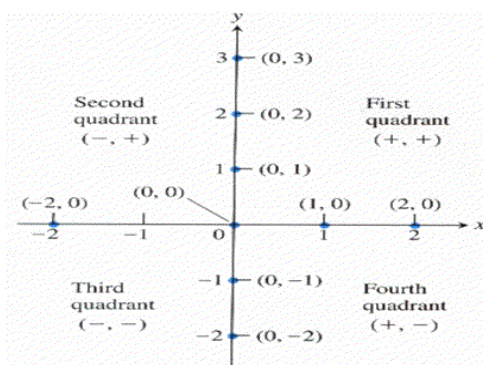
Before giving the concept of Reference Axis System, we recall you the concept of real line and locate some points on the real line as shown in the figure below, also remember that the real number system consist of both Rational and Irrational numbers that is we can write set of real numbers as union of rational and irrational numbers.



Here in the above figure, we have located some of the rational as well as irrational numbers and also note that there are infinite real numbers between every two real numbers.

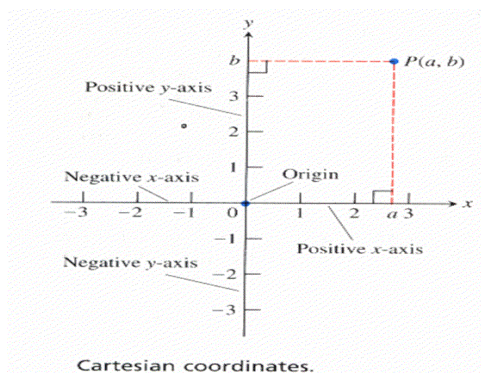
Now if you are working in two dimensions, then you know that we take the two mutually perpendicular lines and call the horizontal line as  $x$ -axis and vertical line as  $y$ -axis and where these lines cut we take that point as origin. Now any point on the  $x$ -axis will be denoted by an order pair whose first element which is also known as abscissa is a real number and other element of the order pair which is also known as ordinate will have 0 values, i.e.  $(x, 0)$

Similarly any point on the y-axis can be representing by an order pair  $(0, y)$ . Some points are shown in the figure below. Also note that these lines divide the plane into four regions: First, Second, Third and Fourth quadrants respectively. We take the positive real numbers at the right side of the origin and negative to the left side, in the case of x-axis. Similarly for y-axis and also shown in the figure.



### Location of a point

Now we will illustrate how to locate the point in the plane using x- and y- axes. Draw two perpendicular lines from the point whose position is to be determined. These lines will intersect at some point on the x-axis and y-axis and we can find out these points. Now the distance of the point of intersection of x-axis and perpendicular line from the origin is the x-coordinate of the point P and similarly the distance from the origin to the point of intersection of y-axis and perpendicular line is the y-coordinate of the point P as shown in the figure below.

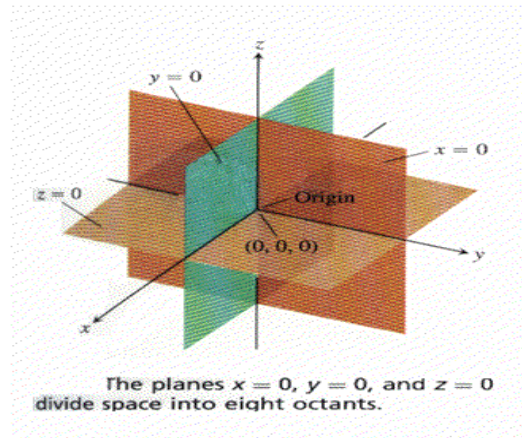


In space, we have three mutually perpendicular lines as reference axes, namely x, y and z axis. Now you can see from the figure below that the planes  $x = 0$ ,  $y = 0$  and  $z = 0$  divide the space into eight octants. Also note that in this case we have  $(0,0,0)$  as origin and any point in the space will have three coordinates.

### Signs of coordinates in different octants

First of all note that the equation  $x = 0$  represents a plane in the 3d space and in this plane every point has its x-coordinate as 0, also that plane passes through the origin as shown in the figure above. Similarly  $y = 0$  and  $z = 0$  also define a plane in 3d space and have properties similar to that of  $x = 0$  such that these planes also pass through the origin and

any point in the plane  $y = 0$  will have y-coordinate as 0 and any point in the plane  $z = 0$  has z-coordinate as 0.



Also remember that when two planes intersect we get the equation of a line and when two lines intersect then we get a plane containing these two lines. Now note that by the intersection of the planes  $x = 0$  and  $z = 0$  we get the line which is our y-axis.

Also by the intersection of  $x = 0$  and  $y = 0$  we get the line which is z-axis, similarly you can easily see that by the intersection of  $z = 0$  and  $y = 0$  we get line which is x-axis.

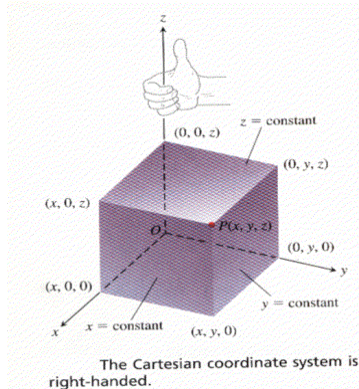
Now these three planes divide the 3d space into eight octants depending on the positive and negative direction of axis. The octant in which every coordinate of any point has positive sign is known as first octant formed by the positive x, y and z –axes. Similarly in second octant every point has x-coordinate as negative and other two coordinates as positive correspond to negative x-axis and positive y and z axis.

Now one octant is that in which every point has x and y coordinate negative and z-coordinate positive, which is known as the third octant. Similarly we have eight octants depending on the sign of coordinates of a point. These are summarized below.

First octant	(+, +, +)	Formed by positive sides of the three axis.
Second octant	(–, +, +)	Formed by –ve x-axis and positive y and z-axis.
Third octant	(–, –, +)	Formed by –ve x and y axis with positive z-axis.
Fourth octant	(+, –, +)	Formed by +ve x and z axis and –ve y-axis.
Fifth octant	(+, +, –)	Formed by +ve x and y axis with -ve z-axis.
Sixth octant	(–, +, –)	Formed by –ve x and z axis with positive y-axis.
Seventh octant	(–, –, –)	Formed by –ve sides of three axis.
Eighth octant	(+, –, –)	Formed by -ve y and z-axis with +ve x-axis.

(Remember that we have two sides of any axis one of positive values and the other is of negative values) Now as we told you that in space we have three mutually perpendicular lines as reference axis. So far you are familiar with the reference axis for 2d which consist of two perpendicular lines namely x-axis and y-axis. For the reference axis of 3d

space we need another perpendicular axis which can be obtained by the cross product of the two vectors, now the direction of that vector can be obtained by Right Hand rule. This is illustrated below with diagram.



### Concept of a Function

Historically, the term, function denotes the dependence of one quantity on other quantity. The quantity  $x$  is called the independent variable and the quantity  $y$  is called the dependent variable. We write it as  $y = f(x)$  and we read  $y$  is a function of  $x$ .

**For example**, the equation  $y = 2x$  defines  $y$  as a function of  $x$  because each value assigned to  $x$  determines unique value of  $y$ .

### Examples of function

- The area of a circle depends on its radius  $r$  by the equation  $A = \pi r^2$  so, we say that  $A$  is a function of  $r$ .
- The volume of a cube depends on the length of its side  $x$  by the equation  $V = x^3$  so, we say that  $V$  is a function of  $x$ .
- The velocity  $V$  of a ball falling freely in the earth's gravitational field increases with time  $t$  until it hits the ground, so we say that  $V$  is function of  $t$ .
- In a bacteria culture, the number  $n$  of present after one day of growth depends on the number  $N$  of bacteria present initially, so we say that  $N$  is function of  $n$ .

### Function of Several Variables

Many functions depend on more than one independent variable.

#### Examples

- 1) The area of a rectangle depends on its length  $l$  and width  $w$  by the equation  $A = l w$ , so we say that  $A$  is a function of  $l$  and  $w$ .
- 2) The volume of a rectangular box depends on the length  $l$ , width  $w$  and height  $h$  by the equation  $V = l w h$  so, we say that  $V$  is a function of  $l$ ,  $w$  and  $h$ .
- 3) The area of a triangle depends on its base length  $l$  and height  $h$  by the equation  $A = \frac{1}{2} l \times h$ , so we say that  $A$  is a function of  $l$  and  $h$ .
- 4) The volume  $V$  of a right circular cylinder depends on its radius  $r$  and height  $h$  by the equation  $V = \pi r^2 h$  so, we say that  $V$  is a function of  $r$  and  $h$ .

## LECTURE No. 2

## VALUES OF FUNCTIONS

**Example 1:** Consider the function  $f(x) = 2x^2 - 1$ , then  $f(1) = 2(1)^2 - 1 = 1$

$$f(4) = 2(4)^2 - 1 = 31, \quad f(-2) = 2(-2)^2 - 1 = 7$$

$$f(t-4) = 2(t-4)^2 - 1 = 2t^2 - 16t + 31$$

These are the values of the function at some points.

**Example 2 :** Now we will consider a function of two variables, so consider the function

$$f(x, y) = x^2y + 1 \text{ then } f(2, 1) = (2)^2 \cdot 1 + 1 = 5, \quad f(1, 2) = (1)^2 \cdot 2 + 1 = 3,$$

$$f(0, 0) = (0)^2 \cdot 0 + 1 = 1, \quad f(1, -3) = (1)^2 \cdot (-3) + 1 = -2,$$

$$f(3a, a) = (3a)^2 \cdot a + 1 = 9a^3 + 1, \quad f(ab, a-b) = (ab)^2 (a-b) + 1 = a^3b^2 - a^2b^3 + 1$$

These are values of the function at some points.

**Example 3:** Now consider the function  $f(x, y) = x + \sqrt[3]{xy}$ , then

$$(a) \quad f(2, 4) = 2 + \sqrt[3]{(2)(4)} = 2 + \sqrt[3]{8} = 2 + 2 = 4$$

$$(b) \quad f(t, t^2) = t + \sqrt[3]{(t)(t^2)} = t + \sqrt[3]{t^3} = t + t = 2t$$

$$(c) \quad f(x, x^2) = x + \sqrt[3]{(x)(x^2)} = x + \sqrt[3]{x^3} = x + x = 2x$$

$$(d) \quad f(2y^2, 4y) = 2y^2 + \sqrt[3]{(2y^2)(4y)} = 2y^2 + \sqrt[3]{8y^3} = 2y^2 + 2y$$

**Example 4:** Now again we take another function of three variables

$$f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2} \text{ then } f\left(0, \frac{1}{2}, \frac{1}{2}\right) = \sqrt{1 - 0 - \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}}$$

**Example 5:** Consider the function  $f(x, y, z) = xy^2z^3 + 3$ , then at certain points we have

$$f(2, 1, 2) = (2)(1)^2(2)^3 + 3 = 19, \quad f(0, 0, 0) = (0)(0)^2(0)^3 + 3 = 3,$$

$$f(a, a, a) = (a)(a)^2(a)^3 + 3 = a^6 + 3, \quad f(t, t^2, -t) = (t)(t^2)^2(-t)^3 + 3 = -t^8 + 3,$$

$$f(-3, 1, 1) = (-3)(1)^2(1)^3 + 3 = -3 + 3 = 0$$

**Example 6:** Consider the function  $f(x, y, z) = x^2y^2z^4$ , where  $x(t) = t^3$ ,  $y(t) = t^2$  and  $z(t) = t$

$$(a) \quad f(x(t), y(t), z(t)) = [x(t)]^2 [y(t)]^2 [z(t)]^4 = [t^3]^2 [t^2]^2 [t]^4 = t^{14}$$

$$(b) \quad f(x(0), y(0), z(0)) = [x(0)]^2 [y(0)]^2 [z(0)]^4 = [0^3]^2 [0^2]^2 [0]^4 = 0$$

**Example 7:** Let us consider the function  $f(x, y, z) = xyz + x$ , then

$$f\left(xy, \frac{y}{x}, xz\right) = (xy)\left(\frac{y}{x}\right)(xz) + xy = xy^2z + xy$$

**Example 8:** Let us consider  $g(x, y, z) = z \sin(xy)$ ,  $u(x, y, z) = x^2z^3$ ,

$$v(x, y, z) = Pxyz, w(x, y, z) = \frac{xy}{z}, \text{ then}$$

$$g(u(x, y, z), v(x, y, z), w(x, y, z)) = w(x, y, z) \sin(u(x, y, z) v(x, y, z))$$

Now by putting the values of these functions from the above equations, we get

$$g(u(x, y, z), v(x, y, z), w(x, y, z)) = \frac{xy}{z} \sin\left[\left(x^2 z^3\right)(Pxyz)\right] = \frac{xy}{z} \sin(Px^3 yz^4)$$

**Example 9:** Consider the function  $g(x, y) = y \sin(x^2 y)$  and  $u(x, y) = x^2 y^3$ ,  $v(x, y) = \pi xy$ ,

$$\text{then } g(u(x, y), v(x, y)) = v(x, y) \sin\left[\left(u(x, y)\right)^2 v(x, y)\right]$$

By putting the values of these functions we get

$$g(u(x, y), v(x, y)) = \pi xy \sin\left[\left(x^2 y^3\right)^2 \pi xy\right] = \pi xy \sin(\pi x^5 y^7)$$

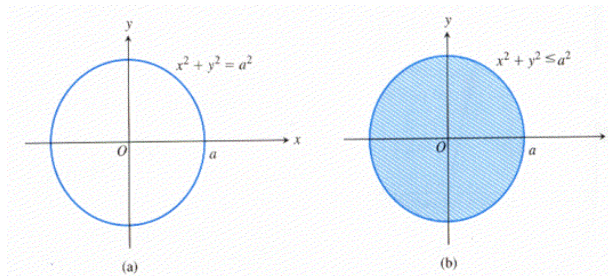
**Function of One Variable:** A function  $f$  of one real variable  $x$  is a rule that assigns a unique real number  $f(x)$  to each point  $x$  in some set  $D$  of the real line.

**Function of two Variables:** A function  $f$  in two real variables  $x$  and  $y$ , is a rule that assigns unique real number  $f(x, y)$  to each point  $(x, y)$  in some set  $D$  of the  $xy$ -plane.

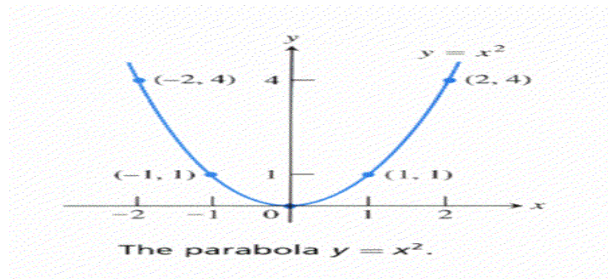
**Function of three variables:** A function  $f$  in three real variables  $x, y$  and  $z$ , is a rule that assigns a unique real number  $f(x, y, z)$  to each point  $(x, y, z)$  in some set  $D$  of three dimensional space.

**Function of  $n$  variables:** A function  $f$  in  $n$  variable real variables  $x_1, x_2, x_3, \dots, x_n$ , is a rule that assigns a unique real number  $w = f(x_1, x_2, x_3, \dots, x_n)$  to each point  $(x_1, x_2, x_3, \dots, x_n)$  in some set  $D$  of  $n$  dimensional space.

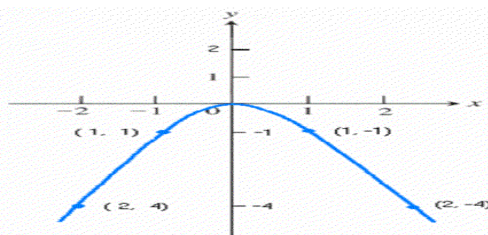
#### Circles and Disks:



#### PARABOLA



Parabola  $y = -x^2$

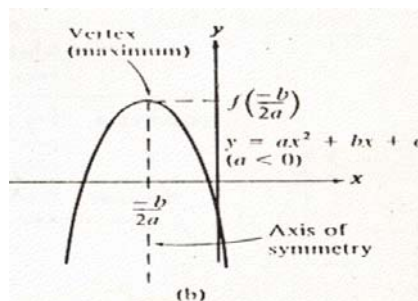
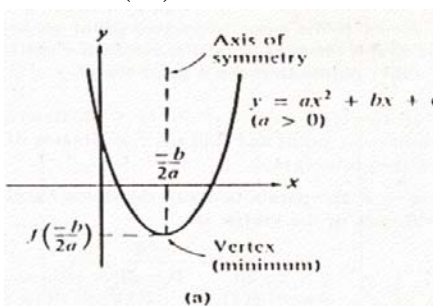


General equation of the Parabola opening upward or downward is of the form

$$y = f(x) = ax^2 + bx + c. \quad \text{Opening upward if } a > 0, \quad \text{Opening downward if } a < 0$$

The x-coordinate of the vertex is given by  $x_0 = -\frac{b}{2a}$ . So the y-coordinate of the vertex

is  $y_0 = f(x_0)$ . The axis of symmetry is  $x = x_0$ .



### Sketching of the graph of parabola $y = ax^2 + bx + c$

**Finding vertex:** x-coordinate of the vertex is given by  $x_0 = -\frac{b}{2a}$

The y-coordinate of the vertex is  $y_0 = ax_0^2 + bx_0 + c$ . Hence vertex is  $V(x_0, y_0)$ .

**Example 10:** Sketch the parabola  $y = -x^2 + 4x$

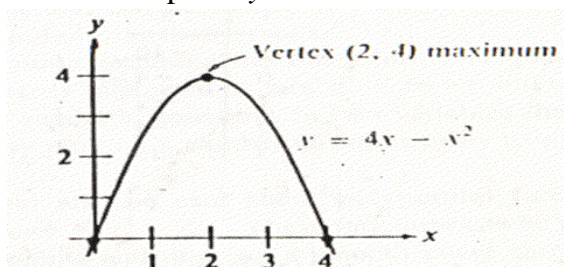
**Solution :** Since  $a = -1 < 0$  because parabola is opening downward. Vertex occurs at

$$x = -\frac{b}{2a} = -\frac{4}{2(-1)} = 2 \quad \text{Axis of symmetry is the vertical line } x = 2.$$

The y-coordinate of the vertex is  $y = -(2)^2 + 4(2) = 4$ . Hence vertex is  $V(2, 4)$ . The zeros of the parabola (i.e. the point where the parabola meets x-axis) are the solutions to  $-x^2 + 4x = 0$ , so  $x = 0$  and  $x = 4$ . Therefore,  $(0,0)$  and  $(4,0)$  lie on the parabola.

Also  $(1,3)$  and  $(3,3)$  lie on the parabola.

Graph of  $y = -x^2 + 4x$



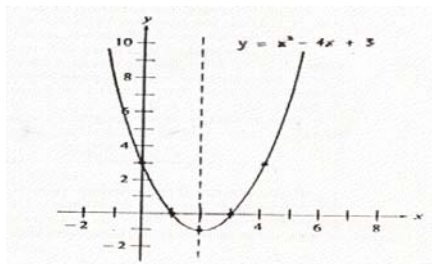
**Example 11:** Sketch the parabola  $y = x^2 - 4x + 3$

**Solution:** Since  $a = 1 > 0$ , parabola is opening upward. Vertex occurs at

$$x = -\frac{b}{2a} = -\frac{(-4)}{2(1)} = 2. \text{ Axis of symmetry is the vertical line } x = 2. \text{ The y co-}$$

ordinate of the vertex is  $y = (2)^2 - 4(2) + 3 = -1$ . Hence vertex is  $V(2, -1)$ . The zeros of the parabola (i.e. the point where the parabola meets x-axis) are the solutions to  $x^2 - 4x + 3 = 0$ , so  $x = 1$  and  $x = 3$ . Therefore  $(1, 0)$  and  $(3, 0)$  lie on the parabola.

Also  $(0, 3)$  and  $(4, 3)$  lie on the parabola. *Graph of  $y = x^2 - 4x + 3$*



Ellipse

ORIENTATION	DESCRIPTION	STANDARD EQUATION
	<ul style="list-style-type: none"> <li>Foci and major axis on the x-axis.</li> <li>Minor axis on the y-axis.</li> <li>Center at the origin.</li> <li>x-intercepts: <math>\pm a</math>.</li> <li>y-intercepts: <math>\pm b</math>.</li> <li><math>a \geq b</math></li> </ul>	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
	<ul style="list-style-type: none"> <li>Foci and major axis on the y-axis.</li> <li>Minor axis on the x-axis.</li> <li>Center at the origin.</li> <li>x-intercepts: <math>\pm b</math>.</li> <li>y-intercepts: <math>\pm a</math>.</li> <li><math>a \geq b</math></li> </ul>	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

ORIENTATION	DESCRIPTION	STANDARD EQUATION	ASYMPTOTE EQUATIONS
	<ul style="list-style-type: none"> <li>Foci on the x-axis.</li> <li>Conjugate axis on the y-axis.</li> <li>Center at the origin.</li> </ul>	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$y = \frac{b}{a}x$ $y = -\frac{b}{a}x$

Hyperbola

### Home Assignments:

In this lecture we recall some basic geometrical concepts which are prerequisite for this course and you can find all these concepts in the chapter # 12 of your book **Calculus By Howard Anton**.

## LECTURE No.3

## ELEMENTS OF THREE DIMENSIONAL GEOMETRY

**Distance formula in three dimensions**

Let  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$  be two points such that  $PQ$  is not parallel to one of the coordinate axis. Then  $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$  Which is known as Distance formula between the points P and Q.

**Example of distance formula**

Let us consider the points  $A(3, 2, 4)$ ,  $B(6, 10, -1)$  and  $C(9, 4, 1)$ , then

$$|AB| = \sqrt{(6-3)^2 + (10-2)^2 + (-1-4)^2} = \sqrt{98} = 7\sqrt{2}$$

$$|AC| = \sqrt{(9-3)^2 + (4-2)^2 + (1-4)^2} = \sqrt{49} = 7$$

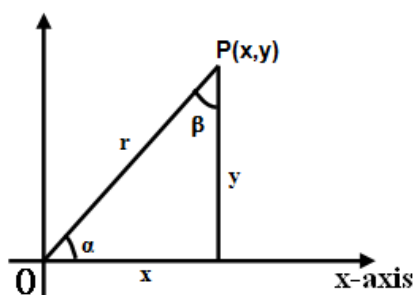
$$|BC| = \sqrt{(9-6)^2 + (4-10)^2 + (1+1)^2} = \sqrt{49} = 7$$

**Mid point of two points**

Let  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$  be two points. If  $R(x, y, z)$  is the middle point of the line segment  $PQ$ , then the coordinates of the middle point  $R(x, y, z)$  are given below:

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}, \quad z = \frac{z_1 + z_2}{2}$$

**Example 2:** Let us consider two points  $A(3, 2, 4)$  and  $B(6, 10, -1)$ , then the coordinates of mid point of  $AB$  are  $\left(\frac{3+6}{2}, \frac{2+10}{2}, \frac{4-1}{2}\right) = \left(\frac{9}{2}, 6, \frac{3}{2}\right)$

**Given a point, finding its Direction Cosines**

From triangle we  
can write  
 $\cos \alpha = x/r$   
 $\cos \beta = y/r$

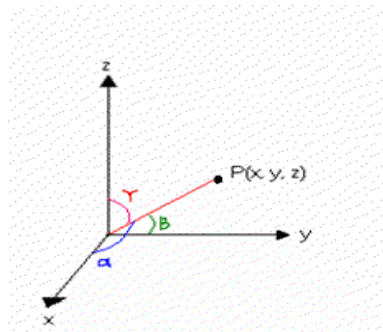
**Direction Angles**

The direction angles  $\alpha, \beta, \gamma$  of a line is defined as

$\alpha$  = Angle between line and the positive x-axis

$\beta$  = Angle between line and the positive y-axis

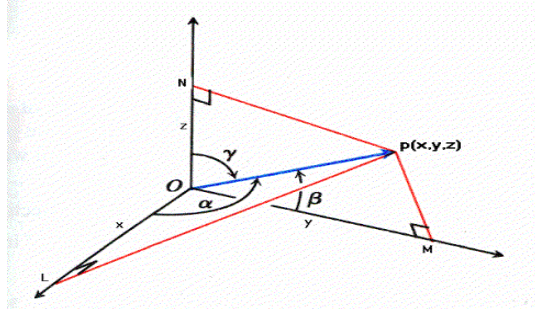
$\gamma$  = Angle between line and the positive z-axis



By definition, each of these angles lie between 0 and  $\pi$ .

**Direction Ratios:** Cosines of direction angles are called direction cosines. Any multiple of direction cosines are called direction numbers or direction ratios of the line  $L$ .

### Direction angles of a Line



The angles which a line makes with positive x, y and z-axis are known as Direction Angles. In the above figure, the blue line has direction angles as  $\alpha$ ,  $\beta$  and  $\gamma$  which are the angles which blue line makes with x, y and z-axes respectively.

### Direction Cosines

Now if we take the cosine of the Direction Angles of a line, then we get the Direction cosines of that line. So the Direction Cosines of the above line are given by

$$\cos \alpha = \frac{x}{OP} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \quad \cos \beta = \frac{y}{OP} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \quad \cos \gamma = \frac{z}{OP} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

Since, by distance formula,  $OP = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2}$

Squaring and adding these equations (1), (2) and (3), we get

$$\begin{aligned} \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma &= \left( \frac{x}{\sqrt{x^2 + y^2 + z^2}} \right)^2 + \left( \frac{y}{\sqrt{x^2 + y^2 + z^2}} \right)^2 + \left( \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)^2 \\ &= \frac{x^2 + y^2 + z^2}{\left( \sqrt{x^2 + y^2 + z^2} \right)^2} = \frac{x^2 + y^2 + z^2}{x^2 + y^2 + z^2} = 1 \end{aligned}$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

### Direction cosines and direction ratios of a line joining two points

For a line joining two points  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$ ,

the direction ratios are  $x_2 - x_1$ ,  $y_2 - y_1$  and  $z_2 - z_1$

and the directions cosines are  $\frac{x_2 - x_1}{|PQ|}$ ,  $\frac{y_2 - y_1}{|PQ|}$  and  $\frac{z_2 - z_1}{|PQ|}$ .

**Example 3:** Find direction cosines and direction ratios for a line joining two points  $P(1, 3, 2)$  and  $Q(7, -2, 3)$ .

**Solution:** For a line joining two points  $P(1, 3, 2)$  and  $Q(7, -2, 3)$ , the direction ratios are

$$x_2 - x_1 = 7 - 1 = 6, \quad y_2 - y_1 = -2 - 3 = -5, \quad z_2 - z_1 = 3 - 2 = 1$$

and the directions cosines are

$$\frac{6}{\sqrt{6^2 + (-5)^2 + 1^2}}, \frac{-5}{\sqrt{6^2 + (-5)^2 + 1^2}}, \frac{1}{\sqrt{6^2 + (-5)^2 + 1^2}}$$

$$\frac{6}{\sqrt{62}}, \frac{-5}{\sqrt{62}}, \frac{1}{\sqrt{62}}$$

### **Intersection of two surfaces**

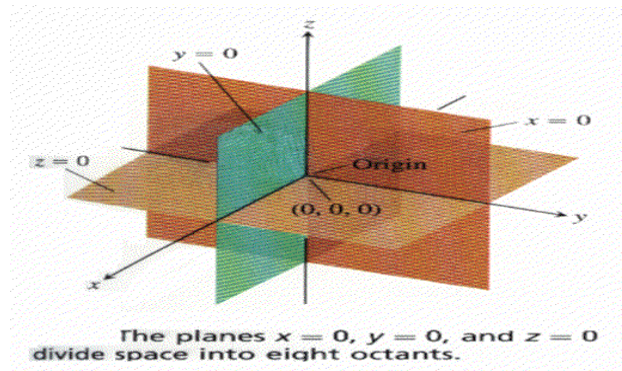
- Intersection of two surfaces is a curve in three dimensional space.
- It is the reason that a curve in three dimensional space is represented by two equations representing the intersecting surfaces.

### **Intersection of Cone and Sphere**



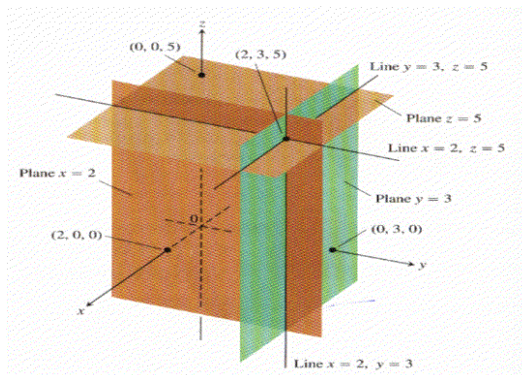
### **Intersection of Two Planes**

If the two planes are not parallel, then they intersect and their intersection is a straight line. Thus, two non-parallel planes represent a straight line given by two simultaneous linear equations in  $x$ ,  $y$  and  $z$  and are known as non-symmetric form of equations of a straight line.



REGION	DESCRIPTION	EQUATION
xy-plane	Consists of all points of the form $(x, y, 0)$	$z = 0$
xz-plane	Consists of all points of the form $(x, 0, z)$	$y = 0$
yz-plane	Consists of all points of the form $(0, y, z)$	$x = 0$
x-axis	Consists of all points of the form $(x, 0, 0)$	$y = 0, z = 0$
y-axis	Consists of all points of the form $(0, y, 0)$	$z = 0, x = 0$
z-axis	Consists of all points of the form $(0, 0, z)$	$x = 0, y = 0$

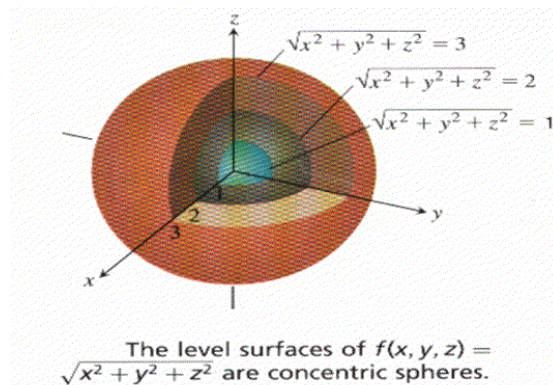
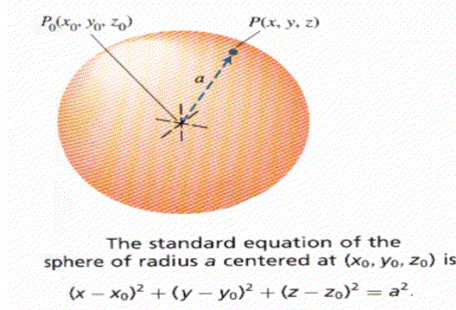
### Planes parallel to Coordinate Planes



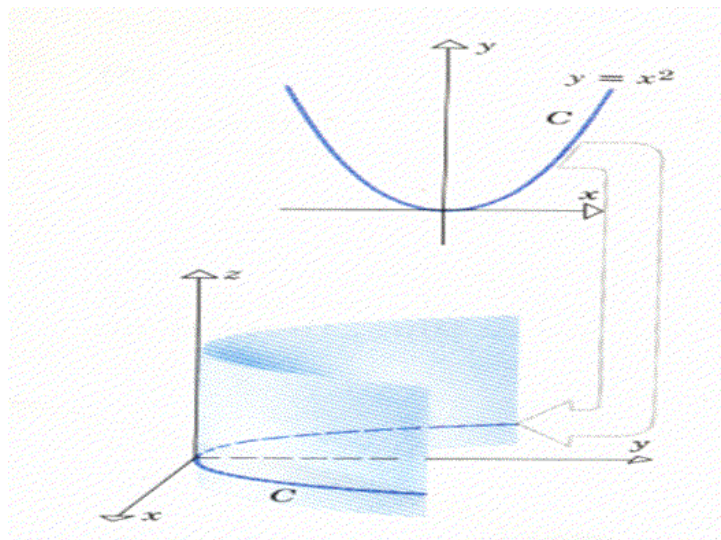
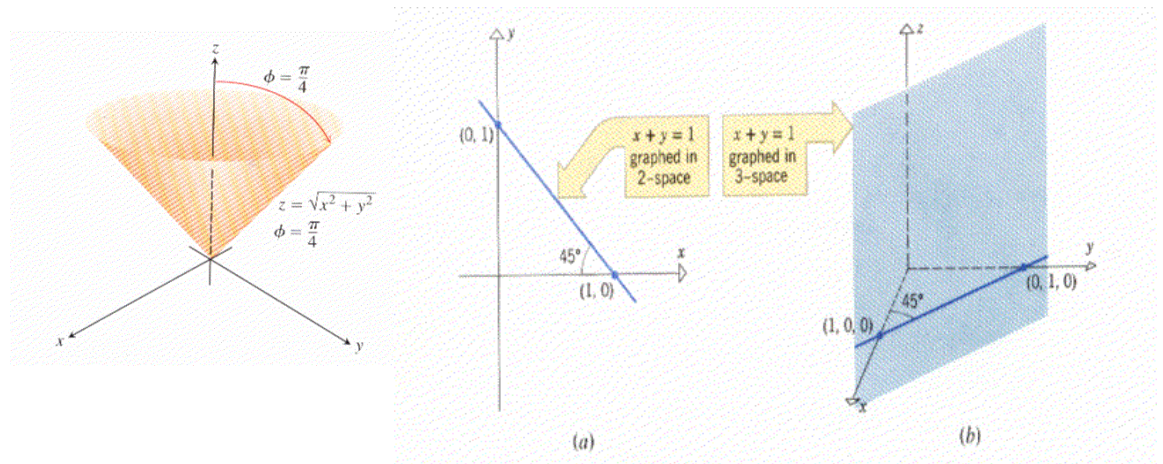
### General Equation of Plane

Any equation of the form  $ax + by + cz + d = 0$  represents a plane, where  $a, b, c, d$  are real numbers.

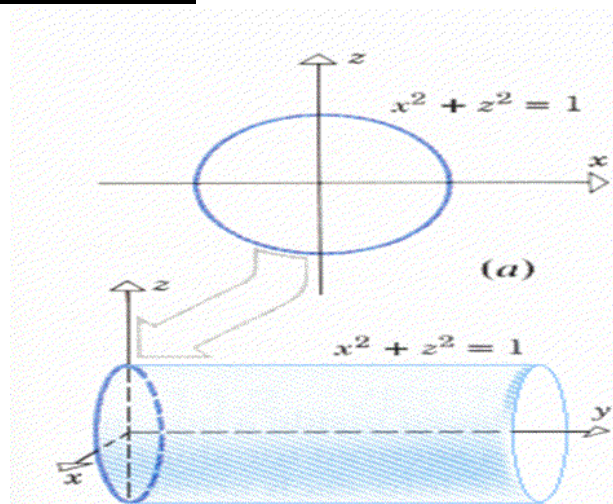
### Sphere



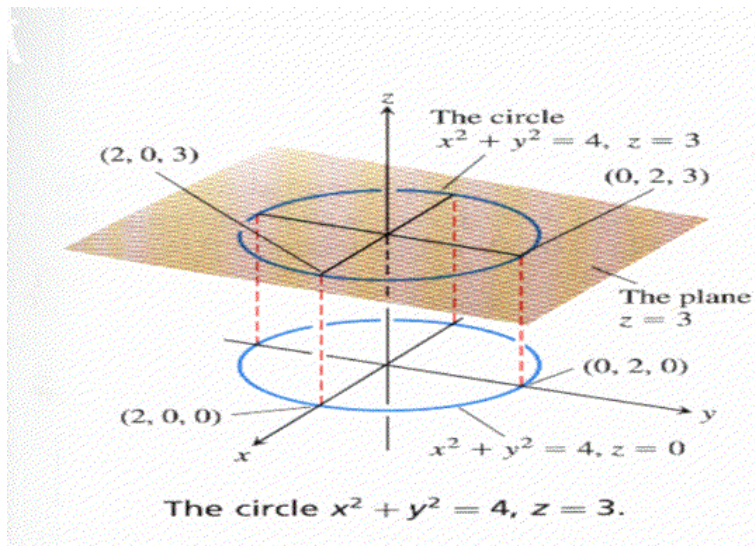
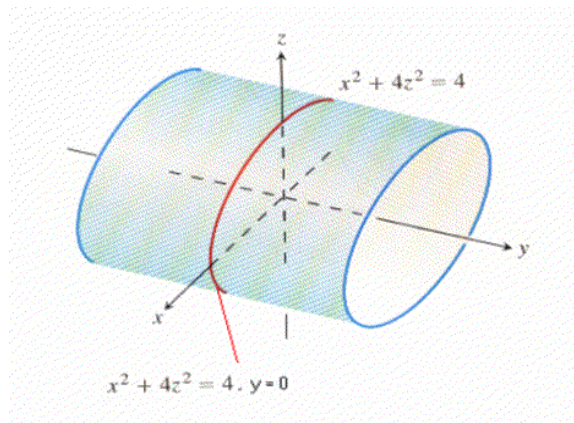
### Right Circular Cone



### Horizontal Circular Cylinder



### Horizontal Elliptic Cylinder



### Overview of Lecture # 3

Chapter # 14

Three Dimensional Space

Page # 657

Book **CALCULUS** by **HOWARD ANTON**

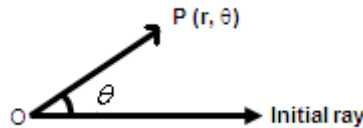
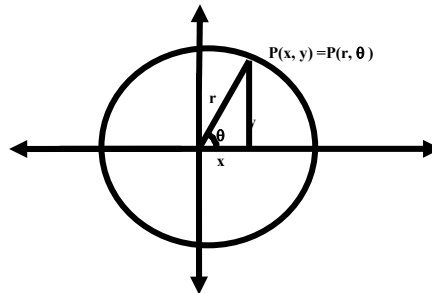
## LECTURE No.4

## POLAR COORDINATES

## Outlines of the lecture:

- Spherical Polar Coordinate
- Cylindrical Polar Coordinate

You know that position of any point in the plane can be obtained by the two perpendicular lines known as  $x$  and  $y$  axes and together we call it as Cartesian coordinates for plane. Beside this coordinate system, we have another coordinate system which can also be used for obtaining the position of any point in the plane. It is called Polar coordinate system. In this coordinate system, we represent position of each particle in the plane by  $r$  and  $\theta$  where  $r$  the distance from a fixed point known as pole is  $O$  and  $\theta$  is the measure of the angle.

Conversion formula from polar to Cartesian coordinates and vice versa

Now we convert the polar coordinates  $P(r, \theta)$  to Cartesian coordinates  $P(x, y)$ .

From above diagram and remembering the trigonometric ratios we can write

$$\frac{x}{r} = \cos \theta \quad \Rightarrow \quad x = r \cos \theta \quad \text{--- (1)}$$

$$\frac{y}{r} = \sin \theta \quad \Rightarrow \quad y = r \sin \theta \quad \text{--- (2)}$$

The equations (1) and (2) are used to convert the polar coordinates  $P(r, \theta)$  to Cartesian coordinates  $P(x, y)$ . Now we convert the Cartesian coordinates  $P(x, y)$  into polar coordinates  $P(r, \theta)$ . Squaring equations (1) and (2), and adding them, we get,

$$x^2 + y^2 = (r \cos \theta)^2 + (r \sin \theta)^2 = r^2 (\cos^2 \theta + \sin^2 \theta)$$

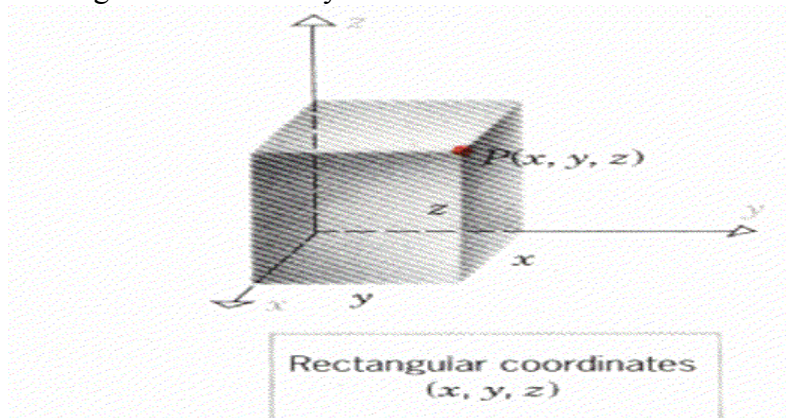
$$x^2 + y^2 = r^2 \quad \text{or} \quad r = \sqrt{x^2 + y^2} \quad \text{---- (3)}$$

Dividing equation (2) by equation (1), we get

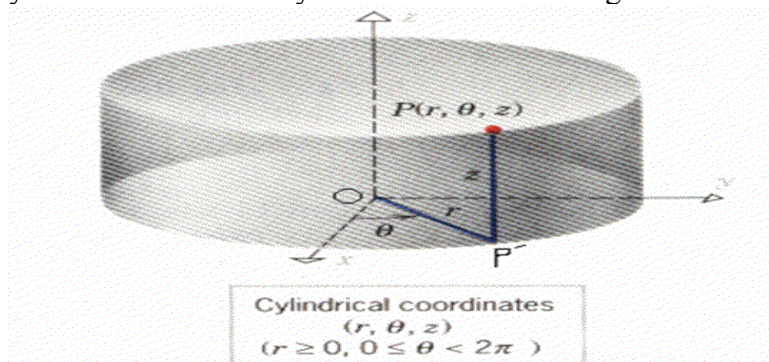
$$\frac{y}{x} = \tan \theta \quad \text{---(4)}$$

The equations (3) and (4) are used to convert the Cartesian coordinates  $P(x, y)$  to polar coordinates  $P(r, \theta)$ .

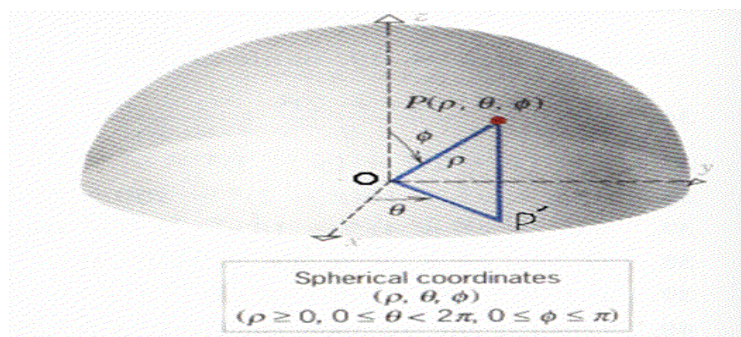
**Rectangular coordinates for three dimensions:** Since you know that the position of any point in the three dimensions can be obtained by the three mutually perpendicular lines known as  $x$ ,  $y$  and  $z$  – axes and also shown in figure below. These coordinate axes are known as Rectangular coordinate system.



**Cylindrical Coordinates:** Beside the Rectangular coordinate system, we have another coordinate system which is used for getting the position of the any particle in space, known as the cylindrical coordinate system as shown in the figure below.



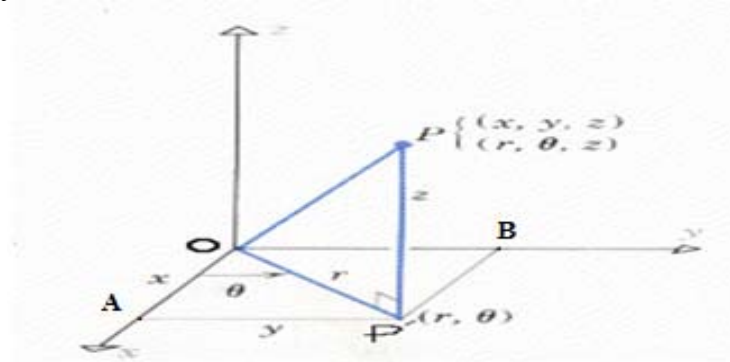
**Spherical Coordinates:** Beside the Rectangular and Cylindrical coordinate systems, we have another coordinate system which is used for getting the position of the any particle in space, known as the spherical coordinate system as shown in the figure below.



### Conversion formulas between Rectangular and Cylindrical coordinates

Now we will find out the relation between the Rectangular coordinate system and Cylindrical coordinate system.

For this, consider any point P in the space and consider the position of this point in both the coordinate systems as shown in the figure below:



In the above figure, we have the projection  $P'(r, \theta)$  of the point  $P(x, y, z)$  in the  $xy$ -plane and write its position in plane polar coordinates and also represent the angle  $\theta$ . Now from that projection, we draw perpendiculars  $P'A$  and  $P'B$  to both of the axes and using the trigonometric ratios, find out the following relations:

$$\frac{x}{r} = \cos \theta, \quad \frac{y}{r} = \sin \theta, \quad z = z$$

$$\text{Therefore,} \quad x = r \cos \theta, \quad y = r \sin \theta, \quad z = z$$

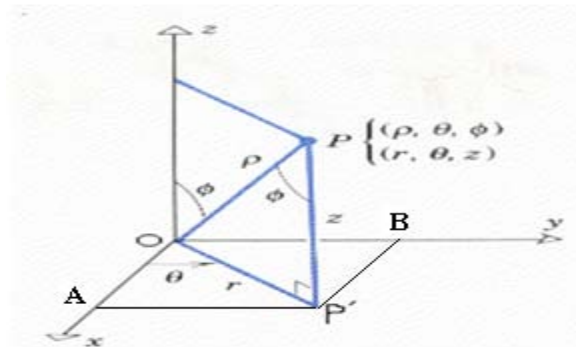
These equations convert the polar coordinates  $P(r, \theta, z)$  to Cartesian coordinates  $P(x, y, z)$ .

$$r = \sqrt{x^2 + y^2}, \quad \tan \theta = \frac{y}{x}, \quad z = z$$

These equations convert the Cartesian coordinates  $P(x, y, z)$  to polar coordinates  $P(r, \theta, z)$ .

### Conversion formulas between cylindrical and spherical coordinates

Now we will find out the relation between spherical coordinate system and cylindrical coordinate system.



First we will find the relation between Planes polar to spherical. From the above figure, you can easily see that from the two right angled triangles we have the following relations:  $(\rho, \theta, \phi) \rightarrow (r, \theta, z)$

In the triangle,  $\cos \theta = \frac{x}{r} \Rightarrow x = r \cos \theta$

$$\sin \theta = \frac{y}{r} \Rightarrow y = r \sin \theta$$

In the triangle,  $\sin \phi = \frac{r}{\rho} \Rightarrow r = \rho \sin \phi \quad \text{----(a)}$

$$\cos \phi = \frac{z}{\rho} \Rightarrow z = \rho \cos \phi \quad \text{----(b)}$$

Therefore,  $r = \rho \sin \phi, \quad \theta = \theta, \quad z = \rho \cos \phi$

Now from these equations we will solve the first and second equation for  $\rho$  and  $\phi$ . Thus we have  $(r, \theta, z) \rightarrow (\rho, \theta, \phi)$

Squaring and adding equations (a) and (b), we get

$$\rho = \sqrt{r^2 + z^2}$$

Divide equation (a) by equation (b),  $\tan \phi = \frac{r}{z}$

Therefore,  $\rho = \sqrt{r^2 + z^2}, \quad \theta = \theta, \quad \tan \phi = \frac{r}{z}$

#### **Conversion formulae between Rectangular and Spherical coordinates $(\rho, \theta, \phi) \rightarrow (x, y, z)$**

Since we know that the relation between Cartesian coordinates and Polar coordinates are

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z \quad \text{----(A)}$$

We also know that the relation between Spherical and cylindrical coordinates are,

$$r = \rho \sin \phi, \quad \theta = \theta, \quad z = \rho \cos \phi \quad \text{----(B)}$$

Now putting this value of  $r$  and  $z$  from (B) in (A), we get

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi \quad \text{----(C)}$$

It is the relation between spherical coordinate system and Cartesian coordinate system.

Now we will find  $(x, y, z) \rightarrow (\rho, \theta, \phi)$

Squaring and adding the equations in (C),

$$\begin{aligned} x^2 + y^2 + z^2 &= (\rho \sin \phi \cos \theta)^2 + (\rho \sin \phi \sin \theta)^2 + (\rho \cos \phi)^2 \\ &= \rho^2 [\sin^2 \phi (\cos^2 \theta + \sin^2 \theta) + \cos^2 \phi] \\ &= \rho^2 [\sin^2 \phi + \cos^2 \phi] \\ &= \rho^2 \end{aligned}$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

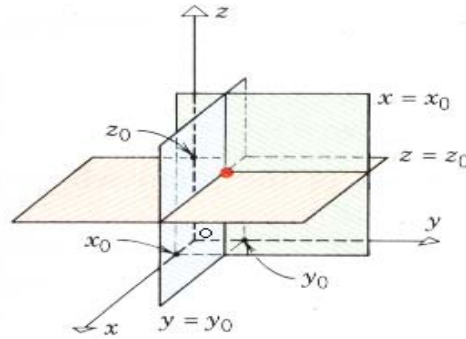
Also,  $\tan \theta = \frac{y}{x}$  And  $\cos \phi = \frac{z}{\rho} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$

### Constant Surfaces in Rectangular Coordinates

The surfaces represented by equations of the form

$$x = x_0, \quad y = y_0, \quad z = z_0$$

where  $x_0, y_0, z_0$  are constants, and are planes parallel to the  $yz$ -plane,  $xz$ -plane and  $xy$ -plane, respectively. Also shown in the figure,



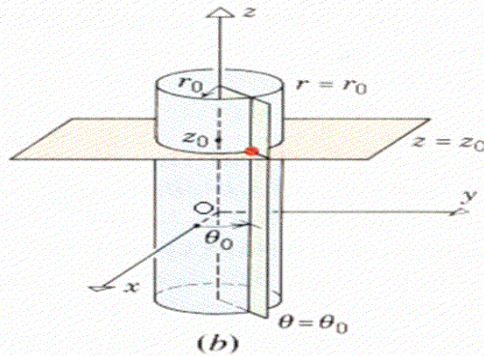
(a)

### Constant Surfaces in Cylindrical Coordinates

The surface  $r = r_0$  is a **right cylinder** of radius  $r_0$  centered on the  $z$ -axis. At each point  $(r, \theta, z)$ , this surface on this cylinder,  $r$  has the value  $r_0$ ,  $z$  is unrestricted and  $0 \leq \theta < 2\pi$ .

The surface  $\theta = \theta_0$  is a **half plane** attached along the  $z$ -axis and making angle  $\theta_0$  with the positive  $x$ -axis. At each point  $(r, \theta, z)$  on the surface,  $\theta$  has the value  $\theta_0$ ,  $z$  is unrestricted and  $r \geq 0$ .

The surfaces  $z = z_0$  is a **horizontal plane**. At each point  $(r, \theta, z)$  this surface  $z$  has the value  $z_0$ , but  $r$  and  $\theta$  are unrestricted as shown in the figure below.



(b)

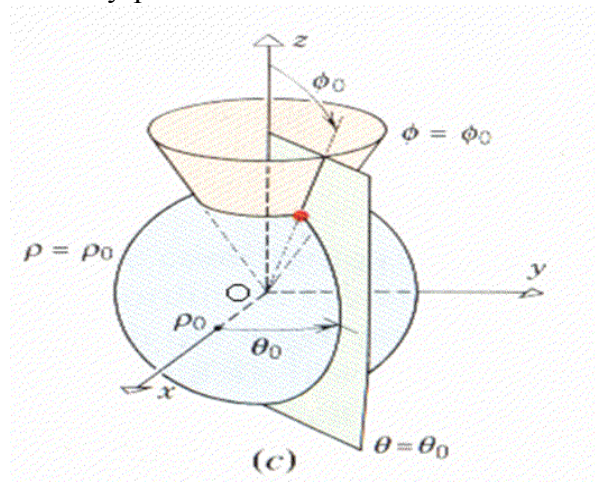
### Constant Surfaces in Spherical Coordinates

The surface  $\rho = \rho_0$  consists of all points whose distance  $\rho$  from origin is  $\rho_0$ . Assuming that  $\rho_0$  to be nonnegative, this is a sphere of radius  $\rho_0$  centered at the origin. The surface

$\theta = \theta_0$  is a **half plane** attached along the  $z$ -axis and making angle  $\theta_0$  with the positive  $x$ -axis. The surface  $\Phi = \Phi_0$  consists of all points from which a line segment to the origin

makes an angle of  $\Phi_0$  with the positive  $z$ -axis. Depending on whether  $0 < \Phi_0 < \frac{\pi}{2}$  or

$\frac{\pi}{2} < \Phi_0 < \pi$ , this will be a cone opening up or opening down. If  $\Phi_0 = \frac{\pi}{2}$ , then the cone is flat and the surface is the xy-plane.



### Spherical Coordinates in Navigation

Spherical coordinates are related to longitude and latitude coordinates used in navigation. Let us consider a right handed rectangular coordinate system with origin at earth's center, positive z-axis passing through the North Pole, and x-axis passing through the prime meridian. Considering earth to be a perfect sphere of radius  $\rho = 4000$  miles, then each point has spherical coordinates of the form  $(4000, \theta, \Phi)$  where  $\Phi$  and  $\theta$  determine the latitude and longitude of the point. Longitude is specified in degree east or west of prime meridian and latitudes is specified in degree north or south of the equator.

### Domain of the Function

- In the above definitions, the set  $D$  is the domain of the function.
- The Set of all values which the function assigns for every element of the domain is called the Range of the function.
- When the range consists of real numbers, the functions are called the real valued function.

### Note:

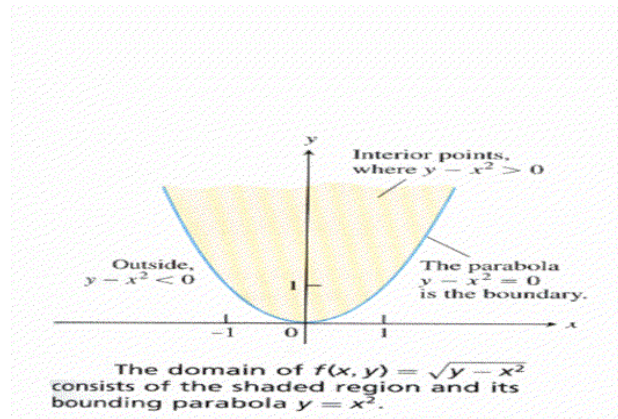
- If a function is of single variable i.e.  $y = f(x)$ , then domain is a subset of real line and its graph is a curve.
- If a function is of two variables i.e.  $z = f(x, y)$ , then domain will be from xy-plane.
- If a function is of three variables i.e.  $w = f(x, y, z)$ , then domain is a subset of space.

### NATURAL DOMAIN

Natural domain consists of all those points at which the formula has no divisions by zero and produces only real numbers.

### Example

Consider the function  $\omega = \sqrt{y - x^2}$ . Then the domain of the function is  $y \geq x^2$  which can be shown in the plane as parabola opening upwards. It includes the shaded area and its boundary is  $y = x^2$  and the range of the function is  $[0, \infty)$ .



### Example

Consider the function  $w = \frac{1}{xy}$

Domain of function  $w = \frac{1}{xy}$  is the whole xy-plane, excluding x-axis and y-axis because at x-axis,  $y=0$  and at y-axis,  $x=0$ .

Domain:  $xy \neq 0 \Rightarrow x \neq 0, y \neq 0$

Domain is entire xy-plane except x-axis and y-axis.

Range is  $(-\infty, 0) \cup (0, \infty)$

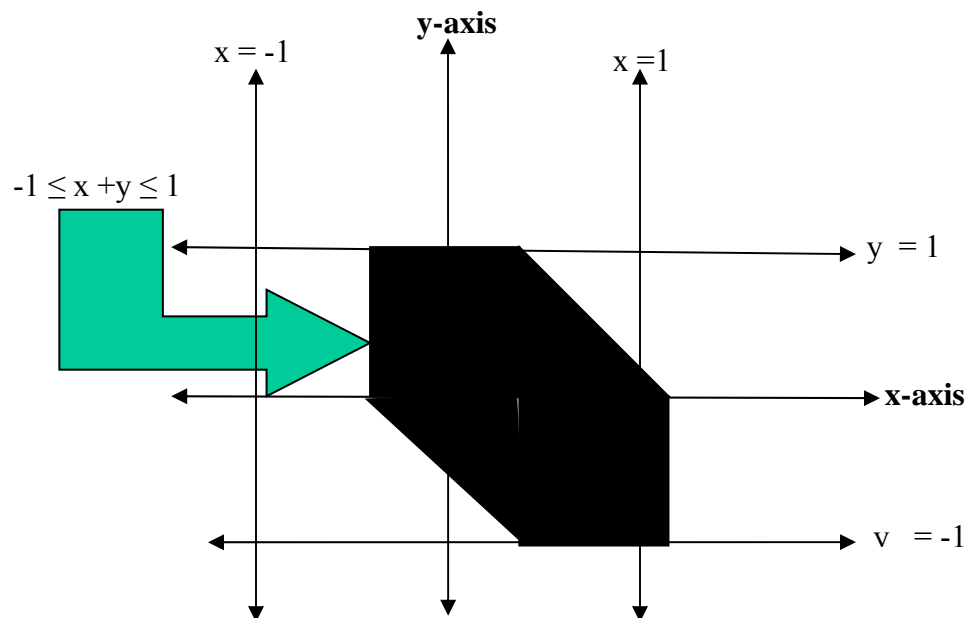
## LECTURE No. 5

## LIMIT OF MULTIVARIABLE FUNCTION

**Example 1:**

$$f(x, y) = \sin^{-1}(x + y)$$

Domain of  $f$  is the region in which  $-1 \leq x + y \leq 1$

**Domains and Ranges**

<i>Functions</i>	<i>Domain</i>	<i>Range</i>
1) $\omega = \sqrt{x^2 + y^2 + z^2}$	Entire space	$[0, \infty)$
2) $\omega = \frac{1}{x^2 + y^2 + z^2}$	Entire space except origin $(x, y, z) \neq (0, 0, 0)$	$(0, \infty)$
3) $\omega = xy \ln z$	Half space, $z > 0$	$(-\infty, \infty)$

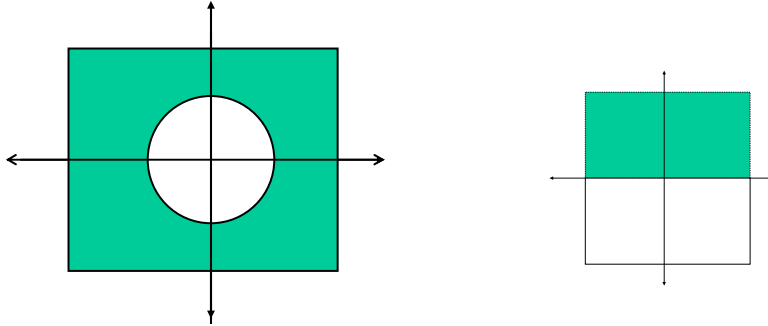
**Examples of domain of a function****Example 2:**  $f(x, y) = xy \sqrt{y - 1}$ 

Domain of  $f$  consists of the region in  $xy$ -plane where  $y \geq 1$ .

(Here we take  $y - 1 \geq 0$  for real values.)

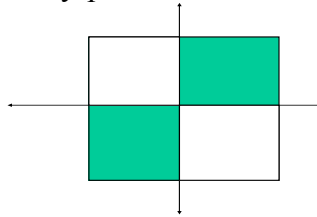
**Example 3:**  $f(x, y) = \sqrt{x^2 + y^2 - 4}$

Domain of  $f$  consists of the region in  $xy$ -plane where  $x^2 + y^2 \geq 4$ . It means that the points of the domain lie outside the circle with radius 2. As shown in the figure



**Example 4:**  $f(x, y) = \ln xy$

For the real values of logarithmic function,  $xy > 0$  which is possible: When  $x < 0, y < 0$  (3rd quadrant) and when  $x > 0, y > 0$  (1st quadrant). Domain of  $f$  consists of region lying in first and third quadrants in  $xy$ -plane as shown below.



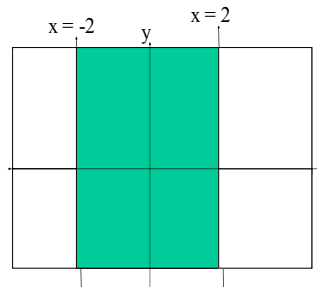
**Example 5:**  $f(x, y, z) = e^{xyz}$

Domain of  $f$  consists of the entire region of three dimensional space.

**Example 6:**  $f(x, y) = \frac{\sqrt{4 - x^2}}{y^2 + 3}$

Here we take  $4 - x^2 \geq 0$  for real values of  $f(x, y)$ .

Domain of  $f$  consists of region in  $xy$ -plane where  $x^2 \leq 4$  which implies that  $-2 \leq x \leq 2$ .



**Example 7:**  $f(x, y, z) = \sqrt{25 - x^2 - y^2 - z^2}$

Here we take  $25 - x^2 - y^2 - z^2 \geq 0$  for real values of  $f(x, y)$ . So,  $x^2 + y^2 + z^2 \leq 5^2$

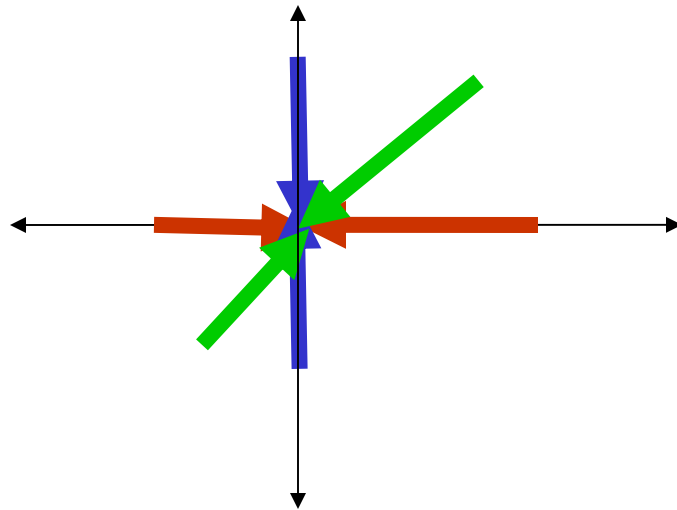
Domain of  $f$  consists of region in three dimensional space occupied by sphere centre at  $(0, 0, 0)$  and radius 5.

**Example 8:**  $f(x, y) = \frac{x^3 + 2x^2y - xy - 2y^2}{x + 2y}$

$f(0, 0)$  is not defined but we see that limit exists.

Approaching to (0,0) through x-axis	$f(x, y)$	Approaching to (0,0) through y-axis	$f(x, y)$
$(0.5, 0)$	0.25	$(0, 0.1)$	-0.1
$(0.25, 0)$	0.0625	$(0, 0.001)$	-0.001
$(0.1, 0)$	0.01	$(0, 0.00001)$	0.00001
$(-0.25, 0)$	0.0625	$(0, -0.001)$	0.001
$(-0.1, 0)$	0.01	$(0, -0.00001)$	0.00001

Approaching to (0,0) through $y = x$	$f(x, y)$
$(0.5, 0.5)$	-0.25
$(0.1, 0.1)$	-0.09
$(0.01, 0.01)$	-0.0099
$(-0.5, -0.5)$	0.75
$(-0.1, -0.1)$	0.11
$(-0.01, -0.01)$	0.0101

**Example 9:**

$$f(x,y) = \frac{xy}{x^2+y^2}$$

$f(0,0)$  is not defined and we see that limit also does not exist.

Approaching to (0,0) through x-axis ( $y = 0$ )	$f(x,y)$	Approaching to (0,0) through $y = x$	$f(x,y)$
( 0.5,0 )	0	( 0.5,0.5 )	0.5
( 0.1,0 )	0	( 0.25,0.25 )	0.5
( 0.01,0 )	0	( 0.1,0.1 )	0.5
( 0.001,0 )	0	( 0.05,0.05 )	0.5
( 0.0001,0 )	0	( 0.001,0.001 )	0.5
( -0.5,0 )	0	( -0.5,-0.5 )	0.5
( -0.1,0 )	0	( -0.25,-0.25 )	0.5
( -0.01,0 )	0	( -0.1,-0.1 )	0.5
( -0.001,0 )	0	( -0.05,-0.05 )	0.5
( -0.0001,0 )	0	( -0.001,-0.001 )	0.5

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} = 0 \text{ (along } y = 0 \text{)}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} = 0.5 \text{ (along } y = x \text{)}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} \text{ does not exist.}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2}$$

Let  $(x, y)$  approach  $(0, 0)$  along the line  $y = x$ .

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

$$f(x, x) = \frac{xx}{x^2 + x^2} = \frac{x^2}{2x^2} = \frac{1}{2} \quad x \neq 0$$

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} = \frac{1}{2} \quad \text{Along the line } y = x$$

Now let  $(x, y)$  approach  $(0, 0)$  along x-axis. On x-axis,  $y = 0$ .

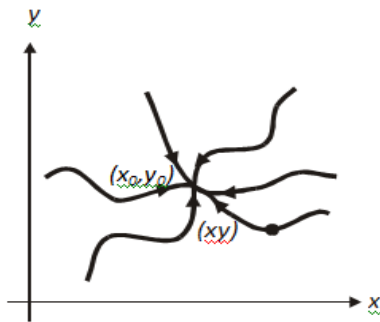
$$f(x, 0) = \frac{x \times 0}{x^2 + 0^2} = \frac{0}{x^2} = 0 \quad x \neq 0$$

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} = 0 \quad \text{Along the line } x - \text{axis.}$$

Therefore  $f(x, y)$  assumes two different values, as  $(x, y)$  approaches  $(0, 0)$  along two

different paths. So  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2}$  does not exist.

We can approach a point in space through infinite paths some of them are shown in the figure below:



**Rule for Non-Existence of a Limit**

If in  $\lim_{(x,y) \rightarrow (a,b)} f(x, y)$ , we get two or more different values, as  $(x, y)$  approaches  $(a, b)$

along two different paths, then  $\lim_{(x,y) \rightarrow (a,b)} f(x, y)$  does not exist.

The paths along which  $(a, b)$  is approached may be straight *lines or plane curves through*  $(a, b)$

**Example 10**

$$\begin{aligned} \lim_{(x,y) \rightarrow (2,1)} \frac{x^3 + 2x^2y - x - 2y^2}{x + 2y} &= \frac{\lim_{(x,y) \rightarrow (2,1)} (x^3 + 2x^2y - x - 2y^2)}{\lim_{(x,y) \rightarrow (2,1)} (x + 2y)} \\ &= \frac{(2^3 + 2(2)^2(1) - (2) - 2(1)^2)}{(2 + 2(1))} = \frac{8 + 8 - 2 - 2}{4} = 3 \end{aligned}$$

**Example 11**

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$$

We set  $x = r \cos \theta$ ,  $y = r \sin \theta$ , then

$$\begin{aligned} \frac{xy}{\sqrt{x^2 + y^2}} &= \frac{(r \cos \theta)(r \sin \theta)}{\sqrt{(r \cos \theta)^2 + (r \sin \theta)^2}} \\ &= \frac{(r^2 \cos \theta \sin \theta)}{r \sqrt{\cos^2 \theta + \sin^2 \theta}} = \frac{(r \cos \theta \sin \theta)}{\sqrt{1}} \\ &= r \cos \theta \sin \theta, \quad r > 0 \end{aligned}$$

Since  $r = \sqrt{x^2 + y^2}$ , so  $r \rightarrow 0$  as  $(x, y) \rightarrow (0, 0)$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2 + y^2}} = \lim_{r \rightarrow 0} r \cos \theta \sin \theta = 0 \times \cos \theta \sin \theta = 0$$

**Note** that  $|\cos \theta \sin \theta| < 1$  for all values of  $\theta$ .

**RULES FOR LIMIT**

If  $\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y) = L_1$  and  $\lim_{(x,y) \rightarrow (x_0, y_0)} g(x, y) = L_2$ , then

$$(a) \quad \lim_{(x,y) \rightarrow (x_0, y_0)} cf(x, y) = cL_1 \quad (\text{if } c \text{ is constant})$$

$$(b) \quad \lim_{(x,y) \rightarrow (x_0, y_0)} \{f(x, y) + g(x, y)\} = L_1 + L_2$$

$$(c) \quad \lim_{(x,y) \rightarrow (x_0, y_0)} \{f(x, y) - g(x, y)\} = L_1 - L_2$$

$$(d) \quad \lim_{(x,y) \rightarrow (x_0, y_0)} \{f(x, y)g(x, y)\} = L_1L_2$$

$$(e) \quad \lim_{(x,y) \rightarrow (x_0, y_0)} \frac{f(x, y)}{g(x, y)} = \frac{L_1}{L_2} \quad (\text{if } L_2 \neq 0)$$

$$\lim_{(x,y) \rightarrow (x_0,y_0)} c = c \quad (c \text{ is a constant}), \quad \lim_{(x,y) \rightarrow (x_0,y_0)} x_0 = x_0, \quad \lim_{(x,y) \rightarrow (x_0,y_0)} y_0 = y_0$$

Similar rules are for the function of three variables.

### **Overview of lecture# 5**

In this lecture we recall you all the limit concept which are prerequisite for this course and you can find all these concepts in the chapter # 16 (topic # 16.2) of your Calculus By Howard Anton.

## LECTURE No. 6

## GEOMETRY OF CONTINUOUS FUNCTIONS

**Geometry of continuous functions in one variable or Informal definition of continuity of function of one variable**

A function is continuous if we draw its graph by a pen such that the pen is not raised so that there is no gap in the graph of the function.

**Geometry of continuous functions in two variables or Informal definition of continuity of function of two variables**

The graph of a continuous function of two variables to be constructed from a thin sheet of clay that has been hollowed and pinched into peaks and valleys without creating tears or pinholes.

**Continuity of functions of two variables**

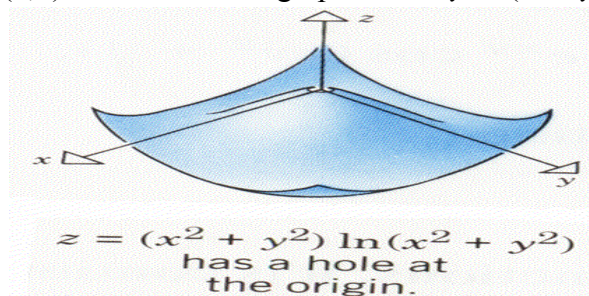
A function  $f$  of two variables is called continuous at the point  $(x_0, y_0)$  if  $f$  satisfies the following conditions:

1.  $f(x_0, y_0)$  is defined.
2.  $\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y)$  exists.
3.  $\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y) = f(x_0, y_0)$

The requirement that  $f(x_0, y_0)$  must be defined at the point  $(x_0, y_0)$  eliminates the possibility of a hole in the surface  $z = f(x, y)$  above the point  $(x_0, y_0)$ .

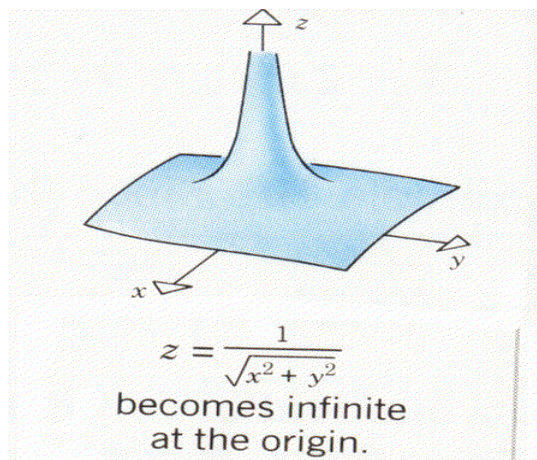
**Justification of three points involving in the definition of continuity**

(1) Consider the function of two variables  $x^2 + y^2 \ln(x^2 + y^2)$ . Now as we know that the Log function is not defined at 0, it means that when  $x = 0$  and  $y = 0$ , our function  $x^2 + y^2 \ln(x^2 + y^2)$  is not defined. Consequently the surface  $z = x^2 + y^2 \ln(x^2 + y^2)$  will have a hole just above the point  $(0,0)$  as shown in the graph of  $x^2 + y^2 \ln(x^2 + y^2)$



(2) The requirement that  $\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y)$  exists ensures us that the surface  $z = f(x, y)$  of the function  $f(x, y)$  doesn't become infinite at  $(x_0, y_0)$  or doesn't oscillate widely.

Consider the function of two variables  $\frac{1}{\sqrt{x^2 + y^2}}$ . Now as we know that the Natural domain of the function is whole the plane except origin. Because at origin, we have  $x = 0$  and  $y = 0$ . In the defining formula of the function, we will have  $\frac{1}{0}$  at that point which is infinity. Thus the limit of the function  $\frac{1}{\sqrt{x^2 + y^2}}$  does not exist at origin. Consequently the surface  $z = \frac{1}{\sqrt{x^2 + y^2}}$  will approach towards infinity when we approach towards origin as shown in the figure above.



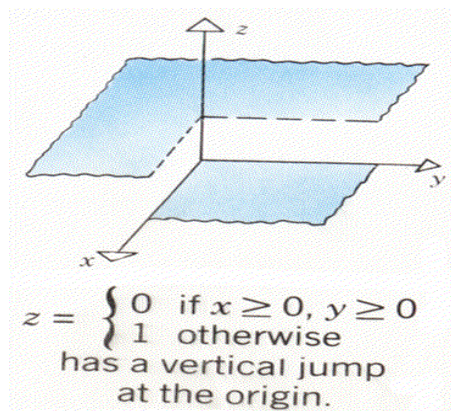
**(3)** The requirement that  $\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y) = f(x_0, y_0)$  ensures us that the surface  $z = f(x, y)$  of the function  $f(x, y)$  doesn't have a vertical jump or step above the point  $(x_0, y_0)$ .

Consider the function of two variables

$$f(x, y) = \begin{cases} 0 & \text{if } x \geq 0 \text{ and } y \geq 0 \\ 1 & \text{otherwise} \end{cases}$$

Now as we know that the Natural domain of the function is whole the plane. But you should note that the function has one value "0" for all the points in the plane for which both  $x$  and  $y$  have nonnegative values. And value "1" for all other points in the plane. Consequently the surface

$$z = f(x, y) = \begin{cases} 0 & \text{if } x \geq 0 \text{ and } y \geq 0 \\ 1 & \text{otherwise} \end{cases} \quad \text{It has a jump as shown in the figure.}$$



**Example 1:** Check whether the limit at  $(0, 0)$  exists or not for the function

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \frac{x^2}{x^2 + y^2}$$

**Solution:** First we will calculate the Limit of the function along  $x$ -axis and we get

$$\lim_{(x,y) \rightarrow (0,0)} f(x, 0) = \lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2 + 0} = \lim_{(x,y) \rightarrow (0,0)} 1 = 1 \quad (\text{Along } x\text{-axis, } y = 0)$$

Now we will find out the limit of the function along  $y$ -axis and we note that the limit is

$$\lim_{(x,y) \rightarrow (0,0)} f(0, y) = \lim_{(x,y) \rightarrow (0,0)} \frac{0^2}{0^2 + y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{0}{y^2} = \lim_{(x,y) \rightarrow (0,0)} 0 = 0 \quad (\text{Along } y\text{-axis, } x = 0)$$

Now we will find out the limit of the function along the line  $y = x$  and we note that

$$\lim_{(x,y) \rightarrow (0,0)} f(x, x) = \lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2 + x^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{2x^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{1}{2} = \frac{1}{2} \quad (\text{Along } y = x)$$

It means that limit of the function  $f(x, y)$  at  $(0, 0)$  doesn't exist because it has different values along different paths. Thus the function cannot be continuous at  $(0, 0)$ . And also note that the function is not defined at  $(0, 0)$  and hence it doesn't satisfy two conditions of the continuity.

**Example 2:** Check the continuity of the function at  $(0, 0)$

$$f(x, y) = \begin{cases} \frac{\sin(x^2 + y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$$

**Solution:** First we will note that the function is defined on the point where we have to check the Continuity; that is, the function has value at  $(0, 0)$ . Next we will find out the Limit of the

function at  $(0, 0)$  and in evaluating this limit, we use the result  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$  and note that

$$\begin{aligned}\lim_{(x,y) \rightarrow (0,0)} f(x,y) &= \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} \\ &= 1 = f(0, 0)\end{aligned}$$

This shows that  $f$  is continuous at  $(0,0)$

### CONTINUITY OF FUNCTION OF THREE VARIABLES

A function  $f$  of three variables is called *continuous at a point*  $(x_0, y_0, z_0)$  if

1.  $f(x_0, y_0, z_0)$  is defined.
2.  $\lim_{(x,y,z) \rightarrow (x_0, y_0, z_0)} f(x, y, z)$  exists.
3.  $\lim_{(x,y,z) \rightarrow (x_0, y_0, z_0)} f(x, y, z) = f(x_0, y_0, z_0)$ .

**EXAMPLE 3:** Check the continuity of the function

$$f(x, y, z) = \frac{y+1}{x^2 + y^2 - 1}$$

**Solution:** First of all, note that the given function is not defined on the cylinder  $x^2 + y^2 - 1 = 0$ .

Thus the function is not continuous on the cylinder  $x^2 + y^2 - 1 = 0$

However,  $f(x, y, z)$  is continuous at **all other points** of its domain.

### RULES FOR CONTINUOUS FUNCTIONS

- 1) If  $g$  and  $h$  are continuous functions of one variable, then  $f(x, y) = g(x)h(y)$  is a continuous function of  $x$  and  $y$ .
- 2) If  $g$  is a continuous function of one variable and  $h$  is a continuous function of two variables, then their composition  $f(x, y) = g(h(x, y))$  is a continuous function of  $x$  and  $y$ .
- 3) A composition of continuous functions is continuous.
- 4) A sum, difference, or product of continuous functions is continuous.
- 5) A quotient of continuous function is continuous, except where the denominator is zero.

### EXAMPLE OF PRODUCT OF FUNCTIONS TO BE CONTINUED

In general, any function of the form  $f(x, y) = Ax^m x^n$  ( $m$  and  $n$  non-negative integers) is continuous everywhere in the domain because it is the product of continuous functions  $Ax^m$  and  $x^n$ . The function of the form  $f(x, y) = 3x^2 x^5$  is continuous every where in the domain because it is the product of continuous functions  $g(x) = 3x^2$  and  $h(y) = y^5$ .

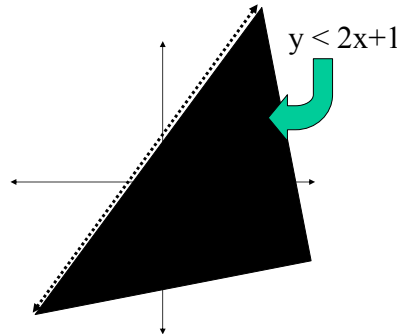
### CONTINUOUS EVERYWHERE

A function  $f$  that is continuous at each point of a region  $R$  in 2-dimensional plane or 3-dimensional space is said to be *continuous on  $R$* . A function that is continuous at every point in 2-dimensional plane or 3-dimensional space is called continuous *everywhere* or simply *continuous*.

**EXAMPLES**

(1)  $f(x, y) = \ln(2x - y + 1)$

The function  $f$  is continuous in the whole region where  $2x > y - 1$ ,  $y < 2x + 1$ . And its region is shown in figure below.



(2)  $f(x, y) = e^{1-xy}$

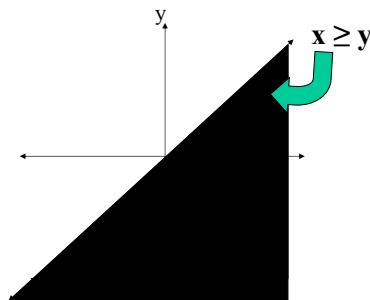
The function  $f$  is continuous in the whole region of  $xy$ -plane.

(3)  $f(x, y) = \tan^{-1}(y - x)$

The function  $f$  is continuous in the whole region of  $xy$ - plane.

(4)  $f(x, y) = \sqrt{y - x}$

The function is continuous where  $x \geq y$

**Partial Derivative**

Let  $f$  a function of  $x$  and  $y$ . If we hold  $y$  constant, say  $y = y_0$  and view  $x$  as a variable, then  $f(x, y_0)$  is a function of  $x$  alone. If this function is differentiable at  $x = x_0$ , then the value of this derivative is denoted by  $f_x(x_0, y_0)$  and is called the Partial derivative of  $f$  with respect of  $x$  at the point  $(x_0, y_0)$ .

Similarly, if we hold  $x$  constant, say  $x = x_0$  and view  $y$  as a variable, then  $f(x_0, y)$  is a function of  $y$  alone. If this function is differentiable at  $y = y_0$ , then the value of this derivative is denoted by  $f_y(x_0, y_0)$  and is called the Partial derivative of  $f$  with respect of  $y$  at the point  $(x_0, y_0)$ .

**Example 4:** Let  $f(x, y) = 2x^3y^2 + 2y + 4x$  be a surface. Find the partial derivatives of  $f$  with respect to  $x$  and  $y$  at point  $(1, 2)$ .

**Solution:** Treating  $y$  as a constant and differentiating with respect to  $x$ , we obtain

$$f_x(x, y) = 6x^2y^2 + 4$$

Treating  $x$  as a constant and differentiating with respect to  $y$ , we obtain

$$f_y(x, y) = 4x^3y + 2$$

Substituting  $x = 1$  and  $y = 2$  in these partial-derivative formulas yields.

$$f_x(1, 2) = 6(1)^2(2)^2 + 4 = 28$$

$$f_y(1, 2) = 4(1)^3(2) + 2 = 10$$

**Example 5:** Let  $z = 4x^2 - 2y + 7x^4y^5$  be a surface. Find the partial derivatives of  $z$  with respect to  $x$  and  $y$ .

**Solution:**  $z = 4x^2 - 2y + 7x^4y^5$        $\frac{\partial z}{\partial x} = 8x + 28x^3y^5$ ,       $\frac{\partial z}{\partial y} = -2 + 35x^4y^4$

**Example 6:** Let  $z = f(x, y) = x^2 \sin^2 y$  be a surface. Find the partial derivatives of  $z$  with respect to  $x$  and  $y$ .

**Solution:**  $z = f(x, y) = x^2 \sin^2 y$

Then to find the derivative of  $f$  with respect to  $x$ , we treat  $y$  as a constant.

Therefore,  $\frac{\partial z}{\partial x} = f_x = 2x \sin^2 y$

Then to find the derivative of  $f$  with respect to  $y$ , we treat  $x$  as a constant.

$$\frac{\partial z}{\partial y} = f_y = x^2 (2 \sin y \cos y) = x^2 \sin 2y$$

**Example 7:** Let  $z = \ln \left( \frac{x^2 + y^2}{x + y} \right)$  be a surface. Find the partial derivatives of  $z$  with respect to  $x$  and  $y$ .

**Solution:** By using the properties of the  $\ln$ , we can write it as

$$\begin{aligned} z &= \ln(x^2 + y^2) - \ln(x + y) \\ \frac{\partial z}{\partial x} &= \frac{1}{x^2 + y^2} \cdot 2x - \frac{1}{x + y} \\ &= \frac{2x^2 + 2xy - x^2 - y^2}{(x^2 + y^2)(x + y)} \\ &= \frac{x^2 + 2xy - y^2}{(x^2 + y^2)(x + y)} \end{aligned}$$

Similarly by symmetry,

$$\frac{\partial z}{\partial y} = \frac{y^2 + 2xy - x^2}{(x^2 + y^2)(x + y)}$$

**Example 8:** Find the partial derivatives of  $z = x^4 \sin(xy^3)$  with respect to  $x$  and  $y$ .

**Solution:**

$$\begin{aligned}
 z &= x^4 \sin(xy^3) \\
 \frac{\partial z}{\partial x} &= \frac{\partial}{\partial x} [x^4 \sin(xy^3)] \\
 &= x^4 \frac{\partial}{\partial x} [\sin(xy^3)] + \sin(xy^3) \frac{\partial}{\partial x} (x^4) \\
 &= x^4 \cos(xy^3) y^3 + \sin(xy^3) 4x^3 \\
 \frac{\partial z}{\partial x} &= x^4 y^3 \cos(xy^3) + 4x^3 \sin(xy^3) \\
 \frac{\partial z}{\partial y} &= \frac{\partial}{\partial y} [x^4 \sin(xy^3)] \\
 &= x^4 \frac{\partial}{\partial y} [\sin(xy^3)] + \sin(xy^3) \frac{\partial}{\partial y} (x^4) \\
 &= x^4 \cos(xy^3) 3xy^2 + \sin(xy^3) \cdot 0 \\
 &= 3x^5 y^2 \cos(xy^3)
 \end{aligned}$$

**Example 9:** Find the partial derivatives of  $z = \cos(x^5 y^4)$  with respect to  $x$  and  $y$ .

**Solution:**

$$\begin{aligned}
 z &= \cos(x^5 y^4) \\
 \frac{\partial z}{\partial x} &= -\sin(x^5 y^4) \frac{\partial}{\partial x} (x^5 y^4) \\
 &= -5x^4 y^4 \sin(x^5 y^4) \\
 \frac{\partial z}{\partial y} &= -\sin(x^5 y^4) \frac{\partial}{\partial y} (x^5 y^4) \\
 &= -4x^5 y^3 \sin(x^5 y^4)
 \end{aligned}$$

**Example 10:** Find the partial derivatives of  $w = x^2 + 3y^2 + 4z^2 - xyz$  with respect to  $x$ ,  $y$  and  $z$ .

**Solution:**

$$\begin{aligned}
 w &= x^2 + 3y^2 + 4z^2 - xyz \\
 \frac{\partial w}{\partial x} &= 2x - yz \\
 \frac{\partial w}{\partial y} &= 6y - xz \\
 \frac{\partial w}{\partial z} &= 8z - xy
 \end{aligned}$$

## LECTURE No.7

## GEOMETRIC MEANING OF PARTIAL DERIVATIVE

**Geometric meaning of partial derivative**

$$z = f(x, y)$$

Partial derivative of  $f$  with respect to  $x$  is denoted by  $\frac{\partial z}{\partial x}$  or  $f_x$  or  $\frac{\partial f}{\partial x}$ .

Partial derivative of  $f$  with respect to  $y$  is denoted by  $\frac{\partial z}{\partial y}$  or  $f_y$  or  $\frac{\partial f}{\partial y}$ .

**Partial Derivatives**

Let  $z = f(x, y)$  be a function of two variables  $x$  and  $y$  defined on a certain domain  $D$ .

For a given change  $\Delta x$  in  $x$ , keeping  $y$  as constant, the change  $\Delta z$  in  $z$ , is given by

$$\Delta z = f(x + \Delta x, y) - f(x, y)$$

If the ratio  $\frac{\Delta z}{\Delta x} = \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$  approaches to a finite limit as  $\Delta x \rightarrow 0$ , then this limit is

called Partial derivative of  $f$  with respect to  $x$ .

Similarly for a given change  $\Delta y$  in  $y$ , keeping  $x$  as constant, the change  $\Delta z$  in  $z$ , is given by

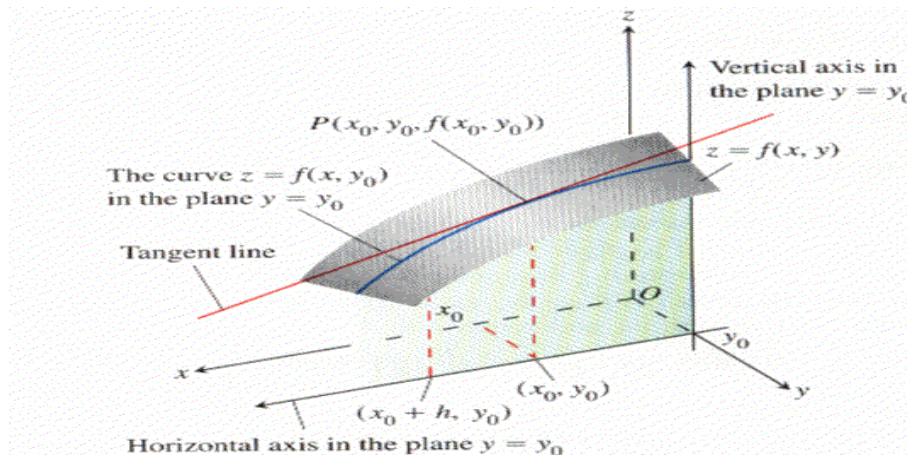
$$\Delta z = f(x, y + \Delta y) - f(x, y)$$

If the ratio  $\frac{\Delta z}{\Delta y} = \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$  approaches to a finite limit as  $\Delta y \rightarrow 0$ , then this limit is

called Partial derivative of  $f$  with respect to  $y$ .

**Geometric Meaning of Partial Derivatives**

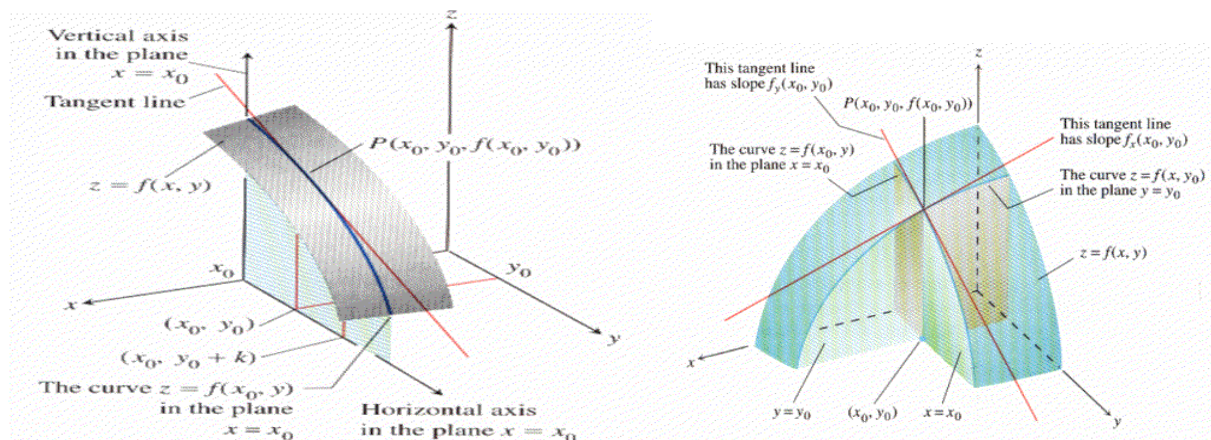
Suppose  $z = f(x, y)$  is a function of two variables  $x$  and  $y$ . The graph of  $f$  is a surface. Let  $P$  be a point on the graph with the coordinates  $(x_0, y_0, f(x_0, y_0))$ .



If a point starting from  $P$ , changes its position on the surface such that  $y$  is constant, then the locus of this point is the curve of intersection of  $z = f(x, y)$  and  $y = \text{constant}$ . On this curve,

$\frac{\partial z}{\partial x}$  is a derivative of  $z = f(x, y)$  with respect to  $x$  with  $y$  constant.

Thus,  $\frac{\partial z}{\partial x}$  = slope of the tangent to this curve at P. Similarly,  $\frac{\partial z}{\partial y}$  is the gradient of the tangent at P to this curve of intersection of  $z = f(x, y)$  and  $x = \text{constant}$ . As shown in the figure below (left). Also together these tangent lines are shown in figure below (right).



### Partial Derivatives of Higher Orders

The partial derivatives  $f_x$  and  $f_y$  of a function  $f$  of two variables  $x$  and  $y$ , being functions of  $x$  and  $y$ , may possess derivatives. In such cases, the second order partial derivatives are defined as below:

$$\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} (f_x) = (f_x)_x = f_{xx} = f_{x^2}$$

$$\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} (f_x) = (f_x)_y = f_{xy}$$

$$\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} (f_y) = (f_y)_x = f_{yx}$$

$$\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} (f_y) = (f_y)_y = f_{yy} = f_{y^2}$$

Thus there are four second order partial derivatives for a function  $z = f(x, y)$ . The partial derivatives  $f_{xy}$  and  $f_{yx}$  are called **Mixed Second partials** and are not equal in general. Partial derivatives of order more than two can be defined in a similar manner.

**Example 1:** Find  $\frac{\partial^2 z}{\partial x \partial y}$  and  $\frac{\partial^2 z}{\partial y \partial x}$  for  $z = \arcsin\left(\frac{x}{y}\right)$

**Solution :**  $z = \arcsin\left(\frac{x}{y}\right)$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} \left( \arcsin\left(\frac{x}{y}\right) \right) = \frac{1}{\sqrt{1 - \left(\frac{x}{y}\right)^2}} \cdot \frac{\partial}{\partial x} \left( \frac{x}{y} \right) = \frac{y}{\sqrt{y^2 - x^2}} \left( \frac{1}{y} \right) = \frac{1}{\sqrt{y^2 - x^2}}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} \left( \arcsin \left( \frac{x}{y} \right) \right) = \frac{1}{\sqrt{1 - \left( \frac{x}{y} \right)^2}} \frac{\partial}{\partial y} \left( \frac{x}{y} \right) = \frac{y}{\sqrt{y^2 - x^2}} \left( \frac{-x}{y^2} \right) = \frac{-x}{y\sqrt{y^2 - x^2}}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = \frac{-1}{2} (y^2 - x^2)^{-\frac{3}{2}} \frac{\partial}{\partial y} (y^2 - x^2) = \frac{-1}{2(y^2 - x^2)^{\frac{3}{2}}} \times 2y = \frac{-y}{(y^2 - x^2)^{\frac{3}{2}}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \frac{-1}{y\sqrt{y^2 - x^2}} - \frac{x}{y} \left[ -\frac{1}{2} \frac{-2x}{(y^2 - x^2)^{\frac{3}{2}}} \right] = \frac{-y^2 + x^2 - x^2}{y(y^2 - x^2)^{\frac{3}{2}}} = \frac{-y}{(y^2 - x^2)^{\frac{3}{2}}}$$

Here, you can see that  $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$

**Example 2:** Find  $\frac{\partial^2 f}{\partial x^2}$ ,  $\frac{\partial^2 f}{\partial y^2}$ ,  $\frac{\partial^2 f}{\partial y \partial x}$  and  $\frac{\partial^2 f}{\partial x \partial y}$  for  $f(x, y) = x \cos y + y e^x$ .

**Solution:**  $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (x \cos y + y e^x) = \cos y + y e^x$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (x \cos y + y e^x) = -x \sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} (\cos y + y e^x) = 0 + y e^x = y e^x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} (\cos y + y e^x) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} (-x \sin y + e^x) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} (-x \sin y + e^x) = -x \cos y$$

### **Laplace's Equation**

For a function  $w = f(x, y, z)$ , the equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

is called Laplace's equation.

**Example 3:** Show that the function  $f(x, y) = e^x \sin y + e^y \cos x$  satisfies the Laplace's equation.

**Solution:**  $f(x, y) = e^x \sin y + e^y \cos x$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (e^x \sin y + e^y \cos x) = e^x \sin y - e^y \sin x$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (e^x \sin y + e^y \cos x) = e^x \cos y + e^y \cos x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} (e^x \sin y - e^y \sin x) = e^x \sin y - e^y \cos x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} (e^x \cos y + e^y \cos x) = -e^x \sin y + e^y \cos x$$

Adding both partial second order derivatives, we have

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (e^x \sin y - e^y \cos x) + (-e^x \sin y + e^y \cos x) = 0$$

### Euler's Theorem

#### The Mixed Derivative Theorem

If  $f(x, y)$  and its partial derivatives  $f_x, f_y, f_{xy}$  and  $f_{yx}$  are defined throughout an open region containing a point  $(a, b)$  and are all continuous at  $(a, b)$ , then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

#### Advantage of Euler's theorem

$$w = xy + \frac{e^y}{y^2 + 1}$$

The symbol  $\frac{\partial^2 w}{\partial x \partial y}$  tells us to differentiate first with respect to  $y$  and then with respect to  $x$ .

However, if we postpone the differentiation with respect to  $y$  and differentiate first with respect to  $x$ , we get the answer more quickly.

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x} \left( xy + \frac{e^y}{y^2 + 1} \right) = y + 0 = y$$

$$\text{and } \frac{\partial^2 w}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial w}{\partial x} \right) = \frac{\partial}{\partial y} (y) = 1$$

### Overview of lecture# 7

Chapter # 16 Partial derivatives

Page # 790 Article # 16.3

## LECTURE No. 8

## MORE ABOUT EULER THEOREM, CHAIN RULE

In general, the order of differentiation in an  $n$ th order partial derivative can be changed without affecting the final result whenever the function and all of its partial derivatives of order less than  $n$  are continuous.

For example, if  $f$  and its partial derivatives of the first, second and third orders are continuous on an open set, then at each point of the set,

$$f_{xyy} = f_{yxy} = f_{yyx}$$

$$\text{or in another notation, } \frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial^3 f}{\partial y \partial x \partial y} = \frac{\partial^3 f}{\partial x \partial y^2}$$

**Order of Differentiation**

For a function  $f(x, y) = y^2 x^4 e^x + 2$

If we are interested to find  $\frac{\partial^5 f}{\partial y^3 \partial x^2}$ , that is, differentiating in the order firstly w.r.t.  $x$

and then w.r.t.  $y$ , then the calculation will involve many steps making the job difficult.

But if we differentiate this function with respect to  $y$  first, and then with respect to  $x$  secondly then the value of this fifth order derivative can be calculated in a few steps.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y^2 x^4 e^x + 2) = x^4 e^x \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial y} 2 = x^4 e^x (2y) + 0 = 2yx^4 e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} (2yx^4 e^x) = 2x^4 e^x \frac{\partial}{\partial y} (y) = 2x^4 e^x (1) = 2x^4 e^x$$

$$\frac{\partial^3 f}{\partial y^3} = \frac{\partial}{\partial y} \left( \frac{\partial^2 f}{\partial y^2} \right) = \frac{\partial}{\partial y} (2x^4 e^x) = 0, \quad \frac{\partial^4 f}{\partial x \partial y^3} = \frac{\partial}{\partial x} \left( \frac{\partial^3 f}{\partial y^3} \right) = \frac{\partial}{\partial x} (0) = 0$$

$$\frac{\partial^5 f}{\partial x^2 \partial y^3} = \frac{\partial}{\partial x} \left( \frac{\partial^4 f}{\partial x \partial y^3} \right) = \frac{\partial}{\partial x} (0) = 0$$

**EXAMPLE 1:** Let  $f(x, y) = \frac{x+y}{x-y}$ . Find  $f_x$  and  $f_y$ .

$$\text{Solution: } f_x = \frac{\partial}{\partial x} f = \frac{\partial}{\partial x} \left( \frac{x+y}{x-y} \right) = \frac{(x-y) \frac{\partial}{\partial x} (x+y) - (x+y) \frac{\partial}{\partial x} (x-y)}{(x-y)^2} = \frac{-2y}{(x-y)^2}$$

$$f_y = \frac{\partial}{\partial y} f = \frac{\partial}{\partial y} \left( \frac{x+y}{x-y} \right) = \frac{(x-y) \frac{\partial}{\partial y} (x+y) - (x+y) \frac{\partial}{\partial y} (x-y)}{(x-y)^2} = \frac{2x}{(x-y)^2}$$

**EXAMPLE 2:** If  $f(x, y) = x^3 e^{-y} + y^3 \sec \sqrt{x}$ , then find the partial derivatives of  $f(x, y)$  with respect to  $x$  and  $y$ .

**Solution:**

$$\begin{aligned} f(x, y) &= x^3 e^{-y} + y^3 \sec \sqrt{x} \\ f_x &= \frac{\partial}{\partial x} f = e^{-y} (3x^2) + y^3 \sec \sqrt{x} \tan \sqrt{x} \times \left( \frac{\partial}{\partial x} \sqrt{x} \right) \\ &= 3x^2 e^{-y} + y^3 \sec \sqrt{x} \tan \sqrt{x} \left( \frac{1}{2\sqrt{x}} \right) \\ f_y &= \frac{\partial}{\partial y} f = x^3 (-e^{-y}) + \sec \sqrt{x} \times (3y^2) = -x^3 e^{-y} + 3y^2 \sec \sqrt{x} \end{aligned}$$

**EXAMPLE 3:** If  $f(x, y) = x^2 y e^{xy}$ , then find the partial derivatives of  $f(x, y)$  with respect to  $x$  and  $y$  at  $(1, 1)$ .

**Solution :**  $f(x, y) = x^2 y e^{xy}$

$$\begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x} f = y \left[ \frac{\partial}{\partial x} (x^2) e^{xy} + x^2 \frac{\partial}{\partial x} (e^{xy}) \right] = y \left[ 2x e^{xy} + x^2 e^{xy} \frac{\partial}{\partial x} (xy) \right] \\ f_x(x, y) &= xy e^{xy} [2 + xy] \\ f_x(1, 1) &= (1)(1) e^{(1)(1)} [2 + (1)(1)] = 3e \\ f_y(x, y) &= \frac{\partial}{\partial y} f(x, y) = x^2 \left[ \frac{\partial}{\partial y} (y) e^{xy} + y \frac{\partial}{\partial y} (e^{xy}) \right] \\ f_y(x, y) &= x^2 \left[ 1 \times e^{xy} + y e^{xy} \frac{\partial}{\partial y} (xy) \right] = x^2 e^{xy} [1 + xy] \\ f_y(1, 1) &= (1)^2 e^{(1)(1)} [1 + (1)(1)] = 2e \end{aligned}$$

**Example 4:** If  $f(x, y) = x^2 \cos(xy)$ , then find the partial derivatives of  $f(x, y)$  with respect to  $x$  and  $y$  at  $(\frac{1}{2}, \pi)$ .

**Solution :**  $f(x, y) = x^2 \cos(xy)$

$$f_x(x, y) = \frac{\partial}{\partial x} (x^2) \times \cos(xy) + x^2 \frac{\partial}{\partial x} \cos(xy) = 2x \cos(xy) - x^2 y \sin(xy)$$

$$f_x\left(\frac{1}{2}, \pi\right) = 2\left(\frac{1}{2}\right) \cos\left(\frac{1}{2} \times \pi\right) - \left(\frac{1}{2}\right)^2 \pi \sin\left(\frac{1}{2} \times \pi\right) = 0 - \left(\frac{1}{4}\right) \pi \times 1 = -\frac{\pi}{4}$$

$$\text{Now } f_y(x, y) = \frac{\partial}{\partial y} (x^2 \cos(xy)) = x^2 \frac{\partial}{\partial y} \cos(xy) = x^2 (-\sin(xy)) \frac{\partial}{\partial y} (xy) = -x^3 \sin(xy)$$

$$f_y\left(\frac{1}{2}, \pi\right) = -\left(\frac{1}{2}\right)^3 \sin\left(\frac{1}{2} \times \pi\right) = -\left(\frac{1}{8}\right) \sin\left(\frac{\pi}{2}\right) = -\frac{1}{8}$$

**EXAMPLE 5:** Let  $w = (4x - 3y + 2z)^5$ . Find  $\frac{\partial^4 w}{\partial z^2 \partial y \partial x}$ .

**Solution :**  $w = (4x - 3y + 2z)^5$

$$\frac{\partial w}{\partial x} = 5(4x - 3y + 2z)^4 \frac{\partial}{\partial x}(4x - 3y + 2z) = 20(4x - 3y + 2z)^4$$

$$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial w}{\partial x} \right) = \frac{\partial}{\partial y} (20(4x - 3y + 2z)^4)$$

$$= 20 \times 4(4x - 3y + 2z)^3 \frac{\partial}{\partial y}(4x - 3y + 2z) = -240(4x - 3y + 2z)^3$$

$$\frac{\partial^3 w}{\partial z \partial y \partial x} = \frac{\partial}{\partial z} \left( \frac{\partial^2 w}{\partial y \partial x} \right) = \frac{\partial}{\partial z} (-240(4x - 3y + 2z)^3)$$

$$= -240 \times 3(4x - 3y + 2z)^2 \frac{\partial}{\partial z}(4x - 3y + 2z) = -1440(4x - 3y + 2z)^2$$

$$\frac{\partial^4 w}{\partial z^2 \partial y \partial x} = \frac{\partial}{\partial z} \left( \frac{\partial^3 w}{\partial z \partial y \partial x} \right) = \frac{\partial}{\partial z} (-1440(4x - 3y + 2z)^2)$$

$$= -1440 \times 2(4x - 3y + 2z) \frac{\partial}{\partial z}(4x - 3y + 2z) = -5760(4x - 3y + 2z)$$

## Chain Rule

### **I - Chain Rule in function of One Variable**

The function  $f(x)$  depends on one variable  $x$ , and  $x$  depends on single variable  $t$ .

Given that  $w = f(x)$  and  $x = g(t)$ , we find  $\frac{dw}{dt}$  as follows:

From  $w = f(x)$ , we get  $\frac{dw}{dx}$ , From  $x = g(t)$ , we get  $\frac{dx}{dt}$

Then  $\frac{dw}{dt} = \frac{dw}{dx} \frac{dx}{dt}$

**Example 6:** Let  $w = x + 4$ ,  $x = \sin t$ . Find  $\frac{dw}{dt}$ , using the chain rule.

**Solution :**  $w = x + 4$ ,  $x = \sin t$

$$\frac{dw}{dx} = \frac{d}{dx}(x + 4) = 1 + 0 = 1, \quad \frac{dx}{dt} = \frac{d}{dx}(\sin t) = \cos t$$

$$\text{By Chain Rule, } \frac{dw}{dt} = \frac{dw}{dx} \frac{dx}{dt} = (1)(\cos t) = \cos t$$

### Chain Rule in function of one variable

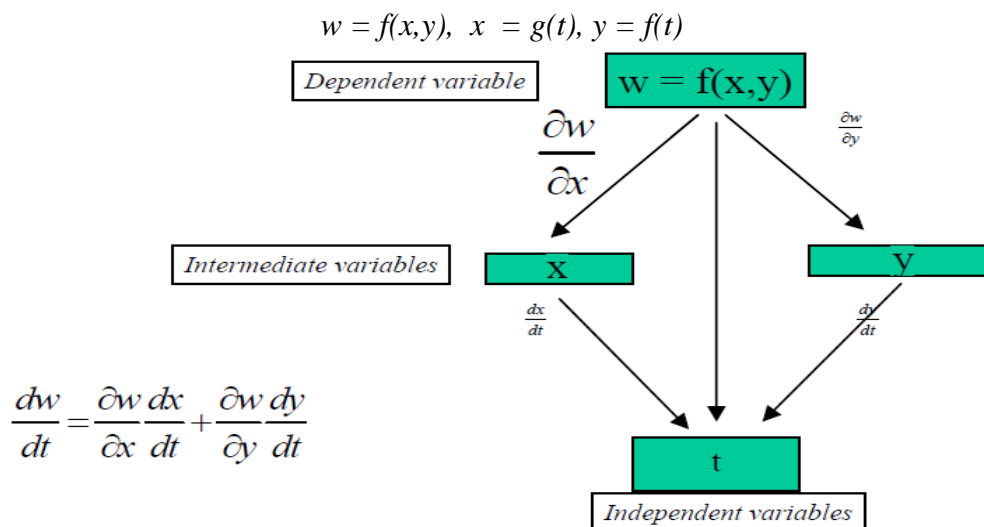
$y$  is a function of  $u$ ,  $u$  is a function of  $v$ ,

$v$  is a function of  $w$ ,  $w$  is a function of  $z$ ,

$z$  is a function of  $x$ . Ultimately,  $y$  is a function of  $x$ . So we can talk about  $\frac{dy}{dx}$ .

By the Chain Rule,  $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dv} \frac{dv}{dw} \frac{dw}{dz} \frac{dz}{dx}$

**II When the function  $f$  is a function of two variable  $x$  and  $y$ . And  $x$  and  $y$  are functions of one variable  $t$ .**



### EXAMPLE BY SUBSTITUTION

Let  $w = xy$ ,  $x = \cos t$ ,  $y = \sin t$ . Find  $\frac{dw}{dt}$  by Substitution method.

**Solution :** By substitution,  $w = xy$ ,  $x = \cos t$ ,  $y = \sin t$

$$w = \cos t \sin t = \frac{1}{2} \times 2 \cos t \sin t = \frac{1}{2} \sin 2t$$

$$\frac{dw}{dt} = \frac{1}{2} (\cos 2t) \times 2 = \cos 2t$$

**EXAMPLE 7 :** Let  $w = xy$ ,  $x = \cos t$ , and  $y = \sin t$ . Find  $\frac{dw}{dt}$  by chain rule.

**Solution :** Given  $w = xy$ ,  $x = \cos t$ , and  $y = \sin t$

$$\frac{\partial w}{\partial x} = \frac{\partial(xy)}{\partial x} = y, \quad \frac{\partial w}{\partial y} = \frac{\partial(xy)}{\partial y} = x, \quad \frac{dx}{dt} = \frac{d \cos t}{dt} = -\sin t, \quad \frac{dy}{dt} = \frac{d \sin t}{dt} = \cos t$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} = (y)(-\sin t) + (x)(\cos t)$$

$$= (\sin t)(-\sin t) + (\cos t)(\cos t) = -\sin^2 t + \cos^2 t = \cos 2t$$

**EXAMPLE 8:** Let  $z = 3x^2 y^3$ ,  $x = t^4$ ,  $y = t^2$ . Find  $\frac{dz}{dt}$ .

**Solution :** Given  $z = 3x^2 y^3$ ,  $x = t^4$ ,  $y = t^2$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(3x^2 y^3) = 3(2x)y^3 = 6xy^3, \quad \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(3x^2 y^3) = 3x^2(3y^2) = 9x^2 y^2$$

$$\frac{dx}{dt} = 4t^3, \quad \frac{dy}{dt} = 2t$$

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = (6xy^3)(4t^3) + (9x^2 y^2)(2t) \\ &= (6(t^4)(t^2)^3)(4t^3) + (9(t^4)^2(t^2)^2)(2t) = 24t^{13} + 18t^{13} = 42t^{13} \end{aligned}$$

**EXAMPLE 9:** Let  $z = \sqrt{1+x-2xy^4}$ ,  $x = \ln t$ ,  $y = t$ . Find  $\frac{dz}{dt}$  by the Chain Rule.

**Solution:** Given  $z = \sqrt{1+x-2xy^4}$ ,  $x = \ln t$ ,  $y = t$

$$\frac{\partial z}{\partial x} = \frac{1}{2}(1+x-2xy^4)^{-\frac{1}{2}} \frac{\partial}{\partial x}(1+x-2xy^4) = \frac{1-2y^4}{2\sqrt{1+x-2xy^4}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}(1+x-2xy^4)^{-\frac{1}{2}} \frac{\partial}{\partial y}(1+x-2xy^4) = \frac{4xy^3}{\sqrt{1+x-2xy^4}}$$

$$\frac{dx}{dt} = \frac{d \ln t}{dt} = \frac{1}{t}, \quad \frac{dy}{dt} = \frac{d}{dt}(t) = 1$$

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = \frac{1-2y^4}{2\sqrt{1+x-2xy^4}} \times \frac{1}{t} + \frac{4xy^3}{\sqrt{1+x-2xy^4}} \times 1 \\ &= \frac{1}{\sqrt{1+x-2xy^4}} \left( \frac{1-2y^4}{2t} + 4xy^3 \right) = \frac{1}{\sqrt{1+\ln t-2t^4(\ln t)}} \left( \frac{1}{2t} - t^3 + 4t^3(\ln t) \right) \end{aligned}$$

**EXAMPLE 10:** Let  $z = \ln(2x^2 + y)$ ,  $x = \sqrt{t}$ ,  $y = t^{\frac{2}{3}}$ . Find  $\frac{dz}{dt}$ , using Chain Rule.

**Solution :**  $z = \ln(2x^2 + y)$   $z = F(x, y)$

$$x = \sqrt{t}, \quad y = t^{\frac{2}{3}} \quad x = g(t), \quad y = f(t)$$

$$\frac{\partial z}{\partial x} = \frac{\partial(\ln(2x^2 + y))}{\partial x} = \frac{1}{2x^2 + y} \frac{\partial(2x^2 + y)}{\partial x} = \frac{1}{2x^2 + y} (4x) = \frac{4x}{2x^2 + y}$$

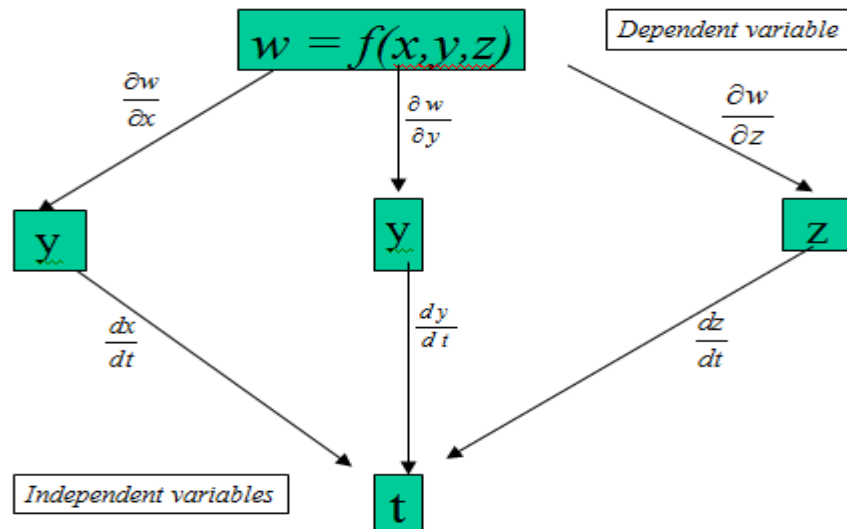
$$\frac{\partial z}{\partial y} = \frac{\partial(\ln(2x^2 + y))}{\partial y} = \frac{1}{2x^2 + y} \frac{\partial(2x^2 + y)}{\partial y} = \frac{1}{2x^2 + y} (0+1) = \frac{1}{2x^2 + y}$$

$$\frac{dx}{dt} = \frac{d\sqrt{t}}{dt} = \frac{1}{2\sqrt{t}}, \quad \frac{dy}{dt} = \frac{dt^{\frac{2}{3}}}{dt} = \frac{2}{3} t^{\frac{2}{3}-1} = \frac{2}{3t^{\frac{1}{3}}}$$

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = \left( \frac{4x}{2x^2 + y} \right) \left( \frac{1}{2\sqrt{t}} \right) + \left( \frac{1}{2x^2 + y} \right) \left( \frac{2}{3t^{\frac{1}{3}}} \right) \\ &= \left( \frac{4\sqrt{t}}{2(\sqrt{t})^2 + t^{\frac{2}{3}}} \right) \left( \frac{1}{2\sqrt{t}} \right) + \left( \frac{1}{2(\sqrt{t})^2 + t^{\frac{2}{3}}} \right) \left( \frac{2}{3t^{\frac{1}{3}}} \right) = \frac{2}{2t + t^{\frac{2}{3}}} + \frac{2}{3t^{\frac{1}{3}}(2t + t^{\frac{2}{3}})} = \frac{6t^{\frac{1}{3}} + 2}{3t^{\frac{1}{3}}(2t + t^{\frac{2}{3}})}\end{aligned}$$

**III When the function  $f$  is a function of three variable  $x$ ,  $y$  and  $z$ . And  $x$ ,  $y$  and  $z$  are functions of one variable  $t$ .**

$$w = f(x, y, z), x = g(t), y = f(t), z = h(t)$$



$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$$

### Overview of Lecture#8

Book Calculus by Howard Anton

( Chapter # 16 - Topic # 16.4, Page # 799)

## LECTURE No. 9

## EXAMPLES

First of all, we revise the example which we did in our 8<sup>th</sup> lecture.  
Consider  $w = f(x, y, z)$ , where  $x = g(t)$ ,  $y = f(t)$ ,  $z = h(t)$ , then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$$

**Example 1:** Consider a function  $w = x^2 + y + z + 4$ , then find  $\frac{dw}{dt}$ .

**Solution:**

$$w = x^2 + y + z + 4$$

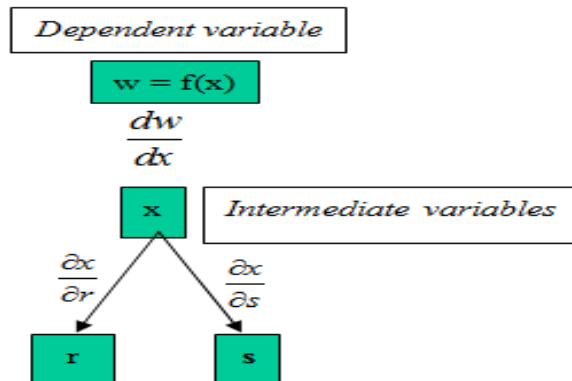
$$x = e^t, \quad y = \cos t, \quad z = t + 4$$

$$\frac{\partial w}{\partial x} = 2x, \quad \frac{\partial w}{\partial y} = 1, \quad \frac{\partial w}{\partial z} = 1$$

$$\frac{dx}{dt} = e^t, \quad \frac{dy}{dt} = -\sin t, \quad \frac{dz}{dt} = 1$$

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} \\ &= (2x)(e^t) + (1)(-\sin t) + (1)(1) \\ &= 2(e^t)(e^t) - \sin t + 1 \\ &= 2e^{2t} - \sin t + 1 \end{aligned}$$

Consider  $w = f(x)$ , where  $x = g(r, s)$ . Now it is clear from the figure that “x” is intermediate variable and we can write  $\frac{\partial w}{\partial r} = \frac{dw}{dx} \frac{\partial x}{\partial r}$  and  $\frac{\partial w}{\partial s} = \frac{dw}{dx} \frac{\partial x}{\partial s}$



**Example 2:** If  $w = \sin x + x^2$ ,  $x = 3r + 4s$ , then find  $\frac{\partial w}{\partial r}$  and  $\frac{\partial w}{\partial s}$ .

**Solution:**  $w = \sin x + x^2$ ,  $x = 3r + 4s$

$$\frac{dw}{dx} = \cos x + 2x, \quad \frac{\partial x}{\partial r} = 3, \quad \frac{\partial x}{\partial s} = 4$$

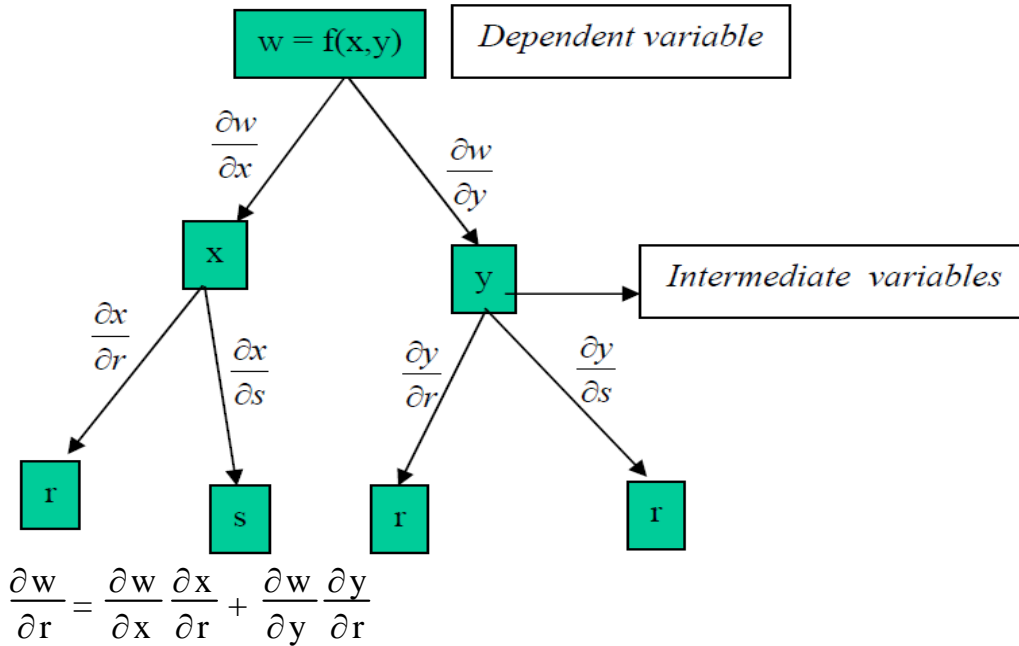
$$\frac{\partial w}{\partial r} = \frac{dw}{dx} \frac{\partial x}{\partial r} = (\cos x + 2x) \cdot 3 = 3 \cos x + 6x = 3 \cos(3r + 4s) + 6(3r + 4s)$$

$$= 3 \cos(3r + 4s) + 18r + 24s$$

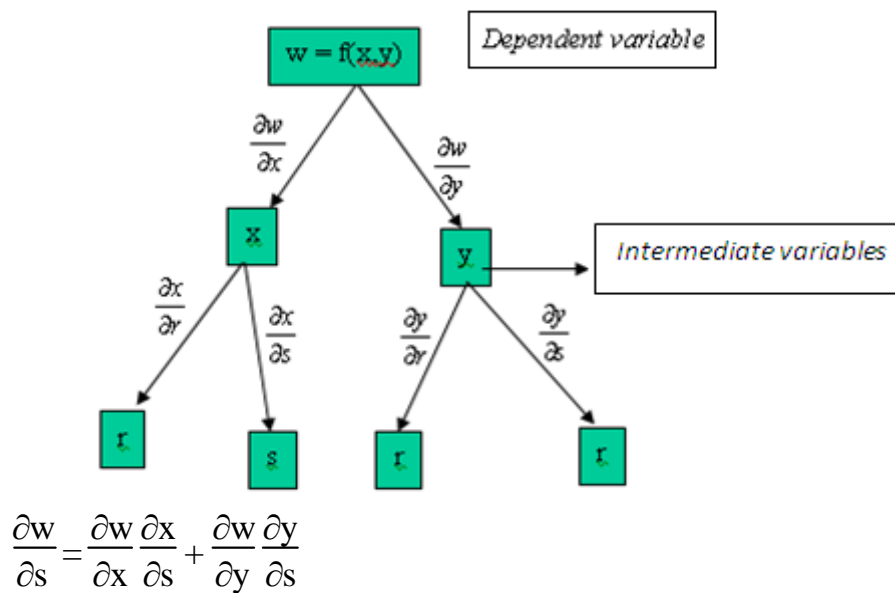
$$\frac{\partial w}{\partial s} = \frac{dw}{dx} \frac{\partial x}{\partial s} = (\cos x + 2x) \cdot 4 = 4 \cos x + 8x = 4 \cos(3r + 4s) + 8(3r + 4s)$$

$$= 4 \cos(3r + 4s) + 24r + 32s$$

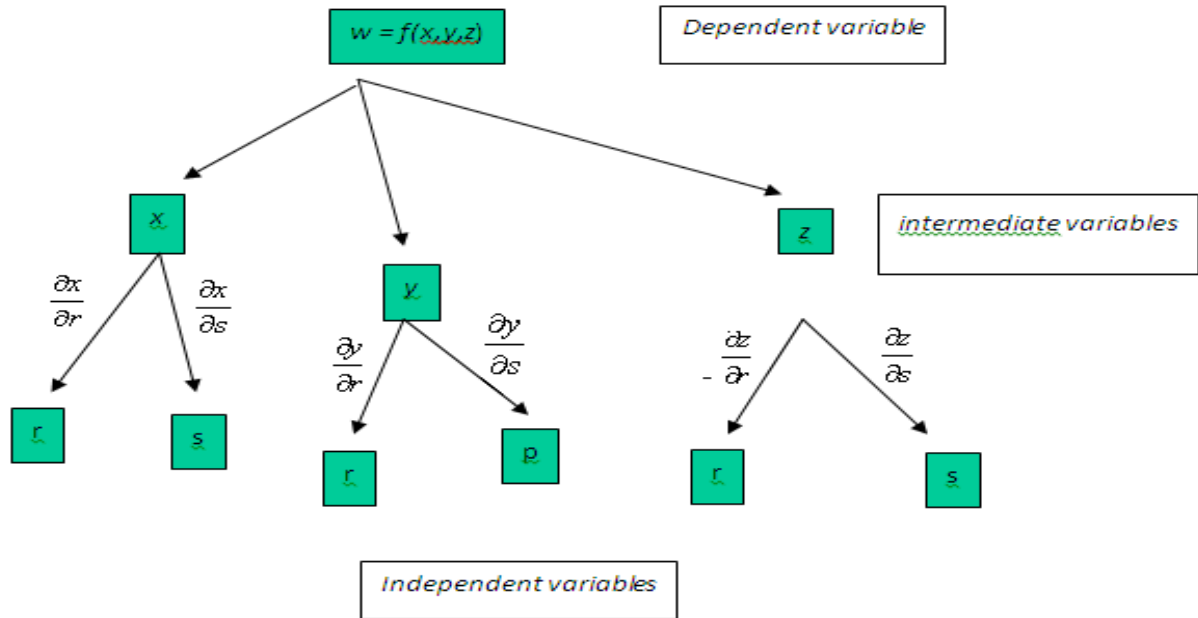
Consider the function  $w = f(x, y)$ , where  $x = g(r, s)$ ,  $y = h(r, s)$



Similarly, if you differentiate the function  $w$  with respect to  $s$  we will get



Consider the function  $w = f(x, y, z)$ , where  $x = g(r, s)$ ,  $y = h(r, s)$ ,  $z = k(r, s)$



Thus we have

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r}$$

Similarly if we differentiate with respect to  $s$  then we have,

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s}$$

**Example 3 :** Consider the function  $w = x + 2y + z^2$ ,  $x = \frac{r}{s}$ ,  $y = r^2 + \ln s$ ,  $z = 2r$ .

**Solution:**

First we calculate  $\frac{\partial w}{\partial x} = 1$ ,  $\frac{\partial w}{\partial y} = 2$ ,  $\frac{\partial w}{\partial z} = 2z$ ,  $\frac{\partial x}{\partial r} = \frac{1}{s}$ ,  $\frac{\partial y}{\partial r} = 2r$ ,  $\frac{\partial z}{\partial r} = 2$

$$\text{Since } \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial r} = (1) \left( \frac{1}{s} \right) + (2)(2r) + (2z)(2) = \frac{1}{s} + 4r + (4r)(2) = \frac{1}{s} + 12r$$

By putting the values from above, we get  $\frac{\partial x}{\partial s} = -\frac{r}{s^2}$ ,  $\frac{\partial y}{\partial s} = \frac{1}{s}$ ,  $\frac{\partial z}{\partial s} = 0$

So we can calculate

$$\begin{aligned} \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s} \\ &= (1) \left( -\frac{r}{s^2} \right) + (2) \left( \frac{1}{s} \right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2} \end{aligned}$$

**Remembering the Different Forms of the Chain Rule:**

The best thing to do is to draw appropriate tree diagram by placing the dependent variable on top, the intermediate variables in the middle, and the selected independent variable at the bottom.

To find the derivative of dependent variable with respect to the selected independent variable, start at the dependent variable and read down each branch of the tree to the independent variable, calculating and multiplying the derivatives along the branch. Then add the products you found for the different branches.

**The Chain Rule for Functions of Many Variables**

Suppose  $\omega = f(x, y, \dots, v)$  is a differentiable function of the variables  $x, y, \dots, v$  (a finite set) and the  $x, y, \dots, v$  are differentiable functions of  $p, q, \dots, t$  (another finite set). Then  $\omega$  is a differentiable function of the variables  $p$  through  $t$  and the partial derivatives of  $\omega$  with respect to these variables are given by equations of the form

$$\frac{\partial \omega}{\partial p} = \frac{\partial \omega}{\partial x} \frac{\partial x}{\partial p} + \frac{\partial \omega}{\partial y} \frac{\partial y}{\partial p} + \dots + \frac{\partial \omega}{\partial v} \frac{\partial v}{\partial p}.$$

The other equations are obtained by replacing  $p$  by  $q, \dots, t$ , one at a time.

One way to remember last equation is to think of the right-hand side as the dot product of two vectors with components.

$\left( \frac{\partial \omega}{\partial x}, \frac{\partial \omega}{\partial y}, \dots, \frac{\partial \omega}{\partial v} \right)$	and	$\left( \frac{\partial x}{\partial p}, \frac{\partial y}{\partial p}, \dots, \frac{\partial v}{\partial p} \right)$
Derivatives of $\omega$ with respect to the intermediate variables		Derivatives of the intermediate variables with respect to the selected independent variable

**Example 4:** For the function  $w = \ln(e^r + e^s + e^t + e^u)$ , find  $w_{rstu}$  where  $w_{rstu} = \frac{\partial^4 w}{\partial u \partial t \partial s \partial r}$ .

**Solution:**  $w = \ln(e^r + e^s + e^t + e^u)$

$$e^w = e^{\ln(e^r + e^s + e^t + e^u)}$$

Take anti-log on both sides

$$e^w = e^r + e^s + e^t + e^u \quad \text{---(1)} \quad \text{since } e^{\ln x} = x$$

Take derivative with respect to  $r$ ,

$$\frac{\partial e^w}{\partial r} = \frac{\partial(e^r + e^s + e^t + e^u)}{\partial r} = (e^r + 0 + 0 + 0)$$

$$e^w w_r = e^r \quad \text{Since} \quad \frac{\partial w}{\partial r} = w_r$$

$$w_r = \frac{e^r}{e^w} = e^{r-w} \quad \text{---(2)}$$

Take derivative with respect to s,

$$\frac{\partial e^w}{\partial s} = \frac{\partial(e^r + e^s + e^t + e^u)}{\partial s} = (0 + e^s + 0 + 0)$$

$$e^w w_s = e^s \quad \text{Since} \quad \frac{\partial w}{\partial s} = w_s$$

$$w_s = \frac{e^s}{e^w} = e^{s-w} \quad \text{---(3)}$$

Similarly, by taking derivative of (1) with respect to u, we get

$$w_u = e^{u-w} \quad \text{---(4)}$$

Similarly, by taking derivative of (1) with respect to t, we get

$$w_t = e^{t-w} \quad \text{---(5)}$$

Now differentiate equation (2) with respect to s,

$$\frac{\partial w_r}{\partial s} = \frac{\partial e^{r-w}}{\partial s} = \frac{e^{r-w} \partial(r-w)}{\partial s} = e^{r-w} (0 - \frac{\partial w}{\partial s}) \quad \text{since } r \text{ is kept constant}$$

$$w_{rs} = -e^{r-w} w_s = -e^{r-w} e^{s-w} \quad \text{by (3)}$$

$$w_{rs} = -e^{r-w+s-w} = -e^{r+s-2w}$$

Now differentiate it with respect to t,

$$\begin{aligned} \frac{\partial w_{rs}}{\partial t} &= -\frac{\partial e^{r+s-2w}}{\partial t} = -e^{r+s-2w} \frac{\partial(r+s-2w)}{\partial t} \\ &= -e^{r+s-2w} (0 + 0 - \frac{\partial 2w}{\partial t}) \quad \text{since } r \text{ is kept constant} \end{aligned}$$

$$\begin{aligned} w_{rst} &= -e^{r+s-2w} (-2w_t) = 2e^{r+s-2w} w_t = 2e^{r+s-2w} e^{t-w} \\ &= 2e^{r+s+t-3w} \quad \text{by (5)} \end{aligned}$$

$$\frac{\partial w_{rst}}{\partial u} = \frac{\partial 2e^{r+s+t-3w}}{\partial u} = 2e^{r+s+t-3w} \frac{\partial(r+s+t-3w)}{\partial u}$$

$$\begin{aligned} w_{rstu} &= 2e^{r+s+t-3w} (0 + 0 + 0 - \frac{\partial 3w}{\partial u}) \\ &= 2e^{r+s+t-3w} (-3w_u) = -6e^{r+s+t-3w} e^{u-w} \\ &= -6e^{r+s+t+u-4w} \quad \text{by (4)} \end{aligned}$$

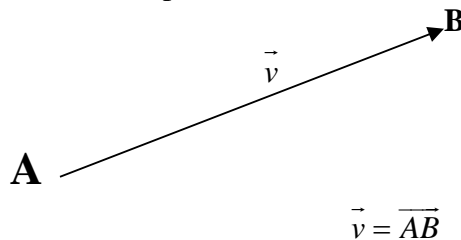
## LECTURE No. 10

## INTRODUCTION TO VECTORS

Some of things we measure are determined by their magnitude, but some times we need magnitude as well as direction to describe the quantities. For example, to describe a force, we need the direction in which that force is acting (Direction) as well as how large it is (Magnitude). Another example is the body's velocity; we have to know where the body is headed as well as how fast it is.

Quantities that have direction as well as magnitude are usually represented by arrows that point the direction of the action and whose lengths give magnitude of the action in term of a suitably chosen unit.

A vector in the plane is a directed line segment.



Vectors are usually described by the single **bold** face Roman letters or letter with an arrow. The vector defined by the directed line segment from point A to point B is written as  $\overrightarrow{AB}$ .

**Magnitude or Length of a Vector :**

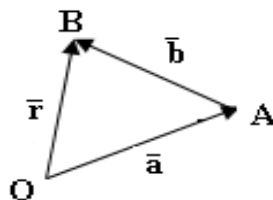
Magnitude of the vector  $\vec{v}$  is denoted by

$$|\vec{v}| = |\overrightarrow{AB}|$$

which is the length of the line segment  $\overrightarrow{AB}$

**Unit vector:** Any vector whose magnitude or length is 1 is a unit vector.

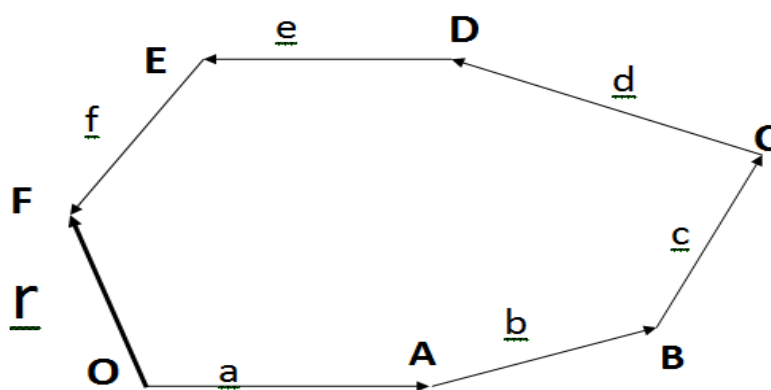
Unit vector in the direction of vector  $\vec{v}$  is denoted by  $\hat{v}$  and is given by  $\hat{v} = \frac{\vec{v}}{|\vec{v}|}$

**Addition of Vectors**

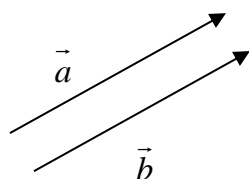
This diagram shows three vectors in two vectors; one vector  $\overrightarrow{OA}$  is connected with tail of vector  $\overrightarrow{AB}$ . The tail of third vector  $\overrightarrow{OB}$  is connected with the tail of  $\overrightarrow{OA}$  and head is connected with the head of vector  $\overrightarrow{AB}$ . This third vector is called Resultant vector  $\vec{r}$ .

The resultant vector  $\vec{r}$  can be written as  $\vec{r} = \vec{a} + \vec{b}$

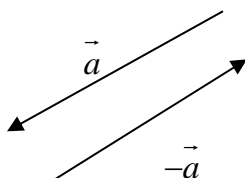
Similarly,  $\vec{r} = \vec{a} + \vec{b} + \vec{c} + \vec{d} + \vec{e} + \vec{f}$



**Equal Vectors:** Two vectors are equal or same vectors if they have same magnitude and direction.  $|\vec{a}| = |\vec{b}|$

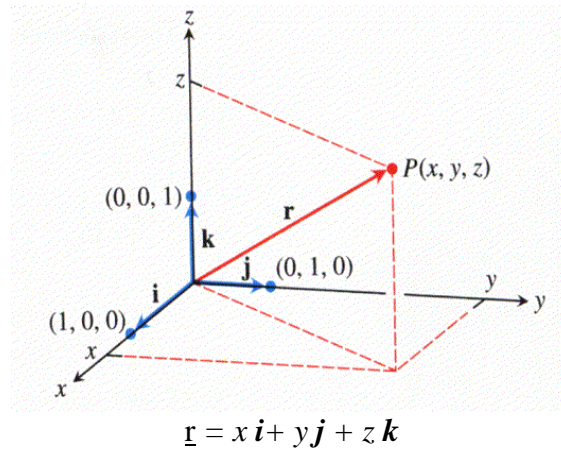


**Opposite Vectors:** Two vectors are opposite vectors if they have same magnitude and opposite directions.



**Parallel Vectors:** Two vectors  $\vec{a}$  and  $\vec{b}$  are parallel if one vector  $\vec{a}$  is scalar multiple of the other  $\vec{b}$ .

$$\vec{b} = \lambda \vec{a} \quad \text{where } \lambda \text{ is a non-zero scalar.}$$



### Addition and subtraction of two vectors in rectangular component:

Let  $\underline{a} = a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}$

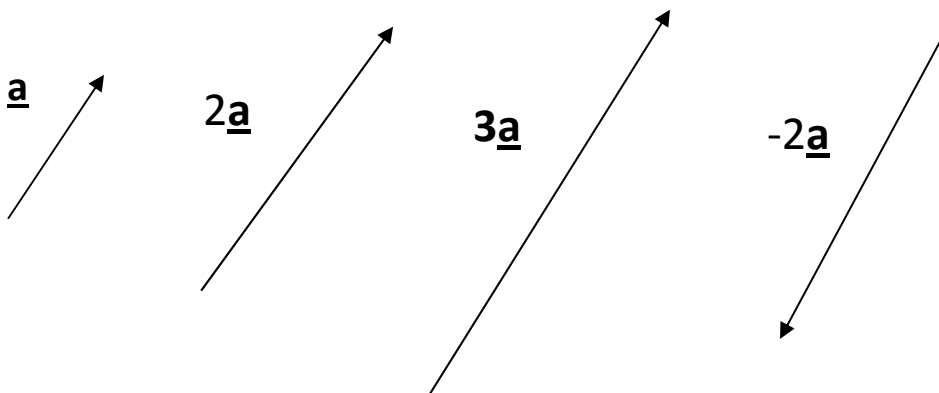
and  $\underline{b} = b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k}$

$$\begin{aligned} \underline{a} + \underline{b} &= (a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}) + (b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k}) \\ &= (a_1 + b_1) \underline{i} + (a_2 + b_2) \underline{j} + (a_3 + b_3) \underline{k} \end{aligned}$$

$$\begin{aligned} \underline{a} - \underline{b} &= (a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}) - (b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k}) \\ &= (a_1 - b_1) \underline{i} + (a_2 - b_2) \underline{j} + (a_3 - b_3) \underline{k} \end{aligned}$$

The  $i$ th component of first vector is added to ( or subtracted from) the  $i$ th component of second vector,  $j$ th component of first vector is added to (or subtracted from) the  $j$ th component of second vector, similarly  $k$ th component of first vector is added to ( or subtracted from) the  $k$ th component of second vector.

### Multiplication of a Vector by a Scalar



Any vector  $\vec{a}$  can be written as  $\vec{a} = |\vec{a}| \hat{a}$

**Scalar Product:** Scalar product (dot product) (“ $\vec{a}$  dot  $\vec{b}$ ”) of vector  $\vec{a}$  and  $\vec{b}$  is the number which is given by the formula:

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$$

where  $\theta$  is the angle between  $\vec{a}$  and  $\vec{b}$ .

In words,  $\vec{a} \cdot \vec{b}$  is the length of  $\vec{a}$  times the length of  $\vec{b}$  times the cosine of the angle between  $\vec{a}$  and  $\vec{b}$ .

**Remark:** This is known as commutative law.  $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

### Some Results of Scalar Product

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$$

1) If  $\vec{a} \perp \vec{b}$ , then it means that  $\vec{a}$  is perpendicular to  $\vec{b}$

$$\text{So } \vec{a} \cdot \vec{b} = 0 \quad \text{since } \theta = 90^\circ, \cos 90^\circ = 0$$

$$\text{Also } \hat{i} \cdot \hat{j} = 0 = \hat{j} \cdot \hat{i}, \quad \hat{j} \cdot \hat{k} = 0 = \hat{k} \cdot \hat{j}, \quad \hat{k} \cdot \hat{i} = 0 = \hat{i} \cdot \hat{k}$$

2) If  $\vec{a} \parallel \vec{b}$  then it means  $\vec{a}$  is parallel to  $\vec{b}$ .

$$\text{So } \vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \quad \text{since } \theta = 0, \cos 0 = 1$$

If we replace  $\vec{b}$  by  $\vec{a}$ , then

$$\vec{a} \cdot \vec{a} = |\vec{a}| |\vec{a}| = |\vec{a}|^2$$

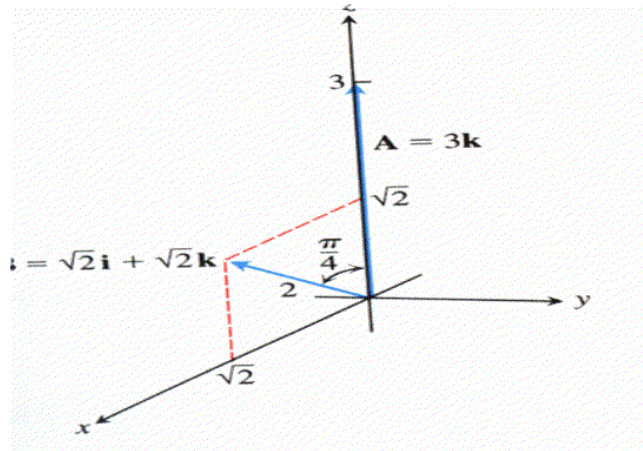
$$\text{So } \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$$

### **Example**

$$\text{If } \vec{a} = 3\hat{k} \text{ and } \vec{b} = \sqrt{2}\hat{i} + \sqrt{2}\hat{k}, \quad \theta = \frac{\pi}{4},$$

$$\text{then } \vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta = |3\hat{k}| |\sqrt{2}\hat{i} + \sqrt{2}\hat{k}| \cos \frac{\pi}{4}$$

$$= (3)(2) \left( \frac{1}{\sqrt{2}} \right) = \frac{6}{\sqrt{2}} = 3\sqrt{2}$$



### EXPRESSION FOR $\mathbf{a} \cdot \mathbf{b}$ IN COMPONENT FORM

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \text{and} \\ \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= a_1\mathbf{i} \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) + a_2\mathbf{j} \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &\quad + a_3\mathbf{k} \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \end{aligned}$$

$$\begin{aligned} &= a_1b_1\mathbf{i} \cdot \mathbf{i} + a_1b_2\mathbf{i} \cdot \mathbf{j} + a_1b_3\mathbf{i} \cdot \mathbf{k} + a_2b_1\mathbf{j} \cdot \mathbf{i} + a_2b_2\mathbf{j} \cdot \mathbf{j} \\ &\quad + a_2b_3\mathbf{j} \cdot \mathbf{k} + a_3b_1\mathbf{k} \cdot \mathbf{i} + a_3b_2\mathbf{k} \cdot \mathbf{j} + a_3b_3\mathbf{k} \cdot \mathbf{k} \\ &= a_1b_1(1) + a_1b_2(0) + a_1b_3(0) + a_2b_1(0) + a_2b_2(1) \\ &\quad + a_2b_3(0) + a_3b_1(0) + a_3b_2(0) + a_3b_3(1) \\ &= a_1b_1 + a_2b_2 + a_3b_3 \end{aligned}$$

In dot product, the  $i$ th component of vector  $\vec{a}$  will multiply with  $i$ th component of vector  $\vec{b}$ ,  
 $j$ th component of vector  $\vec{a}$  will multiply with  $j$ th component of vector  $\vec{b}$  and  
 $k$ th component of vector  $\vec{a}$  will multiply with  $k$ th component of vector  $\vec{b}$ .

### Angle between Two Vectors

The angle  $\theta$  between two vectors  $\vec{a}$  and  $\vec{b}$  is

$$\theta = \cos^{-1} \left( \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \right)$$

Since the values of arc-cosine lie in  $[0, \pi]$ , so the above equation automatically gives the angle made between  $\vec{a}$  and  $\vec{b}$ .

**Example :** Find the angle between the vectors  $\vec{a} = \hat{i} - 2\hat{j} - 2\hat{k}$  and  $\vec{b} = 6\hat{i} + 3\hat{j} + 2\hat{k}$ .

**Solution:**  $\vec{a} \cdot \vec{b} = (\hat{i} - 2\hat{j} - 2\hat{k}) \cdot (6\hat{i} + 3\hat{j} + 2\hat{k}) = (1)(6) + (-2)(3) + (-2)(2)$   
 $= 6 - 6 - 4 = -4$

$$|\vec{a}| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3, \quad |\vec{b}| = \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = 7$$

$$\theta = \cos^{-1} \left( \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \right) = \cos^{-1} \left( \frac{-4}{(3)(7)} \right) = \cos^{-1} \left( \frac{-4}{21} \right) \approx 1.76 \text{ radians}$$

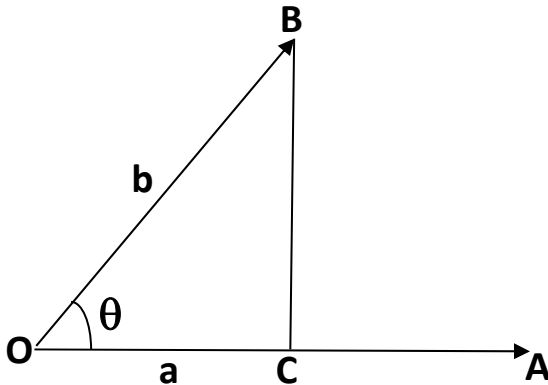
### **Perpendicular ( Orthogonal ) Vectors**

The non-zero vectors  $\vec{a}$  and  $\vec{b}$  are perpendicular if and only if  $\vec{a} \cdot \vec{b} = 0$

This statement has two parts If  $\vec{a}$  and  $\vec{b}$  are perpendicular, then  $\vec{a} \cdot \vec{b} = 0$ . And if  $\vec{a} \cdot \vec{b} = 0$ , then  $\vec{a}$  and  $\vec{b}$  are perpendicular.

### **Vector Projection**

Consider the Projection of a vector  $\vec{b}$  on a vector  $\vec{a}$  making an angle  $\theta$  with each other



From right angle triangle  $OCB$ ,

$$\cos \theta = \frac{\text{base}}{\text{hypotenuse}} = \frac{|\overline{OC}|}{|\vec{b}|}$$

$$\begin{aligned} |\overline{OC}| &= |\vec{b}| \cos \theta \\ &= |\vec{b}| \frac{|\vec{a}|}{|\vec{a}|} \cos \theta = \frac{\vec{b} \cdot \vec{a}}{|\vec{a}|} \end{aligned}$$

$$\text{Projection of } \vec{b} \text{ along } \vec{a} = \vec{b} \cdot \frac{\vec{a}}{|\vec{a}|} \left( \frac{\vec{a}}{|\vec{a}|} \right) \quad \text{where } \left( \frac{\vec{a}}{|\vec{a}|} \right) \text{ is the unit vector along } \vec{a}.$$

$$= \frac{\vec{b} \cdot \vec{a}}{|\vec{a}|^2} \vec{a} = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \vec{a}$$

The number  $|\vec{b}| \cos \theta$  is called the scalar component of  $\vec{b}$  in the direction of  $\vec{a}$  because

$$|\vec{b}| \cos \theta = \vec{b} \cdot \hat{a}$$

**Example :** Find the vector projection of  $\vec{b} = 6\hat{i} + 3\hat{j} + 2\hat{k}$  onto  $\vec{a} = \hat{i} - 2\hat{j} - 2\hat{k}$ .

**Solution:**

$$\text{Projection of } \vec{b} \text{ onto } \vec{a} = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \vec{a}$$

$$\text{Here, } \vec{b} \cdot \vec{a} = (6\hat{i} + 3\hat{j} + 2\hat{k}) \cdot (\hat{i} - 2\hat{j} - 2\hat{k}) = 6 \times 1 + 3(-2) + 2(-2) = 6 - 6 - 4 = -4$$

$$\vec{a} \cdot \vec{a} = (\hat{i} - 2\hat{j} - 2\hat{k}) \cdot (\hat{i} - 2\hat{j} - 2\hat{k}) = 1 \times 1 + (-2)(-2) + (-2)(-2) = 1 + 4 + 4 = 9$$

$$\text{Projection of } \vec{b} \text{ onto } \vec{a} = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \vec{a} = \frac{-4}{9} (\hat{i} - 2\hat{j} - 2\hat{k}) = -\frac{4}{9}\hat{i} + \frac{8}{9}\hat{j} + \frac{8}{9}\hat{k}$$

The scalar component of  $\vec{b}$  in the direction of  $\vec{a}$  is  $|\vec{b}| \cos \theta$ .

$$|\vec{b}| \cos \theta = \frac{\vec{b} \cdot \vec{a}}{|\vec{a}|} = \frac{(6\hat{i} + 3\hat{j} + 2\hat{k}) \cdot (\hat{i} - 2\hat{j} - 2\hat{k})}{\sqrt{(1)^2 + (-2)^2 + (-2)^2}} = \frac{6 \times 1 + 3(-2) + 2(-2)}{\sqrt{9}} = \frac{6 - 6 - 4}{3} = \frac{-4}{3}$$

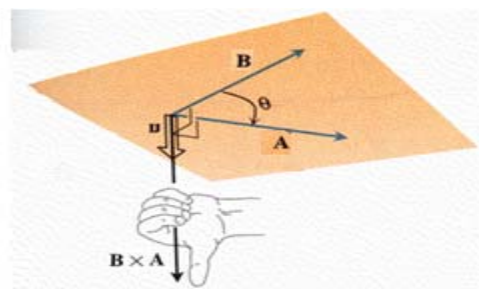
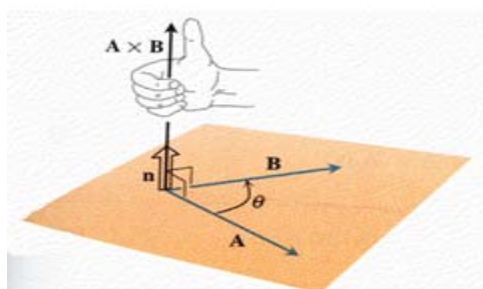
### The Cross Product of Two Vectors in Space

Consider two non-zero vectors  $\vec{a}$  and  $\vec{b}$  in space. The vector product  $\vec{a} \times \vec{b}$  (" $\vec{a}$  cross  $\vec{b}$ ") to be the vector  $\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \hat{n}$  where  $\hat{n}$  is the unit vector determined by the Right Hand rule.

### Right-hand rule

We start with two nonzero nonparallel vectors  $\mathbf{A}$  and  $\mathbf{B}$ . We select a unit vector  $\mathbf{n}$  perpendicular to the plane by the **right handed rule**. This means we choose  $\mathbf{n}$  to be the unit vector that points the way your right thumb points when your fingers curl through the angle  $\theta$  from  $\mathbf{A}$  to  $\mathbf{B}$ .

The vector  $\mathbf{A} \times \mathbf{B}$  is orthogonal to both  $\mathbf{A}$  and  $\mathbf{B}$ .



### Some Results of Cross Product $\vec{a} \times \vec{b}$

As we know that  $\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \hat{n}$

- 1) If  $\vec{a} \parallel \vec{b}$ , then  $\vec{a} \times \vec{b} = \vec{0}$  since  $\sin 0^\circ = 0$

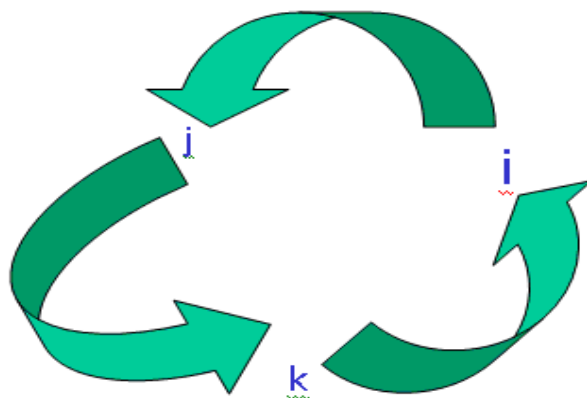
Similarly,  $\vec{a} \times \vec{a} = \vec{0}$  and  $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$

- 2) If  $\vec{a} \perp \vec{b}$ , then  $\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \hat{n}$  since  $\sin 90^\circ = 1$

Similarly,  $\hat{i} \times \hat{j} = \hat{k}$ ,  $\hat{j} \times \hat{i} = -\hat{k}$

$\hat{j} \times \hat{k} = \hat{i}$ ,  $\hat{k} \times \hat{j} = -\hat{i}$

$\hat{k} \times \hat{i} = \hat{j}$ ,  $\hat{i} \times \hat{k} = -\hat{j}$

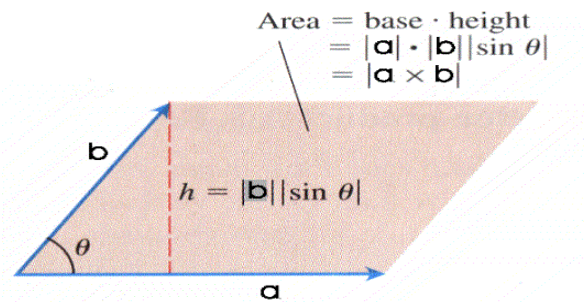


**Note** that the vector product is not commutative.

### The Area of a Parallelogram

Because  $\hat{n}$  is a unit vector and magnitude of  $\vec{a} \times \vec{b}$  is

$$|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta \quad |\hat{n}| = 1 \quad \text{Since } |\hat{n}| = 1$$



This is the area of parallelogram which is determined by  $\vec{a}$  and  $\vec{b}$  where  $\vec{a}$  is the base and  $|\vec{b}| \sin \theta$  is the height of the parallelogram.

### $\mathbf{a \times b}$ from the components of $\mathbf{a}$ and $\mathbf{b}$

$$\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k} \quad \text{and} \quad \vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$$

$$\begin{aligned} \vec{a} \times \vec{b} &= (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \times (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}) \\ &= a_1 \hat{i} \times (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}) + a_2 \hat{j} \times (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}) + a_3 \hat{k} \times (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}) \\ &= a_1 b_1 \hat{i} \times \hat{i} + a_1 b_2 \hat{i} \times \hat{j} + a_1 b_3 \hat{i} \times \hat{k} + a_2 b_1 \hat{j} \times \hat{i} + a_2 b_2 \hat{j} \times \hat{j} + a_2 b_3 \hat{j} \times \hat{k} \\ &\quad + a_3 b_1 \hat{k} \times \hat{i} + a_3 b_2 \hat{k} \times \hat{j} + a_3 b_3 \hat{k} \times \hat{k} \\ &= a_1 b_1 \times 0 + a_1 b_2 \hat{k} + a_1 b_3 (-\hat{j}) + a_2 b_1 (-\hat{k}) + a_2 b_2 (0) + a_2 b_3 \hat{i} \\ &\quad + a_3 b_1 \hat{j} + a_3 b_2 (-\hat{i}) + a_3 b_3 (0) \\ &= (a_2 b_3 - a_3 b_2) \hat{i} - (a_1 b_3 - a_3 b_1) \hat{j} + (a_1 b_2 - a_2 b_1) \hat{k} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \end{aligned}$$

**Example:** Let  $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$  and  $\vec{b} = -4\hat{i} + 3\hat{j} + \hat{k}$ , then find  $\vec{a} \times \vec{b}$ .

**Solution:**

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \hat{i}(1-3) - \hat{j}(2+4) + \hat{k}(6+4) = -2\hat{i} - 6\hat{j} + 10\hat{k}$$

### Over view of Lecture # 10

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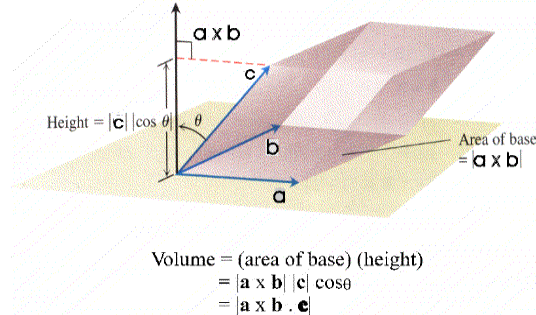
## LECTURE No. 11

## THE TRIPLE SCALAR OR BOX PRODUCT

The product  $(\vec{a} \times \vec{b}) \cdot \vec{c}$  is called the triple scalar product of  $\vec{a}, \vec{b}$  and  $\vec{c}$  ( in that order).

$$\text{As } (\vec{a} \times \vec{b}) \cdot \vec{c} = |\vec{a} \times \vec{b}| |\vec{c}| \cos \theta$$

So the absolute value of the product is the volume of the parallelepiped (parallelogram-sided box) determined by  $\vec{a}, \vec{b}$  and  $\vec{c}$ .



By treating the planes of  $\vec{b}$  and  $\vec{c}$  and of  $\vec{c}$  and  $\vec{a}$  as the base planes of the parallelepiped determined by  $\vec{a}, \vec{b}$  and  $\vec{c}$ .

$$\text{We see that } (\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b}$$

Since the dot product is commutative,  $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$

**Example :** Show that  $\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

**Proof :** Consider  $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$   
 $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$   
 $\vec{c} = c_1 \hat{i} + c_2 \hat{j} + c_3 \hat{k}$

$$\begin{aligned} \vec{a} \cdot (\vec{b} \times \vec{c}) &= \vec{a} \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \cdot \left[ \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \hat{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \hat{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \hat{k} \right] \\ &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \end{aligned}$$

$$\text{So, } \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

**Example :** Let  $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$ ,  $\vec{b} = -2\hat{i} + 3\hat{k}$ ,  $\vec{c} = 7\hat{j} - 4\hat{k}$ . Find  $\vec{a} \cdot (\vec{b} \times \vec{c})$ .

**Solution:**

$$\begin{aligned}\vec{a} \cdot (\vec{b} \times \vec{c}) &= \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = 1 \begin{vmatrix} 0 & 3 \\ 7 & -4 \end{vmatrix} - 2 \begin{vmatrix} -2 & 3 \\ 0 & -4 \end{vmatrix} + (-1) \begin{vmatrix} -2 & 0 \\ 0 & 7 \end{vmatrix} \\ &= 1(0 - 21) - 2(8 - 0) - (-14 - 0) \\ &= -21 - 16 + 14 = -23\end{aligned}$$

When we solve  $\vec{a} \cdot (\vec{b} \times \vec{c})$ , then answer is  $-23$ . If we get negative value, then Absolute value makes it positive and also volume is always positive.

### Gradient of a Scalar Function

$$\bar{\nabla} = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

where  $\bar{\nabla}$  is called “del” operator.

Gradient  $\phi$  is a vector operator defined as

$$\begin{aligned}\text{grad } \phi &= \left( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \phi \\ &= \bar{\nabla} \phi\end{aligned}$$

$\bar{\nabla}$  “del operator” is a vector quantity. Grad means gradient. Gradient is also vector quantity.  $\bar{\nabla} \phi$  is vector and  $\phi$  is scalar quantity.

### Directional Derivative

If  $f(x, y)$  is differentiable at  $(x_0, y_0)$  and if

$\vec{u} = (u_1, u_2)$  is a unit vector, then the

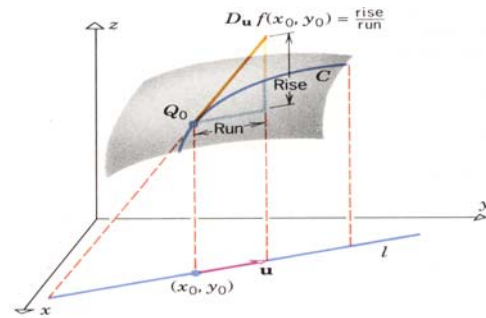
**directional derivative** of  $f(x, y)$  at  $(x_0, y_0)$  in the direction of  $\vec{u}$  is defined by

$$D_u f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

It should be kept in mind that there are infinitely many directional derivatives of  $z = f(x, y)$  at a point  $(x_0, y_0)$ , one for each possible choice of the direction vector  $\vec{u}$ .

### Remarks ( Geometrical Interpretation )

The directional derivative  $D_u f(x_0, y_0)$  can be interpreted algebraically as the instantaneous rate of change in the direction of  $\vec{u}$  at  $(x_0, y_0)$  of  $z = f(x, y)$  with respect



to the distance parameters described above, or geometrically as the rise over the run of the tangent line to the curve  $C$  at the point  $Q_0$ .

**NOTE :** Formula for the directional derivative can be written in the following compact form, using gradient notation:  $D_u f(x, y) = \bar{\nabla} f(x, y) \cdot \hat{u}$

The dot product of the gradient of  $f$  with the unit vector  $\hat{u}$  produces the directional derivative of  $f$  in the direction of  $\vec{u}$ .

**Example :** Find the directional derivative of  $f(x, y) = 3x^2y$  at  $(1, 2)$  in the direction of  $\vec{a} = 3\hat{i} + 4\hat{j}$ .

**Solution :** Given  $f(x, y) = 3x^2y$ ,  $(1, 2)$ ,  $\vec{a} = 3\hat{i} + 4\hat{j}$

$$f_x(x, y) = 6xy, \quad f_x(1, 2) = 6(1)(2) = 12$$

$$f_y(x, y) = 3x^2, \quad f_y(1, 2) = 3(1)^2 = 3$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{3\hat{i} + 4\hat{j}}{\sqrt{3^2 + 4^2}} = \frac{3\hat{i} + 4\hat{j}}{\sqrt{25}} = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}$$

$$\begin{aligned} D_a f(x, y) &= \bar{\nabla} f(1, 2) \cdot \hat{a} = (f_x(1, 2)\hat{i} + f_y(1, 2)\hat{j}) \cdot \left(\frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}\right) \\ &= (12\hat{i} + 3\hat{j}) \cdot \left(\frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}\right) = 12\left(\frac{3}{5}\right) + 3\left(\frac{4}{5}\right) = \frac{48}{5} \end{aligned}$$

**Example :** Find the directional derivative of  $f(x, y) = 2x^2 + y^2$  at  $P_0(-1, 1)$  in the direction of  $\vec{u} = 3\hat{i} - 4\hat{j}$ .

**Solution :** Given  $f(x, y) = 2x^2 + y^2$ ,  $P_0(-1, 1)$ ,  $\vec{u} = 3\hat{i} - 4\hat{j}$ .

$$f_x(x, y) = 4x, \quad f_x(-1, 1) = 4(-1) = -4$$

$$f_y(x, y) = 2y, \quad f_y(-1, 1) = 2(1) = 2$$

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{3\hat{i} - 4\hat{j}}{\sqrt{3^2 + 4^2}} = \frac{3\hat{i} - 4\hat{j}}{\sqrt{25}} = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$$

$$\begin{aligned} D_a f(x, y) &= \bar{\nabla} f(-1, 1) \cdot \hat{u} = (-4\hat{i} + 2\hat{j}) \cdot \left(\frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}\right) \\ &= -4\left(\frac{3}{5}\right) + 2\left(\frac{4}{5}\right) = \frac{-20}{5} = -4 \end{aligned}$$

**Remarks:**

If  $\vec{u} = u_1\hat{i} + u_2\hat{j}$  is a unit vector making an angle  $\theta$  with the positive x-axis, then  $u_1 = \cos \theta$  and  $u_2 = \sin \theta$ . So  $D_u f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$  can be written in the form  $D_u f(x_0, y_0) = f_x(x_0, y_0) \cos \theta + f_y(x_0, y_0) \sin \theta$

**Example:** Find the directional derivative of  $e^{xy}$  at  $(-2, 0)$  in the direction of the unit vector  $\vec{u}$  that makes an angle of  $\frac{\pi}{3}$  with the positive x-axis.

**Solution:** Given  $f(x, y) = e^{xy}$ ,  $(-2, 0)$ ,  $\hat{u} = \cos \frac{\pi}{3} \hat{i} + \sin \frac{\pi}{3} \hat{j}$ .

$$\begin{aligned} f_x(x, y) &= ye^{xy}, & f_x(-2, 0) &= (0)e^{(-2)(0)} = 0 \\ f_y(x, y) &= xe^{xy}, & f_y(-2, 0) &= (-2)e^{(-2)(0)} = -2 \end{aligned}$$

$$\begin{aligned} D_u f(x, y) &= \bar{\nabla} f(-2, 0) \cdot \hat{u} = (0\hat{i} - 2\hat{j}) \cdot \left( \cos \frac{\pi}{3} \hat{i} + \sin \frac{\pi}{3} \hat{j} \right) \\ &= (0\hat{i} - 2\hat{j}) \cdot \left( \frac{1}{2} \hat{i} + \frac{\sqrt{3}}{2} \hat{j} \right) = 0 \left( \frac{1}{2} \right) - 2 \left( \frac{\sqrt{3}}{2} \right) \\ &= -\sqrt{3} \end{aligned}$$

### Gradient of Function

If  $f$  is a function of  $x$  and  $y$ , then gradient of  $f$  is defined as

$$\nabla f(x, y) = f_x(x, y) \hat{i} + f_y(x, y) \hat{j}$$

### Directional Derivative

Formula for the directional derivative can be written in the following compact form using the gradient

$$D_u f(x, y) = \nabla f(x, y) \cdot \hat{u}$$

The dot product of the gradient  $\nabla f$  with the unit vector  $\hat{u}$  produces the directional derivative of  $f$  in the direction of  $\hat{u}$ .

**Example:** Find the directional derivative of  $f(x, y) = 2xy - 3y^2$  at  $P_0(5, 5)$  in the direction of  $\vec{u} = 4\hat{i} + 3\hat{j}$ .

**Solution:**

$$\begin{aligned} \text{Given } f(x, y) &= 2xy - 3y^2, & P_0(5, 5), & \vec{u} = 4\hat{i} + 3\hat{j} \\ f_x(x, y) &= 2y, & f_x(5, 5) &= 2(5) = 10 \\ f_y(x, y) &= 2x - 6y, & f_y(5, 5) &= 2(5) - 6(5) = -20 \end{aligned}$$

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{4\hat{i} + 3\hat{j}}{\sqrt{4^2 + 3^2}} = \frac{4\hat{i} + 3\hat{j}}{\sqrt{25}} = \frac{4}{5}\hat{i} + \frac{3}{5}\hat{j}$$

$$\begin{aligned} D_u f(x, y) &= \bar{\nabla} f(5, 5) \cdot \hat{u} = (10\hat{i} - 20\hat{j}) \cdot \left( \frac{4}{5}\hat{i} + \frac{3}{5}\hat{j} \right) \\ &= 10 \left( \frac{4}{5} \right) - 20 \left( \frac{3}{5} \right) = \frac{-20}{5} = -4 \end{aligned}$$

**Example:** Find the directional derivative of  $f(x, y) = xe^y + \cos(xy)$  at the point  $(2, 0)$  in the direction of  $\vec{a} = 3\hat{i} - 4\hat{j}$ .

**Solution:**

$$\text{Given } f(x, y) = xe^y + \cos(xy), \quad (2, 0), \quad \vec{a} = 3\hat{i} - 4\hat{j}$$

$$f_x(x, y) = e^y - y \sin(xy), \quad f_x(2, 0) = e^0 - (0) \sin(2 \times 0) = 1 - 0 = 1$$

$$f_y(x, y) = xe^y - x \sin(xy), \quad f_y(2, 0) = 2e^0 - (2) \sin(2 \times 0) = 2 - 0 = 2$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{3\hat{i} - 4\hat{j}}{\sqrt{3^2 + 4^2}} = \frac{3\hat{i} - 4\hat{j}}{\sqrt{25}} = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$$

$$\begin{aligned} D_a f(x, y) &= \bar{\nabla} f(2, 0) \cdot \hat{a} \\ &= (\hat{i} + 2\hat{j}) \cdot \left( \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j} \right) \\ &= 1\left(\frac{3}{5}\right) - 2\left(\frac{4}{5}\right) = \frac{-5}{5} = -1 \end{aligned}$$

### Properties of Directional Derivatives

$$D_u f = \bar{\nabla} f \cdot \hat{u} = |\bar{\nabla} f| \cos \theta$$

1. The function  $f$  increases most rapidly when  $\cos \theta = 1$  or  $\theta = 0$  or when  $\hat{u}$  is in the direction of  $\bar{\nabla} f$ . That is, at each point P in its domain,  $f$  increases most rapidly in the direction of gradient vector  $\bar{\nabla} f$  at P. The derivative in this direction is

$$D_u f = \bar{\nabla} f \cdot \hat{u} = |\bar{\nabla} f| \cos 0 = |\bar{\nabla} f|$$

2. The function  $f$  decreases most rapidly when  $\cos \theta = -1$  or  $\theta = \pi$  or when  $\hat{u}$  is in the opposite direction of  $\bar{\nabla} f$ . That is, at each point P in its domain,  $f$  decreases most rapidly in the direction of gradient vector  $-\bar{\nabla} f$  at P. The derivative in this direction is

$$D_u f = \bar{\nabla} f \cdot \hat{u} = |\bar{\nabla} f| \cos \pi = -|\bar{\nabla} f|$$

3. Any direction of  $\hat{u}$  which is orthogonal to the gradient vector  $\bar{\nabla}f$  is the direction of zero change in  $f$  because  $\theta = \frac{\pi}{2}$  and  $\cos \frac{\pi}{2} = 0$

$$D_u f = \bar{\nabla}f \cdot \hat{u} = |\bar{\nabla}f| \cos \frac{\pi}{2} = |\bar{\nabla}f| \cdot 0 = 0$$

**Example:** Find the directions of rapid increase, rapid decrease and no change for the function  $f(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$ .

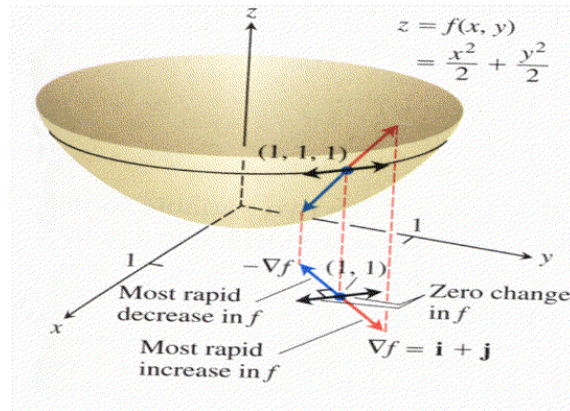
**Solution:**  $f(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$

$$\begin{aligned}\bar{\nabla}f(x, y) &= \left( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} \right) \left( \frac{x^2}{2} + \frac{y^2}{2} \right) \\ &= \hat{i} \frac{\partial}{\partial x} \left( \frac{x^2}{2} + \frac{y^2}{2} \right) + \hat{j} \frac{\partial}{\partial y} \left( \frac{x^2}{2} + \frac{y^2}{2} \right) \\ &= \hat{i} \left( \frac{2x}{2} + 0 \right) + \hat{j} \left( 0 + \frac{2y}{2} \right) \\ &= x\hat{i} + y\hat{j} \\ \bar{\nabla}f(1, 1) &= \hat{i} + \hat{j}\end{aligned}$$

(a) Its direction of rapid increase is  $\hat{u} = \frac{\hat{i} + \hat{j}}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

(b) The function  $f$  decreases most rapidly in the direction of gradient vector  $-\bar{\nabla}f$  at (1,1) which is  $-\hat{u} = -\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$

(c) The direction of zero change of the function  $f$  is orthogonal to gradient vector  $-\bar{\nabla}f$  at (1,1) which is  $\hat{u} = -\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$ ,  $-\hat{u} = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$



## LECTURE No. 12

## TANGENT PLANES TO THE SURFACES

Normal line to the surfaces

If  $C$  is a smooth parametric curve in three dimensions, then tangent line to  $C$  at the point  $P_0$  is the line through  $P_0$  along the unit tangent vector to the  $C$  at the  $P_0$ . The concept of a tangent plane builds on this definition.

If  $P_0(x_0, y_0, z_0)$  is a point on the Surface  $S$ , and if the tangent lines at  $P_0$  to all the smooth curves that pass through  $P_0$  and lies on the surface  $S$  all lie in a common plane, then we shall regard that plane to be the **tangent plane** to the surface  $S$  at  $P_0$ .

Its normal (the straight line through  $P_0$  and perpendicular to the tangent) is called the **surface normal** of  $S$  at  $P_0$ .

Different forms of equation of straight line in two dimensional space

## 1. Slope intercept form of the Equation of a line

$$y = mx + c$$

where  $m$  is the slope and  $c$  is  $y$  intercept

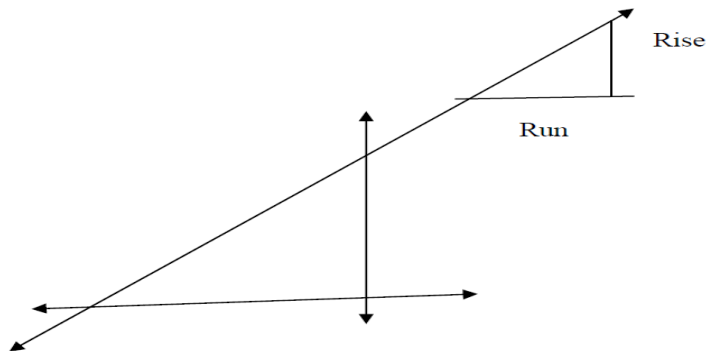
## 2. Point-Slope Form

Let  $m$  be the slope and  $P_0(x_0, y_0)$  be the point of required line, then

$$y - y_0 = m(x - x_0)$$

$$m = \text{slope of line} = \frac{\text{Rise}}{\text{Run}} = \frac{b}{a}$$

$$y - y_0 = \frac{b}{a}(x - x_0)$$



## 3. General Equation of straight line

$$Ax + By + C = 0$$

**Parametric equation of a line**

Parametric equation of a line in two dimensional space passing through the point  $(x_0, y_0)$  and parallel to the vector  $a\mathbf{i} + b\mathbf{j}$  is given by

$$x = x_0 + at, \quad y = y_0 + bt$$

Eliminating  $t$  from both equations, we get

$$\frac{x - x_0}{a} = t, \quad \frac{y - y_0}{b} = t$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}$$

$$y - y_0 = \frac{b}{a}(x - x_0)$$

**Parametric vector form:**

$$\mathbf{r}(t) = (x_0 + at)\mathbf{i} + (y_0 + bt)\mathbf{j}$$

**Equation of line in three dimensional**

Parametric equation of a line in three dimensional space passing through the point  $(x_0, y_0, z_0)$  and parallel to the vector  $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$  is given by

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$

Eliminating  $t$  from these equations we get

$$\frac{x - x_0}{a} = t, \quad \frac{y - y_0}{b} = t, \quad \frac{z - z_0}{c} = t$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

**Example:** Find Parametric equations for the straight line through the point A (2,4,3) and parallel to the vector  $\mathbf{v} = 4\mathbf{i} + 0\mathbf{j} - 7\mathbf{k}$ .

**Solution:**

$$x_0 = 2, y_0 = 4, z_0 = 3$$

$$\text{and } a = 4, b = 0, c = -7$$

The required parametric equations of the straight line are

$$x = 2 + 4t,$$

$$y = 4 + 0t,$$

$$z = 3 - 7t$$

**Different forms of the equation of curve**

Curves in the plane are defined in different ways

**(1) Explicit form:**  $y = f(x)$

**Example:**  $y = \sqrt{9-x^2} \quad -3 \leq x \leq 3$

**(2) Implicit form:**  $F(x, y) = 0$

**Example:**  $x^2 + y^2 = 9 \quad -3 \leq x \leq 3, \quad 0 \leq y \leq 3$

**(3) Parametric form:**  $x = f(t)$  and  $y = g(t)$

**Example:**

$$x = 3 \cos \theta, \quad y = 3 \sin \theta \quad 0 \leq \theta \leq \pi$$

$$x^2 + y^2 = 9 \cos^2 \theta + 9 \sin^2 \theta$$

$$= 9(\cos^2 \theta + \sin^2 \theta) = 9(1)$$

$$x^2 + y^2 = 9$$

**(4) Parametric vector form:**  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}, \quad a \leq t \leq b.$

**Example:**  $\mathbf{r}(t) = 3 \cos t \mathbf{i} + 3 \sin t \mathbf{j} \quad 0 \leq t \leq \pi$

### Equation of a plane

A plane can be completely determined if we know its one point and direction of perpendicular (normal) to it.

Let a plane passing through the point  $P_0(x_0, y_0, z_0)$  and the direction of normal to it is along the vector  $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

Let  $P(x, y, z)$  be any point on the plane, then the line lies on it so that  $\mathbf{n} \perp \overline{P_0P}$   
( $\perp$  means “perpendicular to”)

$$\overline{P_0P} = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$$

Therefore,  $\mathbf{n} \cdot \overline{P_0P} = 0$

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot ((x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}) = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

which is the required equation of the plane.

**NOTE :** Here we use the theorem:

Let  $\vec{a}$  and  $\vec{b}$  be two vectors. If  $\vec{a}$  and  $\vec{b}$  are perpendicular,  
then  $\vec{a} \cdot \vec{b} = 0$

Since  $\mathbf{n}$  and  $\overrightarrow{P_0P}$  are perpendicular vector, so  $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$

### **REMARKS**

Point normal form of equation of plane is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

We can write this equation as

$$ax - ax_0 + by - by_0 + cz - cz_0 = 0$$

$$ax + by + cz - ax_0 - by_0 - cz_0 = 0$$

$$ax + by + cz + d = 0$$

$$\text{where } d = -ax_0 - by_0 - cz_0$$

, which is the equation of plane

**Example:** An equation of the plane passing through the point (3, -1, 7) and perpendicular to the vector  $\mathbf{n} = 4\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$ .

A point-normal form of the equation is

$$4(\mathbf{x} - 3) + 2(\mathbf{y} + 1) - 5(\mathbf{z} - 7) = 0$$

$$4\mathbf{x} + 2\mathbf{y} - 5\mathbf{z} + 25 = 0$$

Which is the same form of the equation of plane  $\mathbf{ax} + \mathbf{by} + \mathbf{cz} + \mathbf{d} = 0$

The general equation of straight line

$$\text{is } ax + by + c = 0$$

Let  $(x_1, y_1)$  and  $(x_2, y_2)$  be two points on this line then

$$ax_1 + by_1 + c = 0$$

$$ax_2 + by_2 + c = 0$$

Subtracting above equation

$$a(x_2 - x_1) + b(y_2 - y_1) = 0$$

$$\mathbf{v} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}$$

is a vector in the direction of line

$$\phi(x, y) = ax + by$$

$$\phi_x = a, \quad \phi_y = b$$

$$\nabla\phi = a\mathbf{i} + b\mathbf{j} = \mathbf{n}$$

$$\nabla\phi \cdot \mathbf{r} = 0$$

Then  $\mathbf{n}$  and  $\mathbf{v}$  are perpendicular

The general equation of plane is

$$ax + by + cz + d = 0$$

For any two points  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  lying on this plane we have

$$ax_1 + by_1 + cz_1 + d = 0 \quad (1)$$

$$ax_2 + by_2 + cz_2 + d = 0 \quad (2)$$

Subtracting equation (1) from (2)

have

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0$$

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot [(x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}] = 0$$

Here we use the definition of dot product of two vectors.

$$\phi = ax + by + cz$$

$$\phi_x = a, \quad \phi_y = b, \quad \phi_z = c$$

$$\nabla\phi = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

$$\text{Where } \mathbf{v} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$

$\nabla\phi$  is always normal to the plane.

### Gradients and Tangents to Surfaces

$$f(x, y) = c$$

$$z = f(x, y), \quad z = c$$

If a differentiable function  $f(x, y)$  has a constant value  $c$  along a smooth curve, having parametric equations:

$$x = g(t), \quad y = h(t), \quad \mathbf{r} = g(t)\mathbf{i} + h(t)\mathbf{j}$$

Differentiating both sides of  $f(x, y) = c$  with respect to  $t$ ,

$$\frac{d}{dt} f(g(t), h(t)) = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x} \frac{dg}{dt} + \frac{\partial f}{\partial y} \frac{dh}{dt} = 0$$

$$\left( \frac{\partial f}{\partial x} \mathbf{i} + \frac{dh}{dt} \mathbf{j} \right) \cdot \left( \frac{dg}{dt} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} \right) = 0$$

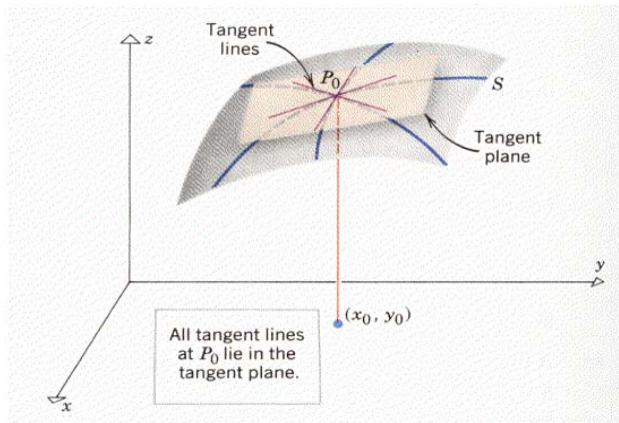
$$\nabla f \cdot \frac{d\mathbf{r}}{dt} = 0$$

$\nabla f$  is normal to the tangent vector  $\frac{d\mathbf{r}}{dt}$ . So it is normal to the curve through  $(x_0, y_0)$ .

### **Tangent Plane and Normal Line**

Consider all the curves through the point  $\mathbf{P}_0(x_0, y_0, z_0)$  on a surface  $\mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 0$ . The plane containing all the tangents to these curves at the point  $P_0(x_0, y_0, z_0)$  is called the **tangent plane to the surface at the point  $P_0$** .

The straight lines perpendicular to all these tangent lines at  $P_0$  is called the **normal line to the surface at  $P_0$**  if  $f_x, f_y, f_z$  are all continuous at  $P_0$  and not all of them are zero, then gradient  $f$  (i.e  $f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k}$ ) at  $P_0$  gives the direction of this normal vector to the surface at  $P_0$ .



### **Tangent plane**

Let  $P_0(x_0, y_0, z_0)$  be any point on the Surface  $f(x, y, z) = 0$ . If  $f(x, y, z)$  is differentiable at  $p_0(x_0, y_0, z_0)$  then the tangents plane at the point  $P_0(x_0, y_0, z_0)$  has the equation

**Example:** Find the equation of tangent plane to the surface  $9x^2 + 4y^2 - z^2 = 36$  at point  $P(2, 3, 6)$ .

**Solution:**

$$9x^2 + 4y^2 - z^2 = 36 \quad P(2, 3, 6).$$

$$f(x, y, z) = 9x^2 + 4y^2 - z^2 - 36$$

$$f_x = 18x, \quad f_y = 8y, \quad f_z = -2z$$

$$f_x(P) = 36, \quad f_y(P) = 24, \quad f_z(P) = -12$$

Equations of Tangent Plane to the surface through P is

$$36(x - 2) + 24(y - 3) - 12(z - 6) = 0$$

$$3x + 2y - z - 6 = 0$$

**Example:** Find the equation of tangent plane to the surface  $z = x \cos y - y e^x$  at point  $(0, 0, 0)$ .

**Solution:**

$$z = x \cos y - y e^x \quad (0, 0, 0).$$

$$\cos y - y e^x - z = 0$$

$$f(x, y, z) = \cos y - y e^x - z$$

$$f_x(0, 0, 0) = (\cos y - y e^x)_{(0,0)} = 1 - 0.1 = 1$$

$$f_y(0, 0, 0) = (-x \sin y - e^x)_{(0,0)} = 0 - 1 = -1.$$

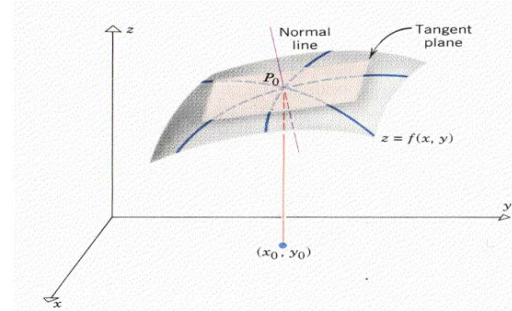
$$f_z(0, 0, 0) = -1$$

The tangent plane is

$$f_x(0, 0, 0)(x - 0) + f_y(0, 0, 0)(y - 0) + f_z(0, 0, 0)(z - 0) = 0$$

$$1(x - 0) - 1(y - 0) - 1(z - 0) = 0,$$

$$x - y - z = 0.$$



## LECTURE No. 13

## ORTHOGONAL SURFACE

In this Lecture we will study the following topics

- Normal line
- Orthogonal Surface
- Total differential for function of one variable
- Total differential for function of two variables

## Normal line

Let  $P_0(x_0, y_0, z_0)$  be any point on the surface  $f(x, y, z) = 0$ . If  $f(x, y, z)$  is differentiable at  $P_0(x_0, y_0, z_0)$  then the normal line at the point  $P_0(x_0, y_0, z_0)$  has the equation

$$x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t$$

Here  $f_x$  means that the function  $f(x, y, z)$  is partially differentiable with respect to  $x$  And  $f_x(P_0)$  means that the function  $f(x, y, z)$  is partially differentiable with respect to  $x$  at the point  $P_0(x_0, y_0, z_0)$

$f_y$  means that the function  $f(x, y, z)$  is partially differentiable with respect to  $y$  And  $f_y(P_0)$  means that the function  $f(x, y, z)$  is partially differentiable with respect to  $y$  at the point  $P_0(x_0, y_0, z_0)$

Similarly,  $f_z$  means that the function  $f(x, y, z)$  is partially differentiable with respect to  $z$  And  $f_z(P_0)$  means that the function  $f(x, y, z)$  is partially differentiable with respect to  $z$  at the point  $P_0(x_0, y_0, z_0)$

**Example:** Find the Equation of the tangent plane and normal of the surface  $f(x, y, z) = x^2 + y^2 + z^2 - 4$  at the point  $P(1, -2, 3)$

**Solution:**

$$f(x, y, z) = x^2 + y^2 + z^2 - 4, \quad P(1, -2, 3)$$

$$f_x = 2x, \quad f_y = 2y, \quad f_z = 2z$$

$$f_x(P_0) = 2, \quad f_y(P_0) = -4, \quad f_z(P_0) = 6$$

Equation of the tangent plane to the surface at  $P$  is

$$2(x-1) - 4(y+2) + 6(z-3) = 0$$

$$x - 2y + 3z - 14 = 0$$

Equation of the normal line of the surface through  $P$  is

$$\frac{(x-1)}{2} = \frac{(y+2)}{-4} = \frac{(z-3)}{6}$$

$$\frac{(x-1)}{1} = \frac{(y+2)}{-2} = \frac{(z-3)}{3}$$

**Example :** Find the equation of the tangent plane and normal plane

**Solution:**

$$4x^2 - y^2 + 3z^2 = 10 \quad P(2, -3, 1)$$

$$f(x, y, z) = 4x^2 - y^2 + 3z^2 - 10$$

$$f_x = 8x, \quad f_y = -2y, \quad f_z = 6z$$

$$f_x(P) = 16, \quad f_y(P) = 6, \quad f_z(P) = 6$$

Equations of Tangent Plane to the surface through P is

$$16(x - 2) + 6(y + 3) + 6(z - 1) = 0$$

$$8x + 3y + 3z = 10$$

Equations of the normal line to the surface through P are

$$\frac{x - 2}{16} = \frac{y + 3}{6} = \frac{z - 1}{6}$$

$$\frac{x - 2}{8} = \frac{y + 3}{3} = \frac{z - 1}{3}$$

**Example**

$$z = \frac{1}{2} x^7 y^2$$

$$f(x, y, z) = \frac{1}{2} x^7 y^2 - z$$

$$f_x = \frac{7}{2} x^6 y^2, \quad f_y = -x^7 y, \quad f_z = -1$$

$$f_x(2, 4, 4) = \frac{7}{2} (2)^6 (4)^2 = 14$$

$$f_y(2, 4, 4) = -(2)^7 (4) = -2$$

$$f_z(2, 4, 4) = -1$$

Equation of Tangent at (2, 4, 4) is given by

$$f_x(2, 4, 4)(x - 2) + f_y(2, 4, 4)(y - 4) + f_z(2, 4, 4)(z - 4) = 0$$

$$14(x - 2) + (-2)(y - 4) - (z - 4) = 0$$

$$14x - 2y - z - 16 = 0$$

The normal line has equation

$$x = 2 + f_x(2, 4, 4)t, \quad y = 4 + f_y(2, 4, 4)t, \quad z = 4 + f_z(2, 4, 4)t$$

$$x = 2 + 14t, \quad y = 4 - 2t, \quad z = 4 - t$$

## ORTHOGONAL SURFACES

Two surfaces are said to be orthogonal at a point of their intersection if their normals at that point are orthogonal. They are said to intersect orthogonally if they are orthogonal at every point common to them.

### CONDITION FOR ORTHOGONAL SURFACES

Let  $(x, y, z)$  be any point of intersection of

$$f(x, y, z) = 0 \text{ ---- (1)}$$

$$\text{and } g(x, y, z) = 0 \text{ ---- (2)}$$

Direction ratios of a line normal to (1) are  $f_x, f_y, f_z$

Similarly, direction ratios of a line normal to (2)

are  $g_x, g_y, g_z$

The two normal lines are orthogonal if and only if

$$f_x g_x + f_y g_y + f_z g_z = 0$$

### **Example**

Show that given two surfaces are orthogonal or not

$$f(x, y, z) = x^2 + y^2 + z - 16$$

$$g(x, y, z) = x^2 + y^2 - 63z$$

$$f(x, y, z) = x^2 + y^2 + z - 16 \text{ ---- (1)}$$

$$g(x, y, z) = x^2 + y^2 - 63z \text{ ---- (2)}$$

$$\text{Adding (1) and (2), } x^2 + y^2 = \frac{63}{4}, \quad z = \frac{1}{4} \text{ ---- (3)}$$

$$f_x = 2x, \quad f_y = 2y, \quad f_z = 1$$

$$g_x = 2x, \quad g_y = 2y, \quad g_z = -63$$

$$f_x g_x + f_y g_y + f_z g_z = 4(x^2 + y^2) - 63 = 4\left(\frac{63}{4}\right) - 63 \quad \text{using (3)}$$

$$= 0$$

Since they satisfy the condition of orthogonality, so they are orthogonal.

### Differentials of a functions

For a function  $y = f(x)$

$$dy = f'(x) dx$$

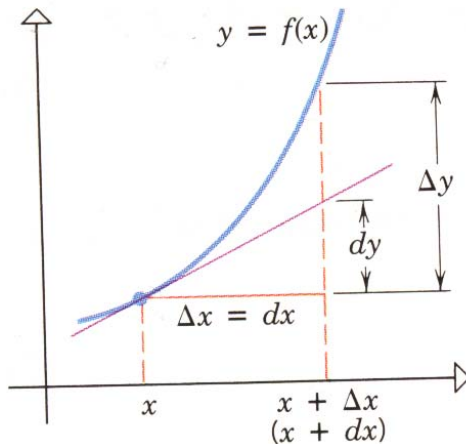
is called the differential of functions  $f(x)$

$dx$  the differential of  $x$  is the same as the actual change in  $x$

i.e.  $dx = \nabla x$  where as  $dy$  the

differential of  $y$  is the approximate change in the value of the functions which is different from the actual change  $\nabla y$  in the value of the functions.

**Distinction between the increments  $\Delta y$  and the differential  $dy$**



### Approximation to the curve

If  $f$  is differentiable at  $x$ , then the tangent line to the curve  $y = f(x)$  at  $x_0$  is a reasonably good approximation to the curve  $y = f(x)$  for value of  $x$  near  $x_0$ . Since the tangent line passes

through the point  $(x_0, f(x_0))$  and has slope  $f'(x_0)$ , the point-slope form of its equation is

$$y - f(x_0) = f'(x_0)(x - x_0) \quad \text{or}$$

$$y = f(x_0) + f'(x_0)(x - x_0)$$

### EXAMPLE

$$f(x) = \sqrt{x}$$

$$x = 4 \text{ and } dx = \Delta x = 3, y = \sqrt{3}$$

$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$$

$$= \sqrt{7} - \sqrt{4} \approx .65$$

If  $y = \sqrt{x}$ , then

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \quad \text{so } dy = \frac{1}{2\sqrt{x}} dx$$

$$= \frac{1}{2\sqrt{x}} (3) = \frac{3}{4} = .75$$

**Example**

**Using differentials approximation  
for the value of  $\cos 61^\circ$ .**

Let  $y = \cos x$  and  $x = 60^\circ$

then  $dx = 61^\circ - 60^\circ = 1^\circ$

$$\Delta y \approx dy = -\sin x \, dx = -\sin 60^\circ (1^\circ)$$

$$= \frac{\sqrt{3}}{2} \left( \frac{1}{180} \pi \right)$$

Now  $y = \cos x$

$$y + \Delta y = \cos (x + \Delta x) = \cos (x + dx)$$

$$= \cos (60^\circ + 1^\circ) = \cos 61^\circ$$

$$\cos 61^\circ = y + \Delta y = \cos x + \Delta y$$

$$\approx \cos 60^\circ - \frac{\sqrt{3}}{2} \left( \frac{1}{180} \pi \right)$$

$$\cos 61^\circ \approx \frac{1}{2} - \frac{\sqrt{3}}{2} \left( \frac{1}{180} \pi \right)$$

$$= 0.5 - 0.01511 = 0.48489$$

$$\cos 61^\circ \approx 0.48489$$

**Example**

**A box with a square base has its height twice its width. If the width of the box is 8.5 inches with a possible error of  $\pm 0.3$  inches**

Let  $x$  and  $h$  be the width and the height of the box respectively, then its volume

$V$  is given by

$$V = x^2 h$$

Since  $h = 2x$ , so (1) take the form

$$V = 2x^3$$

$$dV = 6x^2 \, dx$$

Since  $x = 8.5$ ,  $dx = \pm 0.3$ , so

putting these values in (2), we have

$$dV = 6 (8.5)^2 (\pm 0.3) = \pm 130.05$$

This shows that the possible error in the volume of the box is  $\pm 130.05$ .

**TOTAL DIFFERENTIAL**

If we move from  $(x_0, y_0)$  to a point  $(x_0 + dx, y_0 + dy)$  nearby, the resulting differential in  $f$  is

$$df = f_x(x_0, y_0) \, dx + f_y(x_0, y_0) \, dy$$

This change in the linearization of

$f$  is called the total differential of  $f$ .

## EXACT CHANGE

$$\text{Area} = xy$$

$$\begin{aligned} x = 10, y = 8 & \quad \text{Area} = 80 \\ x = 10.03, y = 8.02 & \quad \text{Area} = 80.4406 \\ \text{Exact Change in area} &= 80.4406 - 80 \\ &= 0.4406 \end{aligned}$$

### Example

**A rectangular plate expands in such a way that its length changes from 10 to 10.03 and its breadth changes from 8 to 8.02.**

Let  $x$  and  $y$  the length and breadth of the rectangle respectively, then its area is

$$A = xy$$

$$dA = A_x dx + A_y dy = ydx + xdy$$

By the given conditions

$$x = 10, dx = 0.03, y = 8, dy = 0.02.$$

$$dA = 8(0.03) + 10(0.02) = 0.44$$

Which is an exact change.

### Example

The volume of a rectangular parallelepiped is given by the formula  $V = xyz$ . If this solid is compressed from above so that  $z$  is decreased by 2% while  $x$  and  $y$  each is increased by 0.75% approximately

$$V = xyz$$

$$dV = V_x dx + V_y dy + V_z dz$$

$$dV = yzdx + xzdy + zydz \quad (1)$$

$$dx = \frac{0.75}{100} x, dy = \frac{0.75}{100} y, dz = -\frac{2}{100} z$$

Putting these values in (1), we have

$$\begin{aligned} dV &= \frac{0.75}{100} xyz + \frac{0.75}{100} xyz - \frac{2}{100} xyz \\ &= -\frac{0.5}{100} xyz = -\frac{0.5}{100} V \end{aligned}$$

This shows that there is 0.5 % decrease in the volume.

**Example**

A formula for the area  $\Delta$  of a triangle is

$$\Delta = \frac{1}{2} ab \sin C. \text{ Approximately what error is}$$

made in computing  $\Delta$  if  $a$  is taken to be 9.1 instead of 9,  $b$  is taken to be 4.08 instead of 4 and  $C$  is taken to be  $30^\circ 3'$  instead of  $30^\circ$ .

By the given conditions

$$a = 9, b = 4, C = 30^\circ,$$

$$da = 9.1 - 9 = 0.1,$$

$$db = 4.08 - 4 = 0.08$$

$$dC = 30^\circ 3' - 30^\circ = \left(\frac{3}{60}\right)^\circ$$

$$= \frac{3}{60} \times \frac{\pi}{180} \text{ radians}$$

Putting these values in (1), we have

$$\Delta = \frac{1}{2} ab \sin C$$

$$d\Delta = \frac{\partial}{\partial a} \left( \frac{1}{2} ab \sin C \right) da + \frac{\partial}{\partial b} \left( \frac{1}{2} ab \sin C \right) db$$

$$+ \frac{\partial}{\partial C} \left( \frac{1}{2} ab \sin C \right) dC$$

$$d\Delta = \frac{1}{2} b \sin C da + \frac{1}{2} a \sin C db$$

$$+ \frac{1}{2} ab \cos C dC$$

$$d\Delta = \frac{1}{2} 4 \sin 30^\circ (0.1) + \frac{1}{2} 9 \sin 30^\circ (0.08) + \frac{1}{2} 36 \cos 30^\circ \left( \frac{\pi}{3600} \right)$$

$$= 2 \left( \frac{1}{2} \right) (0.1) + \frac{9}{2} \left( \frac{1}{2} \right) (0.08) + 18 \left( \frac{\sqrt{3}}{2} \right) \left( \frac{3.14}{3600} \right) = 0.293$$

$$\% \text{ age change in area} = \frac{0.293}{\Delta} \times 100 = \frac{0.293}{9} \times 100 = 3.25\%$$

## LECTURE No.14

## EXTREMEA OF FUNCTIONS OF TWO VARIABLES

In this lecture, we shall find the techniques for finding the highest and lowest points on the graph of a function or, equivalently, the largest and smallest values of the function.

The graphs of many functions form hills and valleys. The tops of the hills are relative maxima and the bottoms of the valleys are called relative minima. Just as the top of a hill on the earth's terrain need not be the highest point on the earth, so a relative maximum need not be the highest point on the entire graph.

**Absolute maximum**

A function  $f$  of two variables on a subset of  $\mathbf{R}^2$  is said to have an  **$D$  absolute (global) maximum** value on  $D$  if there is some point  $(x_0, y_0)$  of  $D$  such that value of  $f$  on  $D$

$$f(x_0, y_0) \geq f(x, y) \text{ for all } (x, y) \in D$$

In such a case  $f(x_0, y_0)$  is the **absolute maximum**

**Relative extremum and absolute extremum**

If  $f$  has a relative maximum or a relative minimum at  $(x_0, y_0)$ , then we say that  $f$  has a relative extremum at  $(x_0, y_0)$ , and if  $f$  has an absolute maximum or absolute minimum at  $(x_0, y_0)$ , then we say that  $f$  has an absolute extremum at  $(x_0, y_0)$ .

**Absolute minimum**

A function  $f$  of two variables on a subset  $D$  of  $\mathbf{R}^2$  is said to have an **absolute (global) minimum** value on  $D$  if there is some point  $(x_0, y_0)$  of  $D$  such that

$$f(x_0, y_0) \leq f(x, y) \text{ for all } (x, y) \in D.$$

In such a case  $f(x_0, y_0)$  is the **absolute minimum** value of  $f$  on  $D$ .

**Relative (local) maximum**

The function  $f$  is said to have a relative (local) maximum at some point  $(x_0, y_0)$  of its domain  $D$  if there exists an open disc  $K$  centered at  $(x_0, y_0)$  and of radius  $r$

$$K = \{(x, y) \in \mathbf{R}^2 : (x - x_0)^2 + (y - y_0)^2 < r^2\}$$

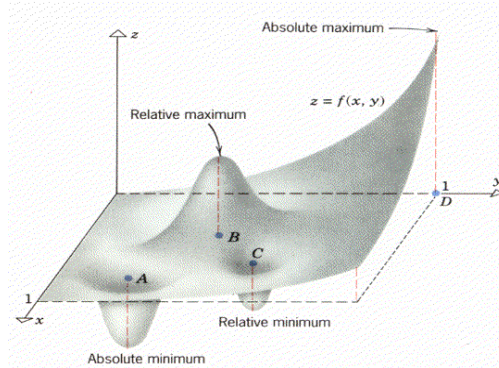
With  $K \in D$  such that

$$f(x_0, y_0) \geq f(x, y) \text{ for all } (x, y) \in K$$

### **Relative ( Local ) Minimum**

The function  $f$  is said to have a **relative ( local ) minimum** at some point  $(x_0, y_0)$  of  $D$  if there exists an open disc  $K$  centered at  $(x_0, y_0)$  and of radius  $r$  with  $K \subset D$  such that

$$f(x_0, y_0) \leq f(x, y) \quad \text{for all } (x, y) \in K$$



### **Extreme Value Theorem**

If  $f(x, y)$  is continuous on a closed and bounded set  $R$ , then  $f$  has both an absolute maximum and on absolute minimum on  $R$ .

### **Remarks**

If any of the conditions the Extreme Value Theorem fail to hold, then there is no guarantee that an absolute maximum or absolute minimum exists on the region  $R$ . Thus, a discontinuous function on a closed and bounded set need not have any absolute extrema, and a continuous function on a set that is not closed and bounded also need not have any absolute extrema.

### **Extreme values or extrema of $f$**

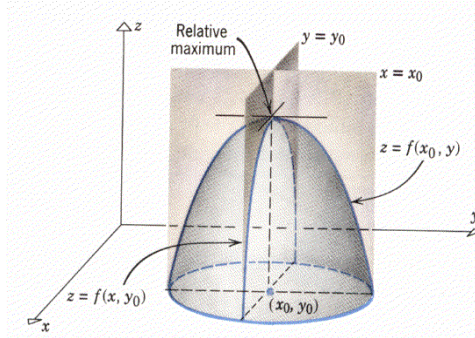
The maximum and minimum values of  $f$  are referred to as extreme values of extrema of  $f$ . Let a function  $f$  of two variables be defined on an open disc

$$K = \{(x, y): (x - x_0)^2 + (y - y_0)^2 < r^2\}.$$

Suppose  $f_x(x_0, y_0)$  and  $f_y(x_0, y_0)$  both exist on  $K$

If  $f$  has relative extrema at  $(x_0, y_0)$ , then

$$f_x(x_0, y_0) = 0 = f_y(x_0, y_0).$$



### Saddle Point

A differentiable function  $f(x, y)$  has a saddle point  $(a, b)$  if in every open disk centered at  $(a, b)$  there are domain points  $(x, y)$  where  $f(x, y) > f(a, b)$  and domain points  $(x, y)$  where  $f(x, y) < f(a, b)$ . The corresponding point  $(a, b, f(a, b))$  on the surface  $z = f(x, y)$  is called a saddle point of the surface.

### Remarks

Thus, the only points where a function  $f(x, y)$  can assume extreme values are critical points and boundary points. As with differentiable functions of a single variable, not every critical point gives rise to a local extremum. A differentiable function of a single variable might have a point of inflection. A differentiable function of two variables might have a saddle point.

### EXAMPLE

Find the critical points of the given function

$$f(x, y) = x^3 + y^3 - 3axy, a > 0.$$

$f_x, f_y$  exist at all points of the domain of  $f$ .

$$f_x = 3x^2 - 3ay, \quad f_y = 3y^2 - 3ax$$

For critical points  $f_x = f_y = 0$ .

$$\text{Therefore, } x^2 - ay = 0 \quad (1)$$

$$\text{and } ax - y^2 = 0 \quad (2)$$

Substituting the value of  $x$  from (2) into (1), we have

$$\frac{y^4}{a^2} - ay = 0$$

$$y(y^3 - a^3) = 0$$

$$y = 0, \quad y = a$$

and so

$$x = 0, \quad x = a.$$

The critical points are  $(0, 0)$  and  $(a, a)$ .

**Overview of lecture # 14**

<b>Topic</b>	<b>Article #</b>	<b>page #</b>
Extrema of Functions of Two Variables	16.9	833
Absolute maximum	16.9.1	833
Absolute manimum	16.9.2	833
Extreme Value Theorem	16.9.3	834
Exercise set	Q#1,3,5,7,9,11,13,15,17	841

**Book**

**CALCULUS by HOWARD ANTON**

## LECTURE No. 15

## EXAMPLES

**Example:** Find the critical point of  $f(x, y) = \sqrt{x^2 + y^2}$ .

$$f(x, y) = \sqrt{x^2 + y^2}$$

$$f_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}}$$

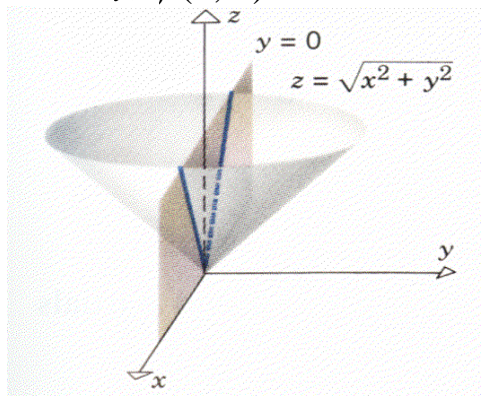
$$f_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}}$$

The partial derivatives exist at all points of the domain of  $f$  except at the origin which is in the domain of  $f$ . Thus  $(0, 0)$  is a critical point of  $f$ .

Now  $f_x(x, y) = 0$  only if  $x = 0$  and  
 $f_y(x, y) = 0$  only if  $y = 0$

The only critical point is  $(0, 0)$  and  $f(0, 0) = 0$

Since  $f(x, y) \geq 0$  for all  $(x, y)$ ,  $f(0, 0) = 0$  is the absolute minimum value of  $f$ .



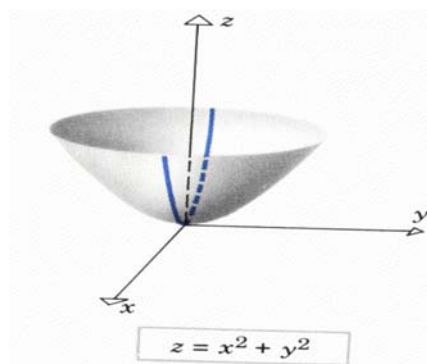
**Example :** Find the critical point of  $f(x, y) = x^2 + y^2$ .

$$z = f(x, y) = x^2 + y^2 \text{ (Paraboloid)}$$

$$f_x(x, y) = 2x, \quad f_y(x, y) = 2y$$

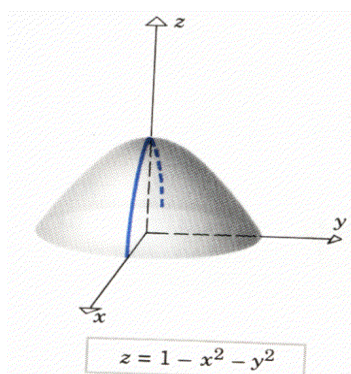
$$\text{when } f_x(x, y) = 0, \quad f_y(x, y) = 0$$

we have  $(0, 0)$  as critical point.



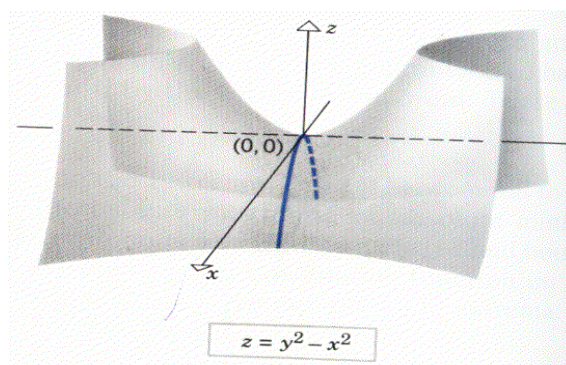
**Example:** Find the critical point of  $z = g(x, y) = 1 - x^2 - y^2$ .

$z = g(x, y) = 1 - x^2 - y^2$  (Paraboloid)  
 $g_x(x, y) = -2x$ ,  $g_y(x, y) = -2y$   
 when  $g_x(x, y) = 0$ ,  $g_y(x, y) = 0$   
 we have  $(0, 0)$  as critical point.



**Example:** Find the critical point of  $z = h(x, y) = y^2 - x^2$ .

$z = h(x, y) = y^2 - x^2$  (Hyperboloid)  
 $h_x(x, y) = -2x$ ,  $h_y(x, y) = 2y$   
 when  $h_x(x, y) = 0$ ,  $h_y(x, y) = 0$   
 we have  $(0, 0)$  as critical point.



**Example:**

$$f(x, y) = \sqrt{x^2 + y^2}$$

$$f_x = \frac{x}{\sqrt{x^2 + y^2}} \quad f_y = \frac{y}{\sqrt{x^2 + y^2}}$$

The point (0,0) is critical point of  $f$  because the partial derivatives do not both exist. It is evident geometrically that  $f_x(0,0)$  does not exist because the trace of the cone in the plane  $y=0$  has a corner at the origin.

The fact that  $f_x(0,0)$  does not exist can also be seen algebraically by noting that  $f_x(0,0)$  can be interpreted as the derivative with respect to  $x$  of the function

$$f(x, 0) = \sqrt{x^2 + 0} = |x| \quad \text{at } x = 0.$$

But  $|x|$  is not differentiable at  $x = 0$ , so  $f_x(0,0)$  does not exist. Similarly,  $f_y(0,0)$  does not exist. The function  $f$  has a relative minimum at the critical point (0,0).

**The Second Partial Derivative Test**

Let  $f$  be a function of two variables with continuous second order partial derivatives in some circle centered at a critical point  $(x_0, y_0)$ , and let

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$$

- (a) If  $D > 0$  and  $f_{xx}(x_0, y_0) > 0$ , then  $f$  has a **relative minimum** at  $(x_0, y_0)$ .
- (b) If  $D > 0$  and  $f_{xx}(x_0, y_0) < 0$ , then  $f$  has a **relative maximum** at  $(x_0, y_0)$ .
- (c) If  $D < 0$ , then  $f$  has a **saddle point** at  $(x_0, y_0)$ .
- (d) If  $D = 0$ , then no conclusion can be drawn.

**REMARKS**

If a function  $f$  of two variables has an absolute extremum (either an absolute maximum or an absolute minimum) at an interior point of its domain, then this extremum occurs at a critical point.

**Example:**

$$f(x, y) = 2x^2 - 4x + xy^2 - 1$$

$$f_x(x, y) = 4x - 4 + y^2, \quad f_{xx}(x, y) = 4$$

$$f_y(x, y) = 2xy, \quad f_{yy}(x, y) = 2x, \quad f_{xy}(x, y) = f_{yx}(x, y) = 2y$$

For the critical points, we set the first partial derivatives equal to zero. Then

$$f_x(x, y) = 0 \Rightarrow 4x - 4 + y^2 = 0 \quad (1)$$

$$f_y(x, y) = 0 \Rightarrow 2xy = 0 \quad (2)$$

$$\Rightarrow x = 0 \text{ or } y = 0$$

$$\text{When } x = 0, \text{ then by (1), } 4(0) - 4 + y^2 = 0 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$$

$$\text{When } y = 0, \text{ then by (1), } 4x - 4 + 0^2 = 0 \Rightarrow 4x - 4 = 0 \Rightarrow x = 1$$

So the critical points are  $(1, 0)$ ,  $(0, 2)$  and  $(0, -2)$ .

Now we check the nature of each point:

$$\begin{aligned} \text{At } (1, 0), \quad f_{xx}(1, 0) &= 4 \\ f_{yy}(1, 0) &= 2(1) = 2 \\ f_{xy}(1, 0) &= 2(0) = 0 \end{aligned}$$

$$D = f_{xx}(1, 0) f_{yy}(1, 0) - [f_{xy}(1, 0)]^2 = 4 \times 2 - 0^2 = 8$$

Since  $D > 0$  and  $f_{xx}(1, 0)$  is positive, so  $f$  has a relative minimum at  $(1, 0)$ .

$$\begin{aligned} \text{At } (0, -2), \quad f_{xx}(0, -2) &= 4 \\ f_{yy}(0, -2) &= 2(0) = 0 \\ f_{xy}(0, -2) &= 2(-2) = -4 \end{aligned}$$

$$D = f_{xx}(0, -2) f_{yy}(0, -2) - [f_{xy}(0, -2)]^2 = 4 \times 0 - (-4)^2 = 0 - 16 = -16$$

Since  $D < 0$ , so  $f$  has a saddle point at  $(0, -2)$ .

$$\text{At } (0, 2), \quad f_{xx}(0, 2) = 4, \quad f_{yy}(0, 2) = 2(0) = 0, \quad f_{xy}(0, 2) = 2(2) = 4$$

$$D = f_{xx}(0, 2) f_{yy}(0, 2) - [f_{xy}(0, 2)]^2 = 4 \times 0 - (4)^2 = 0 - 16 = -16$$

Since  $D < 0$ , so  $f$  has a saddle point at  $(0, 2)$ .

**Example:**

$$f(x, y) = e^{-(x^2 + y^2 + 2x)}$$

$$f_x(x, y) = -2(x+1)e^{-(x^2 + y^2 + 2x)}, \quad f_{xx}(x, y) = [(-2x-2)^2 - 2]e^{-(x^2 + y^2 + 2x)}$$

$$f_y(x, y) = -2ye^{-(x^2 + y^2 + 2x)}, \quad f_{yy}(x, y) = [4y^2 - 2]e^{-(x^2 + y^2 + 2x)}$$

$$f_{xy}(x, y) = -2y(-2x-2)e^{-(x^2 + y^2 + 2x)}$$

For critical points,

$$\begin{aligned} \text{put } f_x(x, y) = 0 &\Rightarrow -2(x+1)e^{-(x^2 + y^2 + 2x)} = 0 \\ &\Rightarrow -2e^{-(x^2 + y^2 + 2x)} \neq 0, \quad x+1 = 0 \Rightarrow x = -1 \end{aligned}$$

$$\begin{aligned} \text{put } f_y(x, y) = 0 &\Rightarrow -2ye^{-(x^2+y^2+2x)} = 0 \\ &\Rightarrow -2e^{-(x^2+y^2+2x)} \neq 0, \quad y = 0 \end{aligned}$$

The critical point is  $(-1, 0)$ .

$$\begin{aligned} f_{xx}(-1, 0) &= \left[ (-2(-1) - 2)^2 - 2 \right] e^{-((-1)^2 + 0^2 + 2(-1))} \\ &= [0 - 2] e^{-(1+2)} = -2e^1 = -2e \end{aligned}$$

$$f_{yy}(-1, 0) = \left[ 4(0)^2 - 2 \right] e^{-((-1)^2 + 0^2 + 2(-1))} = -2e$$

$$f_{xy}(-1, 0) = -2y(-2x - 2)e^{-(x^2+y^2+2x)} = -2(0)(-2(-1) - 2)e^{-((-1)^2 + 0^2 + 2(-1))} = 0e = 0$$

$$\begin{aligned} D &= f_{xx}(-1, 0)f_{yy}(-1, 0) - [f_{xy}(-1, 0)]^2 \\ &= (-2e)(-2e) - (0)^2 = 4e^2 \end{aligned}$$

Since  $D > 0$ , so  $f_{xx}(-1, 0)$  has a maximum point at  $(-1, 0)$ .

### EXAMPLE

$$f(x, y) = 2x^4 + y^2 - x^2 - 2y$$

$$\begin{aligned} f_x(x, y) &= 8x^3 - 2x, & f_y(x, y) &= 2y - 2 \\ f_{xx}(x, y) &= 24x^2 - 2, & f_{yy}(x, y) &= 2, \\ f_{xy}(x, y) &= 0 \end{aligned}$$

For critical points

$$\begin{aligned} f_x(x, y) &= 0, \\ 2x(4x^2 - 1) &= 0, & x &= 0, 1/2, -1/2 \\ f_y(x, y) &= 0, \\ 2y - 2 &= 0, & y &= 1 \end{aligned}$$

Solving above equation we have the critical

$$\text{points } (0, 1), \left(-\frac{1}{2}, 1\right), \left(\frac{1}{2}, 1\right).$$

$$\begin{aligned} f_{xx}(0, 1) &= -2, & f_{yy}(0, 1) &= 2, \\ f_{xy}(0, 1) &= 0 \end{aligned}$$

$$\begin{aligned} D &= f_{xx}(0, 1)f_{yy}(0, 1) - f_{xy}^2(0, 1) \\ &= (-2)(2) - 0 = -4 < 0 \end{aligned}$$

This shows that  $(0, 1)$  is a saddle point.

$$f_{xx}\left(\frac{1}{2}, 1\right) = 4, \quad f_{yy}\left(\frac{1}{2}, 1\right) = 2, \quad f_{xy}\left(\frac{1}{2}, 1\right) = 0$$

$$D = f_{xx}\left(\frac{1}{2}, 1\right)f_{yy}\left(\frac{1}{2}, 1\right) - \left[f_{xy}\left(\frac{1}{2}, 1\right)\right]^2 = (4)(2) - (0)^2 = 8 > 0$$

$$\text{Since } f_{xx}\left(\frac{1}{2}, 1\right) = 4 > 0, \text{ so } f \text{ is minimum at } \left(\frac{1}{2}, 1\right).$$

**Example**

Locate all relative extrema and saddle points of

$$f(x, y) = 4xy - x^4 - y^4.$$

$$f_x(x, y) = 4y - 4x^3, \quad f_y(x, y) = 4x - 4y^3$$

For critical points

$$f_x(x, y) = 0$$

$$4y - 4x^3 = 0 \quad (1)$$

$$y = x^3$$

$$f_y(x, y) = 0$$

$$4x - 4y^3 = 0 \quad (2)$$

$$x = y^3$$

Solving (1) and (2), we have the critical points (0,0), (1, 1), (-1, -1).

$$\text{Now } f_{xx}(x, y) = -12x^2, \quad f_{xx}(0, 0) = 0$$

$$f_{yy}(x, y) = -12y^2, \quad f_{yy}(0, 0) = 0$$

$$f_{xy}(x, y) = 4, \quad f_{xy}(0, 0) = 4$$

$$D = f_{xx}(0, 0) f_{yy}(0, 0) - f_{xy}^2(0, 0)$$

$$= (0)(0) - (4)^2 = -16 < 0$$

This shows that (0,0) is the saddle point.

$$f_{xx}(x, y) = -12x^2, \quad f_{xx}(1, 1) = -12 < 0$$

$$f_{yy}(x, y) = -12y^2, \quad f_{yy}(1, 1) = -12$$

$$f_{xy}(x, y) = 4, \quad f_{xy}(1, 1) = 4$$

$$D = f_{xx}(1, 1) f_{yy}(1, 1) - f_{xy}^2(1, 1)$$

$$= (-12)(-12) - (4)^2 = 128 > 0$$

This shows that f has relative maximum at (1,1).

$$f_{xx}(x, y) = -12x^2, \quad f_{xx}(-1, -1) = -12 < 0$$

$$f_{yy}(x, y) = -12y^2, \quad f_{yy}(-1, -1) = -12$$

$$f_{xy}(x, y) = 4, \quad f_{xy}(-1, -1) = 4$$

$$D = f_{xx}(-1, -1) f_{yy}(-1, -1) - f_{xy}^2(-1, -1)$$

$$= (-12)(-12) - (4)^2 = 128 > 0$$

This shows that f has relative maximum (-1, -1).

**Overview of lecture #15 Book (Calculus by HOWARD ANTON)**

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Graph of f(x,y)	16.9.4	836
The Second Partial Derivative Test	16.9.5	836
Example	5	837

## LECTURE No.16

## EXTREME VALUED THEOREM

**EXTREME VALUED THEOREM**

If the function  $f$  is continuous on the closed interval  $[a, b]$ , then  $f$  has an absolute maximum value and an absolute minimum value on  $[a, b]$

**Remarks**

An absolute extremum of a function on a closed interval must be either a relative extremum or a function value at an end point of the interval. Since a necessary condition for a function to have a relative extremum at a point  $C$  is that  $C$  be a critical point, we may determine the absolute maximum value and the absolute minimum value of a continuous function  $f$  on a closed interval  $[a, b]$  by the following procedure:

1. Find the critical points of  $f$  on  $[a, b]$  and the function values at these critical points.
2. Find the values of  $f(a)$  and  $f(b)$ .
3. The largest and the smallest of the above calculated values are the absolute maximum value and the absolute minimum value respectively

**Example:** Find the absolute extrema of  $f(x) = x^3 + x^2 - x + 1$  on  $[-2, 1/2]$

**Solution:** Since  $f$  is continuous on  $[-2, 1/2]$ , the extreme value theorem is applicable.

For this

$$f'(x) = 3x^2 + 2x - 1$$

This shows that  $f'(x)$  exists for all real numbers, and so the only critical numbers of  $f$  will be the values of  $x$  for which  $f'(x) = 0$ .

Setting  $f'(x) = 0$ , we have

$$(3x - 1)(x + 1) = 0$$

from which we obtain

$$x = -1 \quad \text{and} \quad x = \frac{1}{3}$$

The critical points of  $f$  are  $-1$  and  $\frac{1}{3}$ , and each of these points is in the given closed interval  $(-2, \frac{1}{2})$ . We find the function values at the critical points and at the end points of the interval, which are given below.

$$f(-2) = -1, \quad f(-1) = 2, \quad -$$

$$f\left(\frac{1}{3}\right) = \frac{22}{27}, \quad f\left(\frac{1}{2}\right) = \frac{7}{8}$$

The absolute maximum value of  $f$  on  $(-2, \frac{1}{2})$  is therefore

2, which occurs at  $-1$ , and the absolute min. value of  $f$  on

$(-2, \frac{1}{2})$  is  $-1$ , which occurs at the left end point  $-2$ .

**Example:**

Find the absolute extrema of

$$f(x) = (x - 2)^{2/3} \quad \text{on } [1, 5].$$

Since  $f$  is continuous on  $[1, 5]$ , the extreme-value theorem is applicable.

Differentiating  $f$  with respect to  $x$ , we get

$$f'(x) = \frac{2}{3(x-2)^{1/3}}$$

There is no value of  $x$  for which  $f'(x) = 0$ . However, since  $f'(x)$  does not exist at 2, we conclude that 2 is a critical point of  $f$ ,

so that the absolute extrema occur either at 2 or at one of the end points of the interval. The function values at these points are given below.

$$f(1) = 1, \quad f(2) = 0, \quad f(5) = \sqrt[3]{9}$$

From these values we conclude that the absolute minimum value of  $f$  on  $[1, 5]$  is 0, occurring at 2, and the absolute maximum value of  $f$  on  $[1, 5]$  is  $\sqrt[3]{9}$ , occurring at 5.

**Example:**

Find the absolute extrema of

$$h(x) = x^{2/3} \text{ on } [-2, 3].$$

$$h'(x) = \frac{2}{3} x^{-1/3} = \frac{2}{3x^{1/3}}$$

$h'(x)$  has no zeros but is undefined at  $x = 0$ .

The values of  $h$  at this one critical point and at the endpoints  $x = -2$  and  $x = 3$  are

$$h(0) = 0$$

$$h(-2) = (-2)^{2/3} = 4^{1/3}$$

$$h(3) = (3)^{2/3} = 9^{1/3}.$$

The absolute maximum value is  $9^{1/3}$  assumed at  $x = 3$ ; the absolute minimum is 0, assumed at  $x = 0$ .

**How to Find the Absolute Extrema of a Continuous Function  $f$  of Two Variables on a Closed and Bounded Region  $R$ .**

**Step 1.**

Find the critical points of  $f$  that lie in the interior of  $R$ .

**Step 2.**

Find all boundary points at which the absolute extrema can occur,

**Step 3.**

Evaluate  $f(x, y)$  at the points obtained in the previous steps. The largest of these values is the absolute maximum and the smallest the absolute minimum.

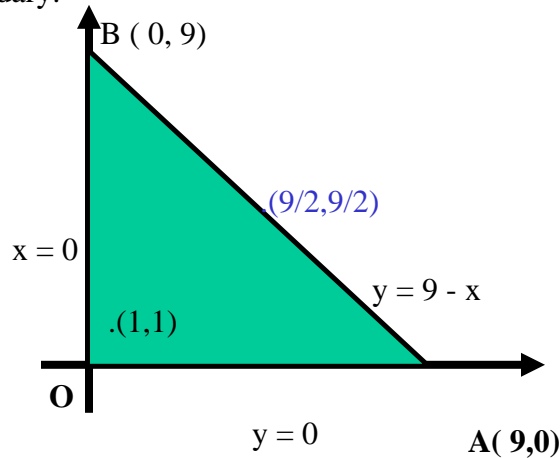
**Example:**

Find the absolute maximum and minimum value of

$$f(x,y) = 2 + 2x + 2y - x^2 - y^2$$

On the triangular plate in the first quadrant bounded by the lines  $x=0, y=0, y=9-x$

Since  $f$  is a differentiable, the only places where  $f$  can assume these values are points inside the triangle having vertices at  $O(0,0)$ ,  $A(9,0)$  and  $B(0,9)$  where  $f_x = f_y = 0$  and points of boundary.

**For interior points:**

We have  $f_x = 2 - 2x = 0$  and  $f_y = 2 - 2y = 0$  yielding the single point  $(1,1)$

For boundary points we take the triangle one side at time :

1. On the segment OA,  $y=0$

$$U(x) = f(x, 0) = 2 + 2x - x^2$$

may be regarded as function of  $x$  defined on the closed interval  $0 \leq x \leq 9$ . Its extreme values may occur at the endpoints  $x=0$  and  $x=9$  which corresponds to points  $(0, 0)$  and  $(9, 0)$  and  $U(x)$  has critical point where

$$U'(x) = 2 - 2x = 0 \text{ Then } x=1$$

On the segment OB,  $x=0$  and

$$V(y) = f(0, y) = 2 + 2y - y^2$$

Using symmetry of function  $f$ , possible points are  $(0,0)$ ,  $(0,9)$  and  $(0,1)$

3. The interior points of AB.

With  $y = 9 - x$ , we have

$$f(x, y) = 2 + 2x + 2(9-x) - x^2 - (9-x)^2$$

$$W(x) = f(x, 9-x) = -61 + 18x - 2x^2$$

$$\text{Setting } w'(x) = 18 - 4x = 0, x = 9/2.$$

At this value of  $x, y = 9 - 9/2$

Therefore we have  $(\frac{9}{2}, \frac{9}{2})$  as a critical point.

$(x, y)$	$(0, 0)$	$(9, 0)$	$(1, 0)$	$\left(\frac{9}{2}, \frac{9}{2}\right)$
$f(x, y)$	2	-61	3	$\frac{-41}{2}$

$(x, y)$	$(0, 9)$	$(0, 1)$	$(1, 1)$
$f(x, y)$	-61	3	4

The absolute maximum is 4 which  $f$  assumes at the point  $(1, 1)$  The absolute minimum is -61 which  $f$  assumes at the points  $(0, 9)$  and  $(9, 0)$

### EXAMPLE

Find the absolute maximum and the absolute minimum values of

$$f(x, y) = 3xy - 6x - 3y + 7$$

on the closed triangular region  $R$  with the vertices  $(0, 0)$ ,  $(3, 0)$  and  $(0, 5)$ .

$$f(x, y) = 3xy - 6x - 3y + 7$$

$$f_x(x, y) = 3y - 6, \quad f_y(x, y) = 3x - 3$$

For critical points

$$f_x(x, y) = 0$$

$$3y - 6 = 0$$

$$y = 2$$

$$f_y(x, y) = 0$$

$$3x - 3 = 0$$

$$x = 1$$

Thus,  $(1, 2)$  is the only critical point in the interior of  $R$ . Next, we want to determine the location of the points on the boundary of  $R$  at which the absolute extrema might occur. The boundary extrema might occur. The boundary each of which we shall treat separately.

#### (i) The line segment between $(0, 0)$ and $(3, 0)$ :

On this line segment we have  $y=0$  so (1) simplifies to a function of the single variable  $x$ ,

$$u(x) = f(x, 0) = -6x + 7, \quad 0 \leq x \leq 3$$

This function has no critical points because  $u'(x) = -6$  is non zero for all  $x$ . Thus, the extreme values of  $u(x)$  occur at the endpoints  $x = 0$  and  $x=3$ , which corresponds to the points  $(0, 0)$  and  $(3, 0)$  on  $R$

#### (ii) The line segment between the $(0, 0)$ and $(0, 5)$

On this line segment we have  $x=0$ , so single variable  $y$ ,

$$v(y) = f(0, y) = -3y + 7, \quad 0 \leq y \leq 5$$

This function has no critical points because  $v'(y) = -3$  is non zero for all  $y$ . Thus, the extreme values of  $v(y)$  occur at the endpoints  $y = 0$  and  $y = 5$  which correspond to the point  $(0, 0)$  and  $(0, 5)$  or  $R$

**(iii) The line segment between  $(3, 0)$  and  $(0, 5)$**

In the  $xy$ -plane, an equation for the line segment

$$y = -\frac{5}{3}x + 5, \quad 0 \leq x \leq 3$$

So (1) simplifies to a function of the single variable  $x$ ,

$$\begin{aligned} w(x) &= f\left(x, -\frac{5}{3}x + 5\right) \\ &= -5x^2 + 14x - 8, \quad 0 \leq x \leq 3 \end{aligned}$$

$$w'(x) = -10x + 14$$

$$w'(x) = 0$$

$$10x + 14 = 0$$

$$x = \frac{7}{5}$$

This shows that  $x = \frac{7}{5}$  is the only critical point of  $w$ . Thus, the extreme values of  $w$

occur either at the critical point  $x = \frac{7}{5}$  or at the endpoints  $x = 0$  and  $x = 3$ . The endpoints correspond to the points  $(0, 5)$  and  $(3, 0)$  of  $R$ , and from (6) the critical point corresponds to  $\left[\frac{7}{5}, \frac{8}{3}\right]$

$(x, y)$	$(0, 0)$	$(3, 0)$	$(0, 5)$	$\left(\frac{7}{5}, \frac{8}{3}\right)$	$(1, 2)$
$f(x, y)$	7	-11	-8	$-\frac{9}{5}$	1

Finally, table list the values of  $f(x, y)$  at the interior critical point and at the points on the boundary where an absolute extremum can occur. From the table we conclude that the absolute maximum value of  $f$  is  $f(0, 0) = 7$  and the absolute minimum values is  $f(3, 0) = -11$ .

**OVER VIEW:**

**Maxima and Minima of functions of two variables.**

**Page # 833**

**Exercise: 16.9**

**Q #26, 27, 28, 29.**

## LECTURE No. 17

## EXAMPLES

**EXAMPLE**

Find the absolute maximum and minimum values of  $f(x,y)=xy-x-3y$  on the closed triangular region  $R$  with vertices  $(0, 0)$ ,  $(0, 4)$ , and  $(5, 0)$ .

$$f(x,y) = xy - x - 3y \quad (1)$$

$$f_x(x, y) = y - 1, \quad f_y(x, y) = x - 3$$

For critical points

$$f_x(x, y) = 0, \quad y - 1 = 0$$

$$y = 1 \quad (2)$$

$$f_y(x, y) = 0, \quad x - 3 = 0$$

$$x = 3 \quad (3)$$

Thus,  $(3, 1)$  is the only critical point in the interior of  $R$ . Next, we want to determine the location of the points on the boundary of  $R$  at which the absolute extrema might occur. The boundary of  $R$  consists of three line segments, each of which we shall treat separately.

**(i) The line segment between  $(0, 0)$  and  $(5, 0)$** 

On this line segment we have  $y = 0$ , so (1) simplifies to a function of the single variable  $x$ ,

$$u(x) = f(x, 0) = -x, \quad 0 \leq x \leq 5 \quad (4)$$

The function has no critical points because the  $u'(x) = -1$  is non zero for all  $x$ . Thus, the extreme values of  $u(x)$  occur at the endpoints  $x=0$  and  $x=5$ , which corresponds to the points  $(0,0)$  and  $(5,0)$  of  $R$ .

**ii) The line segment between  $(0,0)$  and  $(0,4)$** 

On this line segment we have  $x = 0$ , so (1) simplifies to a function of the single variable  $y$ ,

$$v(y) = f(0, y) = -3y, \quad 0 \leq y \leq 4. \quad (5)$$

This function has no critical points because  $v'(y) = -3$  is nonzero for all  $y$ . Thus, the extreme values of  $v(y)$  occur at the endpoints  $y = 0$  and  $y = 4$ , which correspond to the point  $(0,0)$  and  $(0,4)$  of  $R$ .

**iii) The line segment between  $(5,0)$  and  $(0,4)$** 

In the  $xy$ -plan, an equation is

$$y = -\frac{4}{5}x + 4, \quad 0 \leq x \leq 5 \quad (6)$$

so (1) simplifies to a function of the single variable  $x$ ,

$$\begin{aligned}
 w(x) &= f\left(x, -\frac{4}{5}x + 4\right) \\
 &= x\left(-\frac{4}{5}x + 4\right) - x - 3\left(-\frac{4}{5}x + 4\right) \\
 &= -\frac{4}{5}x^2 + \frac{27}{5}x - 12 \\
 w'(x) &= -\frac{8}{5}x + \frac{27}{5} \\
 \text{by } w'(x) &= 0, \text{ we get } x = \frac{27}{8}
 \end{aligned}$$

This shows that  $x = \frac{27}{8}$  is the only critical point of  $w$ . Thus, the extreme values of  $w$  occur either at the critical point  $x = \frac{27}{8}$  or at the endpoints  $x = 0$  and  $x = 5$ . The endpoints correspond to the points  $(0, 4)$  and  $(5, 0)$  of  $R$ , and from (6) the critical point corresponds to  $\left[\frac{27}{8}, \frac{13}{10}\right]$

$(x, y)$	$(0, 0)$	$(5, 0)$	$(0, 4)$	$(27/8, 13/10)$	$(3, 1)$
$f(x, y)$	0	-5	-12	-231/80	-3

Finally, from the table below, we conclude that the absolute maximum value of  $f$  is  $f(0,0) = 0$  and the absolute minimum value is  $f(0,4) = -12$

### **Example**

Find three positive numbers whose sum is 48 and such that their product is as large as possible

Let  $x, y$  and  $z$  be the required numbers, then we have to maximize the product

$$f(x, y) = xy(48 - x - y)$$

Since

$$\begin{aligned}
 f_x &= 48y - 2xy - y^2, & f_y &= 48x - 2xy - x^2 \\
 \text{solving } f_x &= 0, & f_y &= 0
 \end{aligned}$$

$$\text{we get } x=16, y=16, z=16$$

$$\text{Since } x+y+z=48$$

$$f_{xx}(x, y) = -2y, \quad f_{xx}(16, 16) = -32 < 0$$

$$f_{xy}(x, y) = 48 - 2x - 2y, \quad f_{xy}(16, 16) = -16$$

$$f_{yy}(x, y) = -2x, \quad f_{yy}(16, 16) = -32$$

$$\begin{aligned}
 D &= f_{xx}(16, 16)f_{yy}(16, 16) - (f_{xy}(16, 16))^2 \\
 &= (-32)(-32) - (16)^2 = 768 > 0
 \end{aligned}$$

For  $x = 16$ ,  $y = 16$  we have  $z = 16$  since  $x + y + z = 48$   
Thus, the required numbers are 16, 16, 16.

### **Example**

Find three positive numbers whose sum is 27 and such that the sum of their squares is as small as possible

Let  $x, y, z$  be the required numbers, then we have to

$$f(x, y) = x^2 + y^2 + z^2$$

$$= x^2 + y^2 + (27 - x - y)^2$$

Since  $x + y + z = 27$

$$f_x = 4x + 2y - 54, \quad f_y = 2x + 4y - 54,$$

$$f_{xx} = 4, \quad f_{yy} = 4, \quad f_{xy} = 2$$

$$\text{Solving } f_x = 0, \quad f_y = 0$$

$$\text{We get } x = 9, \quad y = 9, \quad z = 9$$

$$\text{Since } x + y + z = 27$$

$$D = f_{xx}(9, 9) f_{yy}(9, 9) - [f_{xy}(9, 9)]^2$$

$$= (4)(4) - 2^2 = 12 > 0$$

This shows that  $f$  is minimum

$x = 9, y = 9, z = 9$ , so the required numbers are 9, 9, 9.

### **Example**

**Find the dimensions of the rectangular box of maximum volume that can be inscribed in a sphere of radius 4.**

**Solution:**

The volume of the parallelepiped with dimensions  $x, y, z$  is

$$V = xyz$$

Since the box is inscribed in the sphere of radius 4, so equation of sphere is

$$x^2 + y^2 + z^2 = 4^2 \text{ from this equation we can write } z = \sqrt{16 - x^2 - y^2} \text{ and putting this value of}$$

“ $z$ ” in above equation we get  $V = xy\sqrt{16 - x^2 - y^2}$ . Now we want to find out the maximum value of this volume, for this we will calculate the extreme values of the function “ $V$ ”. For extreme values we will find out the critical points and for critical points we will solve the equations  $V_x = 0$  and  $V_y = 0$ . Now we have

$$V_x = y\sqrt{16 - x^2 - y^2} + \frac{xy(-2x)}{2\sqrt{16 - x^2 - y^2}}$$

$$\Rightarrow V_x = y \left\{ \frac{-2x^2 - y^2 + 16}{\sqrt{16 - x^2 - y^2}} \right\} \text{ Now } V_x = 0 \Rightarrow y \left\{ \frac{-2x^2 - y^2 + 16}{\sqrt{16 - x^2 - y^2}} \right\} = 0$$

$$\Rightarrow -2x^2 - y^2 + 16 = 0 \Rightarrow 2x^2 + y^2 = 16 \dots\dots\dots(a)$$

Similarly we have

$$V_y = x\sqrt{16-x^2-y^2} + \frac{xy(-2y)}{2\sqrt{16-x^2-y^2}}$$

$$\Rightarrow V_y = x \left\{ \frac{-x^2-2y^2+16}{\sqrt{16-x^2-y^2}} \right\} \text{ Now } V_y = 0 \Rightarrow x \left\{ \frac{-x^2-2y^2+16}{\sqrt{16-x^2-y^2}} \right\} = 0$$

$$\Rightarrow -x^2-2y^2+16=0 \Rightarrow x^2+2y^2=16 \dots\dots\dots(b)$$

$$\text{Solving equations (a) and (b) we get the } x = \frac{4}{\sqrt{3}} \text{ and } y = \frac{4}{\sqrt{3}}$$

$$\text{Now } V_{xx} = \frac{xy(2x^2+3y^2-48)}{(16-x^2-y^2)^{\frac{3}{2}}} \text{ (We obtain this by using quotient rule of differentiation)}$$

$$V_{xx}\left(\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right) = -\frac{16}{\sqrt{3}} < 0$$

$$\text{Also we have to calculate } V_{yy} = \frac{xy(3x^2+2y^2-48)}{(16-x^2-y^2)^{\frac{3}{2}}} \text{ and } V_{yy}\left(\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right) = -\frac{16}{\sqrt{3}} < 0 \text{ Also}$$

$$\text{note that } V_{xy}\left(\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right) = -\frac{8}{\sqrt{3}} \text{ Now as we have the formula for the second order partial}$$

derivative is  $f_{xx} \cdot f_{yy} - (f_{xy})^2$  and putting the values which we calculated above we note

$$\text{that } f_{xx}\left(\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right) \cdot f_{yy}\left(\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right) - (f_{xy}\left(\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right))^2 = +\frac{320}{3} > 0 \text{ Which shows that the}$$

$$\text{function } V \text{ has maximum value when } x = \frac{4}{\sqrt{3}} \text{ and } y = \frac{4}{\sqrt{3}}. \text{ So the dimension of the}$$

$$\text{rectangular box are } x = \frac{4}{\sqrt{3}}, y = \frac{4}{\sqrt{3}} \text{ and } z = \frac{4}{\sqrt{3}}.$$

**Example:** A closed rectangular box with volume of  $16 \text{ ft}^3$  is made from two kinds of materials. The top and bottom are made of material costing Rs. 10 per square foot and the sides from material costing Rs.5 per square foot. Find the dimensions of the box so that the cost of materials is minimized

Let  $x, y, z$ , and  $C$  be the length, width, height, and cost of the box respectively. Then it is clear from that

$$C = 10(xy+xy) + 5(xz+xz) + 5(yz+yz) \dots\dots\dots(1)$$

$$C = 20xy + 10(x+y)z$$

The volume of the box is given by

$$xyz = 16 \dots\dots\dots(2)$$

Putting the value of  $z$  from (2) in

(1), we have

$$C = 20xy + 10(x+y)\frac{16}{xy}$$

$$C = 20xy + \frac{160}{y} + \frac{160}{x}$$

$$C_x = 20y - \frac{160}{x^2}, C_y = 20x - \frac{160}{y^2}$$

For critical points

$$C_x = 0$$

$$20y - \frac{160}{x^2} = 0 \text{ and } C_y = 0$$

$$20x - \frac{160}{y^2} = 0$$

Solving these equations, we have  
 $x = 2, y = 2$ . Thus the critical point  
 is  $(2, 2)$ .

$$C_{xx}(x, y) = \frac{320}{x^3}$$

$$C_{xx}(2, 2) = \frac{320}{8} = 40 > 0$$

$$C_{yy}(x, y) = \frac{320}{y^3}$$

$$C_{yy}(2, 2) = \frac{320}{8} = 40$$

$$C_{xy}(x, y) = 20$$

$$C_{xy}(2, 2) = 20$$

$$C_{xx}(2,2) C_{yy}(2,2) - C_{xy}^2(2,2) = (40)(40) - (20)^2 = 1200 > 0$$

This shows that  $S$  has relative minimum at  $x = 2$  and  $y = 2$ . Putting these values in (2), we have  $z = 4$ , so when its dimensions are  $2 \times 2 \times 4$ .

### **Example**

**Find the dimensions of the rectangular box of maximum volume that can be inscribed in a sphere of radius  $a$ .**

**Solution:**

The volume of the parallelepiped with dimensions  $x, y, z$  is

$$V = xyz$$

Since the box is inscribed in the sphere of radius 4, so equation of sphere is

$$x^2 + y^2 + z^2 = 4^2 \text{ from this equation we can write } z = \sqrt{a^2 - x^2 - y^2} \text{ and putting this value of}$$

“ $z$ ” in above equation we get  $V = xy\sqrt{a^2 - x^2 - y^2}$ . Now we want to find out the maximum value of this volume, for this we will calculate the extreme values of the function “ $V$ ”. For extreme values we will find out the critical points and for critical points we will solve the equations  $V_x = 0$  and  $V_y = 0$ . Now we have

$$V_x = y\sqrt{a^2 - x^2 - y^2} + \frac{xy(-2x)}{2\sqrt{a^2 - x^2 - y^2}}$$

$$\Rightarrow V_x = y \left\{ \frac{-2x^2 - y^2 + a^2}{\sqrt{a^2 - x^2 - y^2}} \right\} \text{ Now } V_x = 0 \Rightarrow y \left\{ \frac{-2x^2 - y^2 + a^2}{\sqrt{a^2 - x^2 - y^2}} \right\} = 0$$

$$\Rightarrow -2x^2 - y^2 + 16 = 0 \Rightarrow 2x^2 + y^2 = a^2 \dots\dots\dots(a)$$

Similarly we have

$$V_y = x\sqrt{a^2 - x^2 - y^2} + \frac{xy(-2y)}{2\sqrt{a^2 - x^2 - y^2}}$$

$$\Rightarrow V_y = x \left\{ \frac{-x^2 - 2y^2 + a^2}{\sqrt{a^2 - x^2 - y^2}} \right\} \text{ Now } V_y = 0 \Rightarrow x \left\{ \frac{-x^2 - 2y^2 + a^2}{\sqrt{a^2 - x^2 - y^2}} \right\} = 0$$

$$\Rightarrow -x^2 - 2y^2 + a^2 = 0 \Rightarrow x^2 + 2y^2 = a^2 \dots\dots\dots(b)$$

Solving equations (a) and (b) we get the  $x = \frac{a}{\sqrt{3}}$  and  $y = \frac{a}{\sqrt{3}}$

$$\text{Now } V_{xx} = \frac{xy(2x^2 + 3y^2 - 3a^2)}{(a^2 - x^2 - y^2)^{\frac{3}{2}}} \text{ (We obtain this by using quotient rule of differentiation)}$$

$$V_{xx}\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right) = -\frac{a^2}{\sqrt{3}} < 0$$

$$\text{Also we have to calculate } V_{yy} = \frac{xy(3x^2 + 2y^2 - 3a^2)}{(a^2 - x^2 - y^2)^{\frac{3}{2}}} \text{ and } V_{yy}\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right) = -\frac{a^2}{\sqrt{3}} \text{ Also note}$$

$$\text{that } V_{xy}\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right) = -\frac{2a}{\sqrt{3}} \text{ Now as we have the formula for the second order partial}$$

derivative is  $f_{xx} \cdot f_{yy} - (f_{xy})^2$  and putting the values which we calculated above we note

$$\text{that } f_{xx}\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right) \cdot f_{yy}\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right) - (f_{xy}\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right))^2 = +\frac{20a^2}{3} > 0 \text{ Which shows that the}$$

function V has maximum value when  $x = \frac{a}{\sqrt{3}}$  and  $y = \frac{a}{\sqrt{3}}$ . So the dimension of the

rectangular box are  $x = \frac{a}{\sqrt{3}}, y = \frac{a}{\sqrt{3}}$  and  $z = \frac{a}{\sqrt{3}}$ .

**Example:** Find the points of the plane  $x + y + z = 5$  in the first octant at which  $f(x, y, z) = xy^2z^2$  has maximum value.

**Solution:** Since we have  $f(x, y, z) = xy^2z^2$  and we are given the plane  $x + y + z = 5$  from this equation we can write  $x = 5 - y - z$ . Thus our function "f" becomes  $f((5 - y - z), y, z) = (5 - y - z)y^2z^2$  Say this function  $u(y, z)$  That is  $u(y, z) = (5 - y - z)y^2z^2$  Now we have to find out extrema of this function. On simplification we get

$$u(y, z) = 5y^2z^2 - y^3z^2 - y^2z^3$$

$$u_y = 10yz^2 - 3y^2z^2 - 2yz^3$$

$$= yz^2(10 - 3y - 2z)$$

$$u_z = 10y^2z - 2y^3 - 3y^2z^2$$

$$= y^2z(10 - 2y - 3z)$$

$$u_y = 0, \quad u_z = 0$$

$$y = 0, \quad z = 0$$

$$10 - 3y - 2z = 0$$

$$10 - 2y - 3z = 0$$

On solving above equations we get  $-10 + 5z = 0 \Rightarrow z = 2$  and  $10 - 3y - 4 = 0 \Rightarrow y = 2$

$$u_{yy} = 10z^2 - 6yz^2 - 2z^3$$

$$u_{zz} = 10y^2 - 2y^3 - 6y^2z$$

$$u_{yz} = 20yz - 6y^2z - 6yz^2$$

at

$$y = 2, \quad z = 2$$

$$u_{yy}(2,2) = 40 - 48 - 16 = -24 < 0$$

$$u_{zz}(2,2) = 40 - 16 - 48 = -24$$

$$u_{yz}(2,2) = 80 - 48 - 48 = -16$$

$$D = u_{yy} u_{zz} - (u_{yz})^2$$

$$= (-24)(-24) - (-16)^2$$

$$= 576 - 256$$

$$= 320 > 0$$

For  $y = 2$  and  $z = 2$

We have  $x = 5 - 2 - 2 = 1$

**Example:** Find all points of the plane  $x+y+z=5$  in the first octant at which  $f(x,y,z)=xy^2z^2$  has a maximum value.

**Solution:**

$$f(x,y,z) = xy^2z^2 = xy^2(5-x-y)^2$$

Since  $x+y+z = 5$

$$f_x = y^2(5-3x-y)(5-x-y),$$

$$f_y = 2xy(5-x-2y)(5-x-y)$$

Solving  $f_x = 0, f_y = 0$ , we get

$$x=1, y=2, z=2 \therefore x+y+z = 5$$

$$f_{xx} = -y^2(5-3x-y) - 3y^2(5-x-5)$$

$$f_{xy} = 2y(5-x-y)(5-3x-y) - y^2(5-3x-y)$$

$$f_{yy} = 2x(5-x-y)(5-x-2y) - 2xy(5-x-2y) - 4xy(5-x-y)$$

$$f_{xx}(1, 2, 2) = -24 < 0$$

$$f_{yy}(1, 2, 2) = -16$$

$$f_{xy}(1, 2, 2) = -8$$

$$f_{xx} f_{yy} - (f_{xy})^2 = (-24)(-16) - (-8)^2$$

$$= 320 > 0$$

Hence “f” has maximum value when  $x = 1$  and  $y = 2$ . Thus the points where the function has maximum value is  $x = 1, y = 2$  and  $z = 2$ .

## LECTURE No. 18

## REVISION OF INTEGRATION

**Example 1:** Consider the following integral  $\int_0^1 (xy + y^2) dx$

Integrating with respect to  $x$ , keeping  $y$  constant, we get

$$\begin{aligned}\int_0^1 (xy + y^2) dx &= y \int_0^1 x dx + y^2 \int_0^1 1 dx \\ &= y \left[ \frac{x^2}{2} \right]_0^1 + y^2 \left[ x \right]_0^1 = y \left( \frac{1}{2} - 0 \right) + y^2 (1 - 0) \\ \int_0^1 (xy + y^2) dx &= \frac{y}{2} + y^2\end{aligned}$$

**Example 2:** Consider the following integral  $\int_0^1 (xy + y^2) dy$

Integrating with respect to  $y$ , keeping  $x$  constant, we get

$$\begin{aligned}\int_0^1 (xy + y^2) dy &= x \int_0^1 y dy + \int_0^1 y^2 dy \\ &= x \left[ \frac{y^2}{2} \right]_0^1 + \left[ \frac{y^3}{3} \right]_0^1 = \frac{x}{2} (1^2 - 0^2) + \frac{1}{3} (1^3 - 0^3) \\ \Rightarrow \int_0^1 (xy + y^2) dy &= \frac{x}{2} + \frac{1}{3}\end{aligned}$$

**Double Integral**

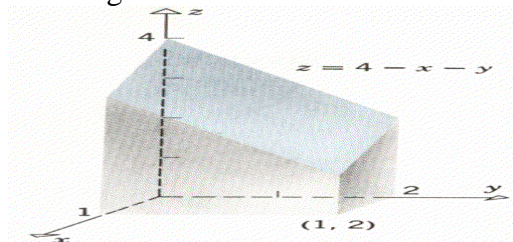
Symbolically, the double integral of two variables  $x$  and  $y$  over the certain region  $R$  of the  $xy$ -plane is denoted by  $\iint_R f(x, y) dx dy$ .

**Example:** Use a double integral to find out the solid bounded above by the plane  $z = 4 - x - y$  and below by the rectangle  $R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 2\}$

**Solution:** We have to find the region “R” out the volume “V” over that is,

$$V = \iint_R (4 - x - y) dA$$

And the solid is shown in the figure below.



$$V = \iint_R (4 - x - y) dA = \int_0^2 \int_0^1 (4 - x - y) dx dy = \int_0^2 \left[ 4x - \frac{x^2}{2} - xy \right]_{x=0}^{x=1} dy$$

$$= \int_0^2 \left( \frac{7}{2} - y \right) dy = \left[ \frac{7}{2}y - \frac{y^2}{2} \right]_0^2 = 5$$

**Example 3:** Evaluate the double integral  $\int_0^1 \int_0^1 (xy + y^2) dx dy$

**Solution :**  $\int_0^1 \int_0^1 (xy + y^2) dx dy = \int_0^1 \left( y \left[ \frac{x^2}{2} \right]_0^1 + y^2 \left[ x \right]_0^1 \right) dy = \int_0^1 \left( \frac{y}{2} + y^2 \right) dy$

$$= \left[ \frac{y^2}{4} + \frac{y^3}{3} \right]_0^1 = \frac{1}{4}(1^2 - 0^2) + \frac{1}{3}(1^3 - 0^3) = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$$

**Example 4:** Evaluate the double integral  $\int_0^1 \int_0^1 (xy + y^2) dy dx$

**Solution:** First we will integrate the given function with respect to  $y$  and our integral becomes

$$\int_0^1 \int_0^1 (xy + y^2) dy dx = \int_0^1 \left( x \left[ \frac{y^2}{2} \right]_0^1 + \left[ \frac{y^3}{3} \right]_0^1 \right) dx$$

$$= \int_0^1 \left( \frac{x}{2}(1^2 - 0^2) + \frac{1}{3}(1^3 - 0^3) \right) dx = \int_0^1 \left( \frac{x}{2} + \frac{1}{3} \right) dx$$

$$\int_0^1 \int_0^1 (xy + y^2) dy dx = \left[ \frac{x^2}{4} + \frac{x}{3} \right]_0^1 = \frac{1}{4}(1^2 - 0^2) + \frac{1}{3}(1 - 0) = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$$

**Remarks:** The example 3 and example 4 show that

$$\int_0^1 \int_0^1 (xy + y^2) dx dy = \int_0^1 \int_0^1 (xy + y^2) dy dx = \frac{7}{12}$$

### Iterated or Repeated Integral

The expression  $\int_c^d \left[ \int_a^b f(x, y) dx \right] dy$  is called iterated or repeated integral. Often the brackets

are omitted and this expression is written as  $\int_c^d \int_a^b f(x, y) dx dy = \int_c^d \left[ \int_a^b f(x, y) dx \right] dy$  in

which  $\int_a^b f(x, y) dx$  yields a function of  $y$ , which is then integrated over the interval  $c \leq y \leq d$ .

Similarly  $\int_a^b \int_c^d f(x, y) dy dx = \int_a^b \left[ \int_c^d f(x, y) dy \right] dx$  in which  $\int_c^d f(x, y) dy$  yields a function of  $x$  which is then integrated over the interval  $a \leq x \leq b$ .

**Example:** Evaluate the integral  $\int_0^1 \int_0^2 (x+3) dy dx$ .

**Solution:** Here we will first integrate with respect to  $y$  and get a function of  $x$  then we will integrate that function with respect to  $x$  to get the required answer. So

$$\begin{aligned} \int_0^1 \int_0^2 (x+3) dy dx &= \int_0^1 (x+3) \left| y \right|_0^2 dx \\ &= \int_0^1 (x+3)(2-0) dx = \int_0^1 2(x+3) dx = 2 \left| \frac{x^2}{2} + 3x \right|_0^1 \\ &= 2 \left( \frac{1}{2}(1^2 - 0^2) + 3(1-0) \right) = 2 \left( \frac{1}{2} + 3 \right) = 7 \end{aligned}$$

Now if we change the order of integration, so we get  $\int_0^2 \int_0^1 (x+3) dx dy$ , then we have

$$\begin{aligned} \int_0^2 \int_0^1 (x+3) dx dy &= \int_0^2 \left( \frac{x^2}{2} + 3x \right) \Big|_0^1 dy \\ &= \int_0^2 \left( \frac{1}{2}(1^2 - 0^2) + 3(1-0) \right) dy = \int_0^2 \left( \frac{1}{2} + 3 \right) dy = \int_0^2 \frac{7}{2} dy = \frac{7}{2} \left| y \right|_0^2 = \frac{7}{2}(2-0) = 7 \end{aligned}$$

Now you note that the values of the integral remain same if we change the order of integration. Actually we have a stronger result which we state as a theorem.

**Theorem:** Let  $R$  be the rectangle defined by the inequalities  $\mathbf{a} < \mathbf{x} < \mathbf{b}$  and  $\mathbf{c} < \mathbf{y} < \mathbf{d}$ . If  $f(x, y)$  is continuous on this rectangle, then

$$\iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx.$$

**Remark:** This powerful theorem enables us to evaluate a double integral over a rectangle by calculating an iterated integral. Moreover, the theorem tells us that the “order of integration in the iterated integral does not matter”.

**Example:** Evaluate the integral  $\int_0^{\ln 2} \int_0^{\ln 3} e^{x+y} dx dy$

**Solution:** First we will integrate the function with respect to  $x$ . Note that we can write  $e^{x+y}$  as  $e^x \cdot e^y$

$$\text{So we have, } \int_0^{\ln 2} e^y \left| e^x \right|_0^{\ln 3} dy = \int_0^{\ln 2} e^y (e^{\ln 3} - e^0) dy = \int_0^{\ln 2} e^y (3-1) dy = 2 \int_0^{\ln 2} e^y dy$$

Here we use the fact that “e” and “ln” are inverse function of each other. So we have

$$e^{\ln 3} = 3. \text{ Thus we get } \int_0^{\ln 2} e^y \left| e^x \right|_0^{\ln 3} dy = 2 \int_0^{\ln 2} e^y dy = 2 \left| e^y \right|_0^{\ln 2} = 2(2-1) = 2$$

**Example:** Evaluate the integral  $\int_0^{\ln 3} \int_0^{\ln 2} e^{x+y} dy dx$

**Solution:** First we will integrate the function with respect to y . Note that we can write

$$e^{x+y} \text{ as } e^x \cdot e^y \text{ So we have, } \int_0^{\ln 3} e^x \left| e^y \right|_0^{\ln 2} dx = \int_0^{\ln 3} e^x (e^{\ln 2} - e^0) dx = \int_0^{\ln 3} e^x (2-1) dx = \int_0^{\ln 3} e^x dx$$

Since “e” and “ln” are inverse function of each other. So we have  $e^{\ln 2} = 2$ .

$$\text{Thus we get } \int_0^{\ln 3} e^x \left| e^y \right|_0^{\ln 2} dx = \int_0^{\ln 3} e^x dx = \left| e^x \right|_0^{\ln 3} = (e^{\ln 3} - e^0) = (3-1) = 2$$

**Note** that in both cases our integral has the same value.

#### Overview:

**Double integrals**

**Page # 854-857**

**Exercise Set 17.1 (page 857): 1, 3, 5, 7, 9, 11, 13, 15, 17, 19**

## LECTURE No. 19

## USE OF INTEGRALS

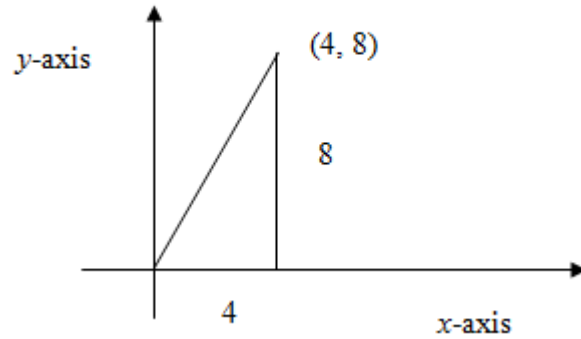
**Area as anti-derivatives**

$$\int_0^4 2x \, dx = 2 \left| \frac{x^2}{2} \right|_0^4 = \left| x^2 \right|_0^4$$

$$= 4^2 - 0^2 = 16$$

$$\text{Area of triangle} = \frac{1}{2} \text{ base} \times \text{altitude}$$

$$= \frac{1}{2} \times 4 \times 8 = 16$$

**Volume as anti-derivatives**

$$\text{Volume} = \int_0^2 \int_0^3 5 \, dy \, dx = \int_0^2 5 \left| y \right|_0^3 \, dx = \int_0^2 5(3-0) \, dx = \int_0^2 15 \, dx = 15 \left| x \right|_0^2 = 15(2-0) = 30$$

Geometrically,  $0 \leq x \leq 2$ ,  $0 \leq y \leq 3$ ,  $0 \leq z \leq 5$

$$\text{Volume} = 2 \times 3 \times 5 = 30$$

The following results are analogous to the result of the definite integrals of a function of single variable.

**THEOREM**

- 1)  $\iint_R c f(x, y) \, dx \, dy = c \iint_R f(x, y) \, dx \, dy$  where  $c$  is a constant.
- 2)  $\iint_R [f(x, y) + g(x, y)] \, dx \, dy = \iint_R f(x, y) \, dx \, dy + \iint_R g(x, y) \, dx \, dy$
- 3)  $\iint_R [f(x, y) - g(x, y)] \, dx \, dy = \iint_R f(x, y) \, dx \, dy - \iint_R g(x, y) \, dx \, dy$

**Example:** Use double integral to find the volume under the surface  $z = 3x^3 + 3x^2y$  and the rectangle  $\{(x, y) : 1 \leq x \leq 3, 0 \leq y \leq 2\}$ .

**Solution:**

$$\text{Volume} = \int_0^2 \int_1^3 (3x^3 + 3x^2y) \, dx \, dy = \int_0^2 \left[ \frac{3x^4}{4} + \frac{3x^3y}{3} \right]_1^3 \, dy = \int_0^2 \left[ \frac{3}{4}(3^4 - 1^4) + y(3^3 - 1^3) \right]_1^3 \, dy$$

$$= \int_0^2 \left[ \frac{3}{4}(81 - 1) + y(27 - 1) \right] \, dy = \int_0^2 (60 + 26y) \, dy = \left[ 60y + \frac{26y^2}{2} \right]_0^2$$

$$= \left[ 60(2 - 0) + 13(2^2 - 0^2) \right]_0^2 = |120 + 52| = 172$$

**Example:** Use double integral to find the volume of solid in the first octant enclosed by the surface  $z = x^2$  and the planes  $x = 2$ ,  $y = 0$ ,  $y = 3$  and  $z = 0$ .

**Solution :** Volume  $= \int_0^2 \int_0^3 x^2 dy dx = \int_0^2 x^2 \left| y \right|_0^3 dx = \int_0^2 x^2 (3-0) dx$

$$= 3 \int_0^2 x^2 dx = 3 \left| \frac{x^3}{3} \right|_0^2 = (2^3 - 0^3) = 8$$

### SOME RESULTS:

- 1)  $\iint_R f(x, y) dA \geq 0$  if  $f(x, y) \geq 0$  on  $R$ .
- 2)  $\iint_R f(x, y) dA \geq \iint_R c g(x, y) dA$  if  $f(x, y) \geq g(x, y)$

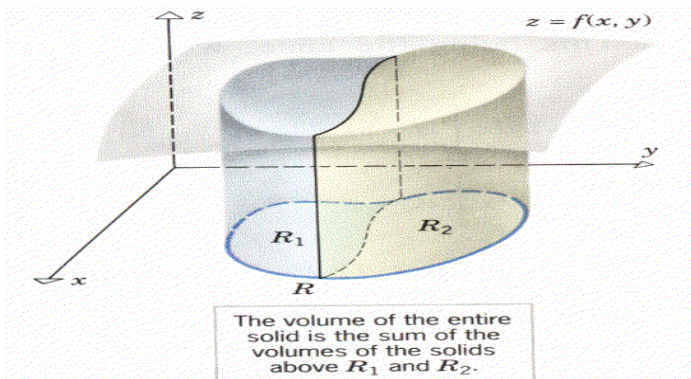
If  $f(x, y)$  is nonnegative on a region  $R$ , then subdividing  $R$  into two regions  $R_1$  and  $R_2$  has the effect to subdividing the solid between  $R$  and  $z = f(x, y)$  into two solids, the sum of whose volumes is the volume of the entire solid.

$$\iint_R f(x, y) dA = \iint_{R_1} f(x, y) dA + \iint_{R_2} f(x, y) dA$$

The volume of the solid  $S$  can also be obtained, using cross sections perpendicular to  $y$ -axis.

$$Vol(S) = \int_c^d A(y) dy \quad \text{-----(1)}$$

Where  $A(y)$  represents the area of the cross section perpendicular to  $y$ -axis, taken at the point  $y$ .



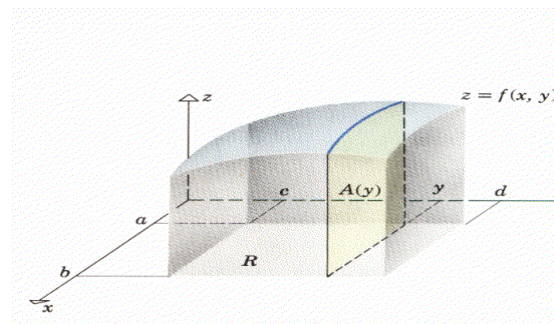
### How to compute cross sectional area

For each fixed  $y$  in the interval  $c \leq y \leq d$ , the function  $f(x, y)$  is a function of  $x$  alone, and  $A(y)$  may be viewed as the area under the graph of this function along the interval  $a < x < b$ ,

Thus 
$$A(y) = \int_a^b f(x, y) dx$$

Substituting this expression in (1), we get

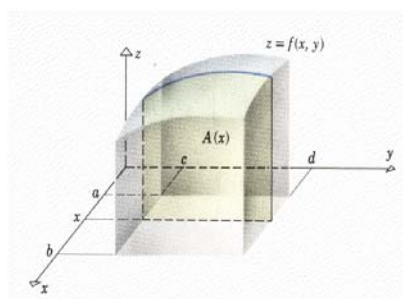
$$\text{Vol}(S) = \int_c^d \left[ \int_a^b f(x, y) dx \right] dy = \int_c^d \int_a^b f(x, y) dx dy \quad \text{-----}(2)$$



Similarly, the volume of the solid  $S$  can also be obtained, perpendicular to  $x$ -axis.

$$\text{Vol}(S) = \int_a^b A(x) dx \quad \text{-----}(3)$$

Where  $A(x)$  represents the area of the cross section perpendicular to  $x$ -axis, taken at the point  $x$ .



For each fixed  $x$  in the interval  $a \leq x \leq b$ , the function  $f(x, y)$  is a function of  $y$  alone,

and  $A(x)$  is given by  $A(x) = \int_c^d f(x, y) dy$ . Substituting this expression in (3), we get

$$\text{Vol}(S) = \int_a^b \left[ \int_c^d f(x, y) dy \right] dx = \int_a^b \int_c^d f(x, y) dy dx \quad \text{-----}(4)$$

By equations (2) and (4),  $\iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx$

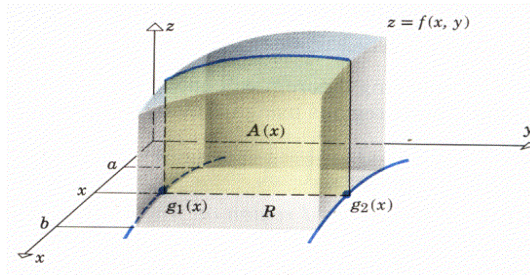
### Double integral for non-rectangular region

**Type I region** is bounded the left and right by the vertical lines  $x = a$  and  $x = b$  and is bounded below and above by continuous curves  $y = g_1(x)$  and  $y = g_2(x)$ , where

$$g_1(x) \leq g_2(x) \quad \text{for } a \leq x \leq b$$

If  $R$  is a type I region on which  $f(x, y)$  is continuous, then

$$\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$



By the method of cross section, the volume of  $S$  is also given by

$$\text{Vol}(S) = \int_a^b A(x) dx \quad \text{----- (5)}$$

where  $A(x)$  is the area of the cross section at the fixed point  $x$  and this cross section area extends from  $g_1(x)$  to  $g_2(x)$  in the  $y$  – direction,

So 
$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y) dy$$

Using it in equation (5), we get 
$$\text{Vol}(S) = \int_a^b \left[ \int_{g_1(x)}^{g_2(x)} f(x, y) dy \right] dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

The volume of  $S$  is also given by 
$$\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

**Type II region** is bounded below and above by horizontal lines  $y = c$  and  $y = d$  and is bounded in the left and right by continuous curves  $x = h_1(y)$  and  $x = h_2(y)$  satisfying

$$h_1(y) \leq h_2(y) \quad \text{for } c \leq y \leq d.$$

If  $R$  is a type II region on which  $f(x, y)$  is continuous,

then 
$$\iint_R f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

Similarly, the partial definite integral  $\int_c^d f(x, y) dy$  with respect to  $y$  is evaluated by

holding  $x$  fixed and integrating with respect to  $y$ . The integral of the form

$$\int_c^d f(x, y) dy \quad \text{produces a function of } x.$$

## LECTURE No.20

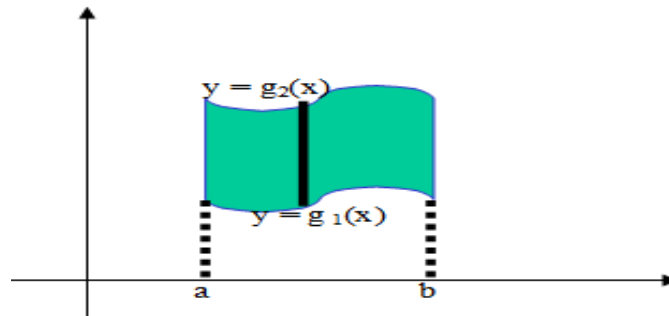
## DOUBLE INTEGRAL FOR NON-RECTANGULAR REGION

**Double Integral for Non-rectangular Region**

**Type I region** is bounded the left and right by vertical lines  $x = a$  and  $x = b$  and is bounded below and above by curves

$$y = g_1(x) \text{ and } y = g_2(x), \quad \text{where } g_1(x) \leq g_2(x) \text{ for } a \leq x \leq b$$

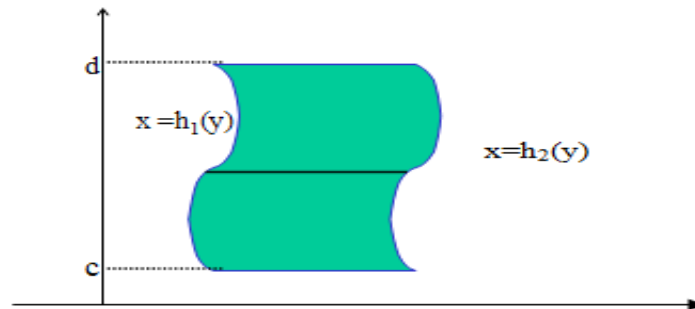
$$\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$



**Type II region** is bounded below and above by the horizontal lines  $y = c$  and  $y = d$  and is bounded on the left and right by the continuous curves  $x = h_1(y)$  and  $x = h_2(y)$

satisfying  $h_1(y) \leq h_2(y)$  for  $c \leq y \leq d$

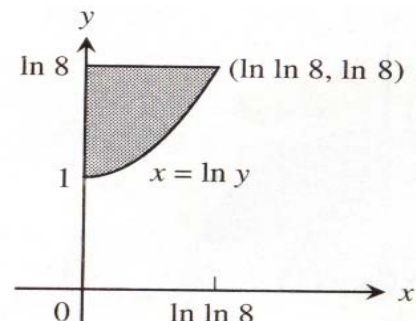
$$\iint_R f(x, y) dx dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$



**Write double integral of the function  $f(x,y)$  on the region whose sketch is given**

$$\int_1^{\ln 8} \int_0^{\ln y} f(x, y) dx dy$$

$$\int_0^{\ln(\ln 8)} \int_{e^x}^{\ln 8} f(x, y) dy dx$$



**Solution:**

$$I = \int_1^{\ln 8} \int_0^{\ln y} f(x, y) dx dy$$

Here, limits of  $x$  are  $0 \leq x \leq \ln y$  ----- (1)

limits of  $y$  are  $1 \leq y \leq \ln 8$  ----- (2)

Take logarithm of each sides of (2),  $0 \leq \ln y \leq \ln(\ln 8)$  ---- (3)  $\because \ln 1 = 0$

Compare (1) and (3),  $0 \leq x \leq \ln y \leq \ln(\ln 8)$  ----- (4)

From (4),  $0 \leq x \leq \ln(\ln 8)$  and  $x \leq \ln y \leq \ln(\ln 8)$

$$e^x \leq y \leq \ln 8$$

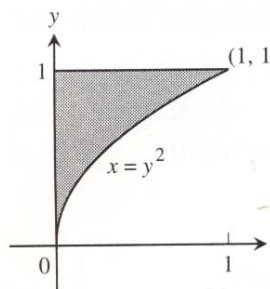
So, 
$$I = \int_0^{\ln(\ln 8)} \int_{e^x}^{\ln 8} f(x, y) dy dx$$

**Write double integral of the function  $f(x, y)$  on the region whose sketch is given**

$$\int_0^1 \int_0^{y^2} f(x, y) dx dy$$

$$\int_0^1 \int_{\sqrt{x}}^1 f(x, y) dy dx$$

Solution : 
$$I = \int_0^1 \int_0^{y^2} f(x, y) dx dy$$



Here, limits of  $x$  are  $0 \leq x \leq y^2 \Rightarrow 0 \leq \sqrt{x} \leq y$  ---- (1) by taking square root

limits of  $y$  are  $0 \leq y \leq 1$  ----- (2)

Compare (1) and (2),  $0 \leq \sqrt{x} \leq y \leq 1$  ----- (3)

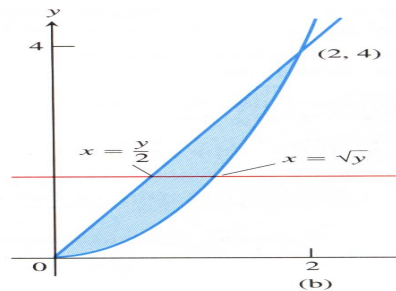
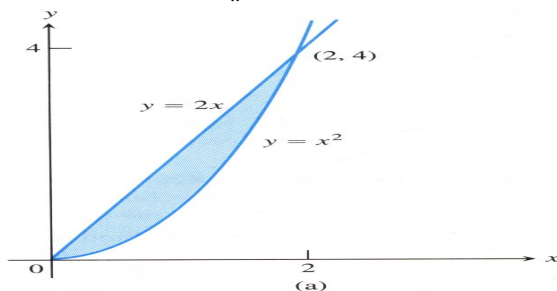
From (3),  $0 \leq \sqrt{x} \leq 1$  and  $\sqrt{x} \leq y \leq 1$

$$0^2 \leq (\sqrt{x})^2 \leq 1^2 \quad \text{and} \quad \sqrt{x} \leq y \leq 1$$

$$0 \leq x \leq 1 \quad \text{and} \quad \sqrt{x} \leq y \leq 1$$

**EXAMPLE:** Draw the region and evaluate an equivalent integral with the order of

integration reversed. 
$$\int_0^2 \int_{x^2}^{2x} (4x + 2) dy dx$$



The region of integration is given by the inequalities  $x^2 \leq y \leq 2x$  and  $0 \leq x \leq 2$ .

$$\text{limits of } y \text{ are } x^2 \leq y \leq 2x \quad \text{-----(1)}$$

$$\text{limits of } x \text{ are } 0 \leq x \leq 2 \quad \text{or} \quad 0 \leq 2x \leq 4 \quad \text{-----(2)}$$

$$\text{Compare (1) and (2),} \quad 0 \leq x^2 \leq y \leq 2x \leq 4 \quad \text{-----(3)}$$

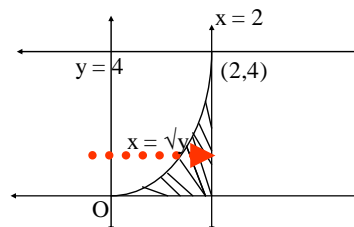
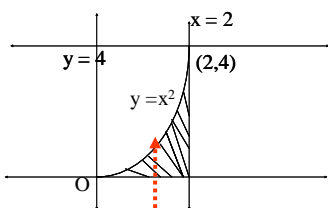
$$\text{From (3), } 0 \leq y \leq 4 \quad \text{and} \quad x^2 \leq y, \quad y \leq 2x$$

$$x \leq \sqrt{y}, \quad \frac{y}{2} \leq x \quad \text{or} \quad \frac{y}{2} \leq x \leq \sqrt{y}$$

$$\begin{aligned} \int_0^2 \int_{x^2}^{2x} (4x+2) dy dx &= \int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} (4x+2) dx dy = \int_0^4 \left[ 4\frac{x^2}{2} + 2x \right]_{\frac{y}{2}}^{\sqrt{y}} dy \\ &= \int_0^4 \left( 2y - \frac{y^2}{2} \right) + (2\sqrt{y} - y) dy = \int_0^4 \left( 2\sqrt{y} + y - \frac{y^2}{2} \right) dy \\ &= \left[ 2\frac{y^{\frac{3}{2}}}{\frac{3}{2}} + \frac{y^2}{2} - \frac{1}{2} \frac{y^3}{3} \right]_0^4 = \left[ \frac{4}{3} y^{\frac{3}{2}} + \frac{y^2}{2} - \frac{y^3}{6} \right]_0^4 = \frac{4}{3}(8) + \frac{1}{2}(16) - \frac{1}{6}(64) = 8 \end{aligned}$$

### EXAMPLE

Evaluate  $I = \int_0^4 \int_{\sqrt{y}}^2 y \cos x^5 dx dy$ . The integral is over the region  $0 \leq y \leq 4$ ,  $x = \sqrt{y}$  and  $x = 2$ .



Solution : For reversing the limits of the integral,

$$\text{limits of } x \text{ are } \sqrt{y} \leq x \leq 2 \quad \text{or} \quad y \leq x^2 \leq 4 \quad \text{-----(1)}$$

$$\text{limits of } y \text{ are } 0 \leq y \leq 4 \quad \text{-----(2)}$$

$$\text{By (1) and (2), } 0 \leq y \leq x^2 \leq 4 \quad \text{-----(3)}$$

$$\text{By (3),} \quad 0 \leq y \leq x^2 \quad \text{and} \quad 0 \leq x^2 \leq 4 \\ 0 \leq x \leq 2$$

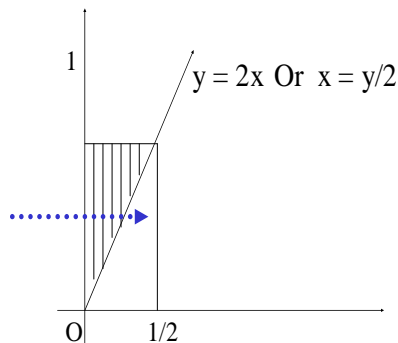
$$\begin{aligned} I &= \int_0^2 \int_0^{x^2} y \cos x^5 dy dx = \int_0^2 \left[ \frac{y^2}{2} \right]_0^{x^2} \cos x^5 dx = \int_0^2 \frac{1}{2} \left( (x^2)^2 - 0^2 \right) \cos x^5 dx \\ &= \frac{1}{2} \int_0^2 x^4 \cos x^5 dx = \frac{1}{2 \times 5} \int_0^2 \cos x^5 (5x^4) dx = \frac{1}{10} \left[ \sin x^5 \right]_0^2 = \frac{1}{10} \sin 32 \end{aligned}$$

**EXAMPLE:** Evaluate  $I = \int_0^{\frac{1}{2}} \int_{2x}^1 e^{y^2} dy dx$ .

**Solution:** The integral cannot be evaluated in the given order, since  $e^{y^2}$  has no anti-derivative. So we shall change the order of integration. The region  $R$  in which integration is performed is given by  $0 \leq x \leq \frac{1}{2}$ ,  $y = 2x$  and  $y = 1$ .

The region is enclosed by  $x = 0$ ,  $x = \frac{y}{2}$  and  $0 \leq y \leq 1$

$$\begin{aligned} I &= \int_0^1 \int_0^{\frac{y}{2}} e^{y^2} dx dy = \int_0^1 e^{y^2} \left| x \right|_0^{\frac{y}{2}} dy \\ &= \int_0^1 \frac{y}{2} e^{y^2} dy = \frac{1}{2 \times 2} \int_0^1 2y e^{y^2} dy \\ &= \frac{1}{4} \left| e^{y^2} \right|_0^1 = \frac{1}{4} (e - 1) \end{aligned}$$



**EXAMPLE:** Evaluate  $I = \int_1^3 \int_0^{\ln x} x dy dx$ .

**Solution:** The region  $R$  in which integration is performed is given by

Limits of  $x$  are  $1 \leq x \leq 3$

$$\ln 1 \leq \ln x \leq \ln 3$$

$$0 \leq \ln x \leq \ln 3 \quad \text{--- (1)}$$

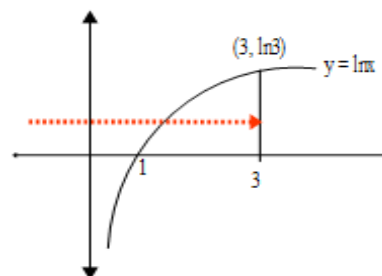
Limits of  $y$  are  $0 \leq y \leq \ln x$  --- (2)

By (1) and (2),  $0 \leq y \leq \ln x \leq \ln 3$  --- (3)

From (3),  $y \leq \ln x \leq \ln 3$  and  $0 \leq y \leq \ln 3$

$$e^y \leq x \leq 3 \quad \text{and} \quad 0 \leq y \leq \ln 3$$

$$\begin{aligned} I &= \int_0^{\ln 3} \int_{e^y}^3 x dx dy = \int_0^{\ln 3} \left| \frac{x^2}{2} \right|_{e^y}^3 dy = \frac{1}{2} \int_0^{\ln 3} (9 - e^{2y}) dy \\ &= \frac{1}{2} \left| 9y - \frac{e^{2y}}{2} \right|_0^{\ln 3} = \frac{1}{2} \left( 9 \ln 3 - \frac{1}{2} (e^{\ln 3^2} - e^0) \right) = \frac{1}{2} (9 \ln 3 - 4) = \frac{9}{2} \ln 3 - 2 \end{aligned}$$



## Over view of Lecture # 20

Book Calculus by Howard Anton

Chapter # 17 Article # 17.2

Page (858-863) Exercise set 17.2 (21, 22, 23, 25, 27, 35, 37, 38 )

## LECTURE No. 21

## EXAMPLES

## Example

$$\int_0^1 \int_{4x}^4 e^{-y^2} dy dx$$

Reversing the order of integration

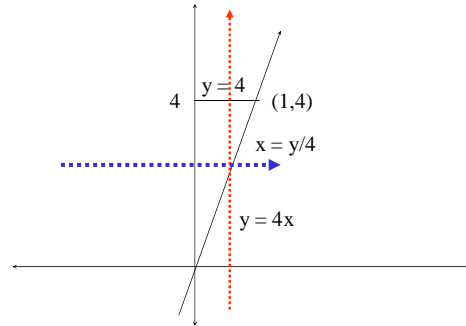
$$\int_0^4 \int_0^{y/4} e^{-y^2} dx dy$$

$$= \int_0^4 \left[ x e^{-y^2} \right]_0^{y/4} dy = \int_0^4 \frac{y}{4} e^{-y^2} dy$$

$$= \frac{-1}{8} \int_0^4 e^{-y^2} (-2y) dy$$

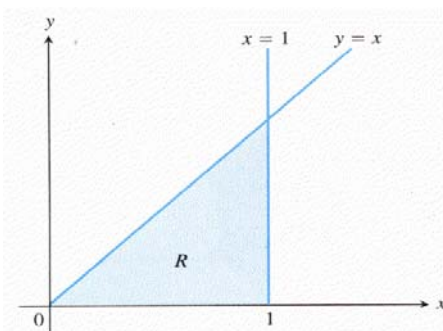
$$= -\frac{1}{8} \left[ e^{-y^2} \right]_0^4 = -\frac{1}{8} [e^{-16} - e^0]$$

$$= \frac{1}{8} \left( 1 - \frac{1}{e^{16}} \right)$$



**Example :** Calculate  $\iint_R \frac{\sin x}{x} dA$ , where R is the triangle in xy-plane bounded by the x-axis, the line  $y = x$  and the line  $x=1$ .

$$\begin{aligned} \text{Solution : } \int_0^1 \left( \int_0^x \frac{\sin x}{x} dy \right) dx &= \int_0^1 \left( \frac{\sin x}{x} \left[ y \right]_0^x \right) dx = \int_0^1 \left( \frac{\sin x}{x} (x-0) \right) dx \\ &= \int_0^1 \sin x dx = -\left[ \cos x \right]_0^1 = -(\cos 1 - \cos 0) \approx 0.46 \end{aligned}$$



**Example**

$$\int_0^2 \int_{y/2}^1 e^{x^2} dx dy$$

Since there is no elementary antiderivative of  $e^{x^2}$ , the integral

$$\int_0^2 \int_{y/2}^1 e^{x^2} dx dy$$

cannot be evaluated by performing the x-integration first.

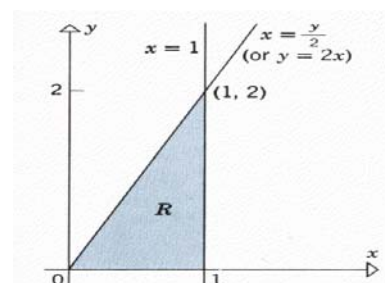
To evaluate this integral, we express it as an equivalent iterated integral with the order of integration reversed. For the inside integration,  $y$  is fixed and  $x$  varies from the line  $x = y/2$  to the line  $x = 1$ . For the outside integration,  $y$  varies from 0 to 2, so the given iterated integral is equal to a double integral over the triangular region  $R$ .

To reverse the order of integration, we treat  $R$  as a type I region, which enables us to write the given integral as

$$\int_0^2 \int_{y/2}^1 e^{x^2} dx dy$$

By changing the order of integration we get,

$$\begin{aligned} \int_0^2 \int_{y/2}^1 e^{x^2} dx dy &= \int_0^1 \int_0^{2x} e^{x^2} dy dx \\ &= \int_0^1 [e^{x^2} y]_{y=0}^{2x} dx \\ &= \int_0^1 2x e^{x^2} dx \\ &= e^{x^2} \Big|_0^1 = e - 1 \end{aligned}$$

**Example**

Use a double integral to find the volume of the solid that is bounded above by the plane  $z = 4 - x - y$  and below by the rectangle  $R = \{(x,y): 0 \leq x \leq 1, 0 \leq y \leq 2\}$

$$\begin{aligned} V &= \iint_R (4 - x - y) dA = \int_0^2 \int_0^1 (4 - x - y) dx dy \\ &= \int_0^2 \left[ 4x - \frac{x^2}{2} - xy \right]_0^1 dy = \int_0^2 \left( \frac{7}{2} - y \right) dy = \left[ \frac{7}{2}y - \frac{y^2}{2} \right]_0^2 = 5 \end{aligned}$$

**Example**

Use a double integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane  $z=4-4x-2y$ . The tetrahedron is bounded above by the plane.

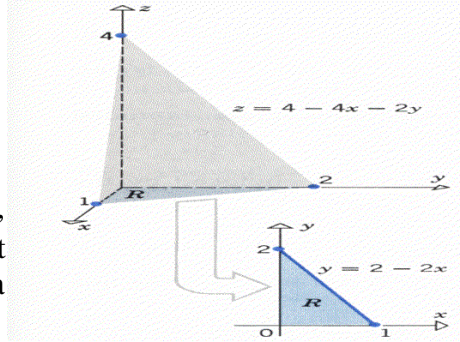
$$z=4-4x-2y \text{ -----(1)}$$

and below by the triangular region R

Thus, the volume is given by

$$V = \iint_R (4 - 4x - 2y) \, dA$$

The region R is bounded by the x-axis, the y-axis, and the line  $y = 2 - 2x$  [set  $z = 0$  in (1)], so that treating R as a type I region yields.



$$\begin{aligned} V &= \iint_R (4 - 4x - 2y) \, dA \\ &= \int_0^1 \int_0^{2-2x} (4 - 4x - 2y) \, dy \, dx \\ &= \int_0^1 [4y - 4xy - y^2]_{y=0}^{2-2x} \, dx \\ &= \int_0^1 (4 - 8x + 4x^2) \, dx \\ &= \frac{4}{3} \end{aligned}$$

**Example :** Find the volume of the solid bounded by the cylinder  $x^2 + y^2 = 4$  and the planes  $y + z = 4$  and  $z = 0$ .

The solid is bounded above by the plane  $z = 4 - y$  and below by the region R within the circle  $x^2 + y^2 = 4$ .

The volume is given by

$$V = \iint_R (4 - y) \, dA$$

Treating R as a type I region we obtain

$$\begin{aligned} V &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - y) \, dy \, dx \\ &= \int_{-2}^2 \left[ 4y - \frac{1}{2} y^2 \right]_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \, dx \\ &= \int_{-2}^2 8\sqrt{4-x^2} \, dx \end{aligned}$$

$$\begin{aligned}
 &= 8 \left| \frac{x \sqrt{4-x^2}}{2} + \frac{4}{2} \sin^{-1} \frac{x}{2} \right|_{-2}^2 \\
 &= 8 | 2 \sin^{-1}(1) - 2 \sin^{-1}(-1) | \\
 &= 8 [2(\frac{\pi}{2}) + 2(\frac{\pi}{2})] \\
 &= 8(2\pi) = 16\pi
 \end{aligned}$$

**Example**

Use double integral to find the volume of the solid that is bounded above by the paraboloid  $z=9x^2 + y^2$ , below by the plane  $z=0$  and laterally by the planes

$$x = 0, \quad y = 0, \quad x = 3, \quad y = 2$$

$$\begin{aligned}
 \text{Volume} &= \int_0^3 \int_0^2 (9x^2 + y^2) dy dx \\
 &= \int_0^3 \left[ 9x^2 y + \frac{y^3}{3} \right]_0^2 dx \\
 &= \int_0^3 \left( 18x^2 + \frac{8}{3} \right) dx \\
 &= \left[ 6x^3 + \frac{8}{3} x \right]_0^3 \\
 &= 6(27) + 8 \\
 &= 170
 \end{aligned}$$

## LECTURE No. 22

## EXAMPLES

**Example :** Evaluate  $\iint_R xy \, dA$ , where  $R$  is the region bounded by the trapezium with the vertices  $(1, 3)$ ,  $(5, 3)$ ,  $(2, 1)$  and  $(4, 1)$ .

**Solution:** Slope of AD =  $\frac{3-1}{1-2} = \frac{2}{-1} = -2$

Since Slope =  $m = \frac{y_2 - y_1}{x_2 - x_1}$

Equation of line AD

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -2(x - 2)$$

$$y - 1 = -2x + 4$$

$$y - 5 = -2x$$

$$x = -\frac{y-5}{2}$$

$$\text{Slope of BC} = \frac{3-1}{5-4} = \frac{2}{1} = 2$$

Equation of line BC:  $y - y_2 = m(x - x_2)$

$$y - 1 = 2(x - 4) \quad \text{or} \quad y + 7 = 2x \quad \Rightarrow \quad x = \frac{y+7}{2}$$

Limits of  $x$  are from  $x = -\frac{y-5}{2}$  to  $x = \frac{y+7}{2}$ .

Limits of  $y$  are from 1 to 3.

$$\begin{aligned} \iint_R xy \, dA &= \int_1^3 \int_{-\frac{y-5}{2}}^{\frac{y+7}{2}} xy \, dx \, dy = \int_1^3 y \left[ \frac{x^2}{2} \right]_{-\frac{y-5}{2}}^{\frac{y+7}{2}} dy = \frac{1}{2} \int_1^3 y \left[ \left( \frac{y+7}{2} \right)^2 - \left( -\frac{y-5}{2} \right)^2 \right] dy \\ &= \frac{1}{2} \int_1^3 y \left\{ \frac{y^2 + 49 + 14y}{4} - \frac{y^2 + 25 - 10y}{4} \right\} dy = \frac{1}{2} \int_1^3 y \left\{ \frac{24 + 24y}{4} \right\} dy \\ &= \int_1^3 (3y + 3y^2) dy = \left[ 3\frac{y^2}{2} + 3\frac{y^3}{3} \right]_1^3 = 12 + 26 = 38 \end{aligned}$$

**EXAMPLE:** Use double integral to find the volume of the wedge cut from the cylinder  $4x^2 + y^2 = 9$  by the plane  $z = 0$  and  $z = y + 3$

**Solution:** Since we can write  $4x^2 + y^2 = 9$  as  $\frac{x^2}{(3/2)^2} + \frac{y^2}{9} = 1$  This is eq of ellipse.

Now the Lower and upper limits for  $x$  are  $x = \frac{-\sqrt{9-y^2}}{2}$  and  $x = \frac{\sqrt{9-y^2}}{2}$

And upper and lower limits for  $y$  are  $-3$  and  $3$  respectively. So the required volume is given by

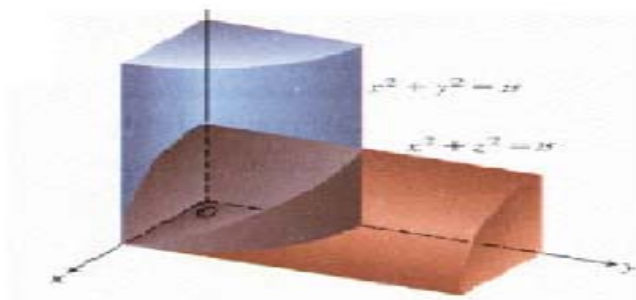
$$\begin{aligned}
 \int_{-3}^3 \int_{\frac{-\sqrt{9-y^2}}{2}}^{\frac{\sqrt{9-y^2}}{2}} (y+3) dx dy &= \int_{-3}^3 \left\{ y \left| x \right|_{\frac{-\sqrt{9-y^2}}{2}}^{\frac{\sqrt{9-y^2}}{2}} + 3 \left| x \right|_{\frac{-\sqrt{9-y^2}}{2}}^{\frac{\sqrt{9-y^2}}{2}} \right\} dy \\
 &= \int_{-3}^3 \left\{ y \left| \frac{\sqrt{9-y^2}}{2} - \left( -\frac{\sqrt{9-y^2}}{2} \right) \right| + 3 \left| \frac{\sqrt{9-y^2}}{2} - \left( -\frac{\sqrt{9-y^2}}{2} \right) \right| \right\} dy \\
 &= \int_{-3}^3 \left\{ y\sqrt{9-y^2} + 3\sqrt{9-y^2} \right\} dy = \frac{-1}{2} \int_{-3}^3 \sqrt{9-y^2} (-2y) dy + 3 \int_{-3}^3 \sqrt{9-y^2} dy \\
 &= \frac{-1}{2} \times \frac{2}{3} \left[ (9-y^2)^{\frac{3}{2}} \right]_{-3}^3 + 3 \left\{ \left| \frac{y}{2} \sqrt{9-y^2} \right|_{-3}^3 + \left| \frac{9}{2} \sin^{-1} \left( \frac{y}{3} \right) \right|_{-3}^3 \right\} \\
 &= 0 + 3 \left\{ \left| \frac{3}{2} \sqrt{0} + \frac{3}{2} \sqrt{0} \right| + \frac{9}{2} \left| \sin^{-1}(1) - \sin^{-1}(-1) \right| \right\} = \frac{27\pi}{2}
 \end{aligned}$$

**EXAMPLE:** Use double integral to find the volume of solid common to the cylinders  $x^2 + y^2 = 25$  and  $x^2 + z^2 = 25$ .

**Solution:** From  $x^2 + y^2 = 25$  or  $y^2 = 25 - x^2$  or  $y = \pm \sqrt{25 - x^2}$

Radius of cylinder is 5, so limits of  $x$  is from 0 to 5.

From  $x^2 + z^2 = 25 \Rightarrow z = \sqrt{25 - x^2}$  Only +ve value taken in first octant.



Volume =  $8 \times$  Area of cylinder in first Octant

$$\begin{aligned}
 &= 8 \iint_R z dA = 8 \int_0^5 \int_0^{\sqrt{25-x^2}} \sqrt{25-x^2} dy dx = 8 \int_0^5 \sqrt{25-x^2} \left| y \right|_0^{\sqrt{25-x^2}} dx \\
 &= 8 \int_0^5 \sqrt{25-x^2} (\sqrt{25-x^2} - 0) dx = 8 \int_0^5 (25-x^2) dx
 \end{aligned}$$

$$= 8 \left| 25x - \frac{x^3}{3} \right|_0^5 = 8 \left[ 25(5-0) - \frac{1}{3}(5^3 - 0^3) \right] = \frac{2000}{3}$$

### **AREA CALCULATED AS A DOUBLE INTEGRAL**

$$V = \iint_R 1 \, dA = \iint_R dA \quad (1)$$

However, the solid has congruent cross sections taken parallel to the  $xy$ -plan so that

$$V = \text{area of base} \times \text{height} = \text{area of } R \times 1 = \text{area of } R$$

Combining this with (1) yields the area formula

$$\text{Area of } R = \iint_R dA \quad (2)$$

**EXAMPLE:** Use a double integral to find the area of the region  $R$  enclosed between the parabola  $y = \frac{x^2}{2}$  and the line  $y = 2x$ .

**Solution:** The required area is between the parabola  $y = \frac{x^2}{2}$  -----(1)

and the line  $y = 2x$  -----(2)

By (1) and (2),  $\frac{x^2}{2} = 2x \Rightarrow x^2 = 4x \Rightarrow x = 0, x = 4$

$$\begin{aligned} \text{Area of } R &= \iint_R dA = \int_0^4 \int_{y=\frac{x^2}{2}}^{y=2x} dy \, dx = \int_0^4 \left| y \right|_{y=\frac{x^2}{2}}^{y=2x} dx = \int_0^4 \left\{ 2x - \frac{x^2}{2} \right\} dx \\ &= \left| 2 \frac{x^2}{2} - \frac{1}{2} \times \frac{x^3}{3} \right|_0^4 = (4^2 - 0^2) - \frac{1}{6}(4^3 - 0^3) = \frac{16}{3} \end{aligned}$$

**EXAMPLE:** Find the area of the region  $R$  enclosed by the parabola  $y = x^2$  and the line  $y = x + 2$ .

**Solution:** The required area is between the parabola  $y = x^2$  -----(1)

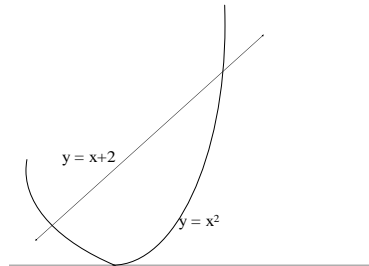
and the line  $y = x + 2$  -----(2)

By (1) and (2),  $x^2 = x + 2$

$$x^2 - x - 2 = 0$$

$$x = 2, x = -1$$

$$\int_{-1}^2 \int_{y=x^2}^{y=x+2} dy \, dx = \int_{-1}^2 \left| y \right|_{y=x^2}^{y=x+2} dx$$



$$= \int_{-1}^2 \left\{ (x+2) - (x^2) \right\} dx = \left| \frac{x^2}{2} \right|_{-1}^2 + \left| 2x \right|_{-1}^2 - \left| \frac{x^3}{3} \right|_{-1}^2 = \frac{1}{2}(4-1) + 2(2+1) - \frac{1}{3}(8+1) = \frac{9}{2}$$

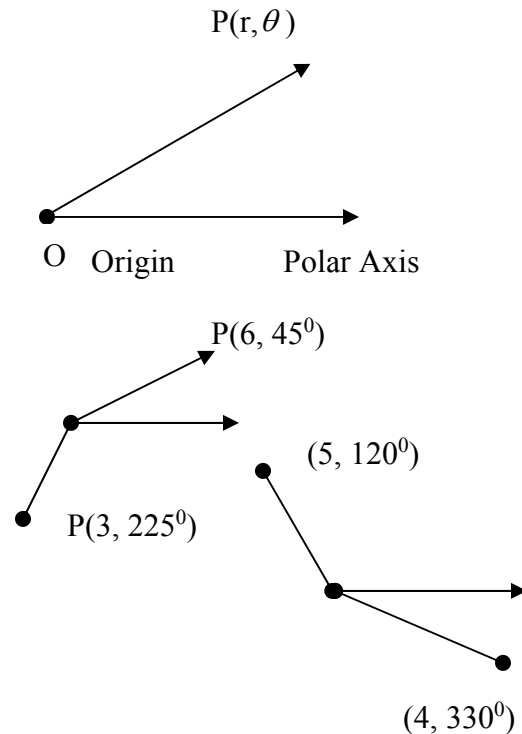
## LECTURE No. 23

## POLAR COORDINATE SYSTEMS

**POLAR COORDINATE SYSTEMS**

To form a polar coordinate system in a plane, we pick a fixed point  $O$ , called the origin or pole, and using the origin as an endpoint we construct a ray, called the polar axis. After selecting a unit of measurement, we may associate with any point  $P$  in the plane a pair of polar coordinates  $(r, \theta)$ , where  $r$  is the distance from  $P$  to the origin and  $\theta$  measures the angle from the polar axis to the line segment  $OP$ .

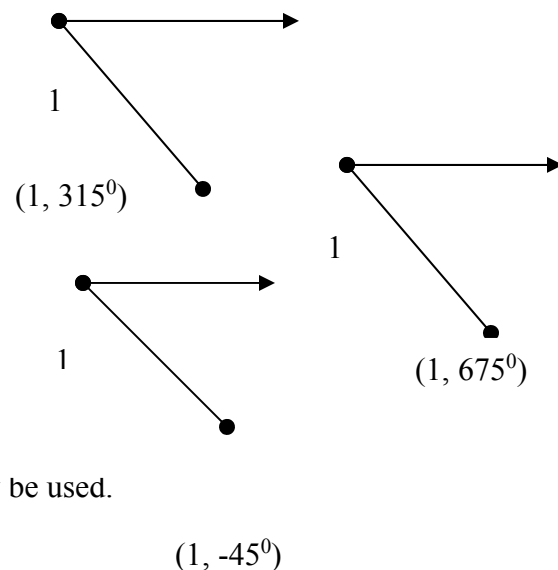
The number  $r$  is called the radial distance of  $P$  and  $\theta$  is called a polar angle of  $P$ . In the points  $(6, 45^\circ)$ ,  $(3, 225^\circ)$ ,  $(5, 120^\circ)$ , and  $(4, 330^\circ)$  are plotted in polar coordinate systems.

**THE POLAR COORDINATES OF A POINT ARE NOT UNIQUE.**

For example, the polar coordinates  $(1, 315^\circ)$ ,  $(1, -45^\circ)$ , and  $(1, 675^\circ)$  all represent the same point

In general, if a point  $P$  has polar coordinate  $(r, \theta)$ , then for any integer  $n=0,1,2,3,\dots$   $(r, \theta + n \cdot 360^\circ)$  and  $(r, \theta + n \cdot 360^\circ)$  are also polar coordinates of  $P$

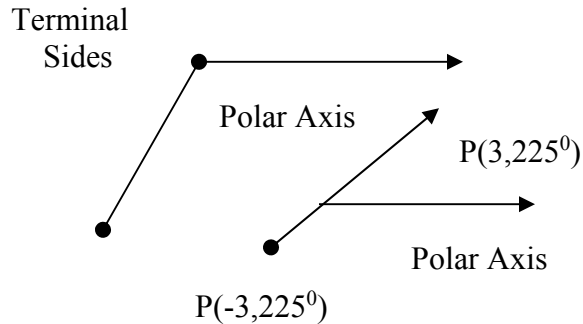
In the case where  $P$  is the origin, the line segment  $OP$  reduces to a point, since  $r = 0$ . Because there is no clearly defined polar angle in this case, we will agree that an arbitrary Polar angle  $\theta$  may be used. Thus, for every  $\theta$  may be used. Thus, for every  $\theta$ , the point  $(0, \theta)$  is the origin.

**NEGATIVE VALUES OF R**

When we start graphing curves in polar coordinates, it will be desirable to allow negative values for  $r$ . This will require a special definition. For motivation, consider the point  $P$

with polar coordinates  $(3, 225^\circ)$  We can reach this point by rotating the polar axis  $225^\circ$  and then moving forward from the origin 3 units along the terminal side of the angle. On the other hand, we can also reach the point P by rotating the polar axis  $45^\circ$  and then moving backward 3 units from the origin along the extension of the terminal side of the angle

This suggests that the point  $(3, 225^\circ)$  might also be denoted by  $(-3, 45^\circ)$  with the minus sign serving to indicate that the point is on the extension of angle's terminal side rather than on terminal side itself.

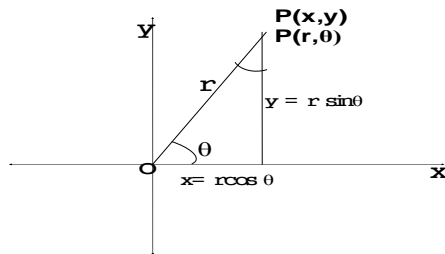


Since the terminal side of the angle  $\theta + 180^\circ$  is the extension of the terminal side other angle  $\theta$ , We shall define.

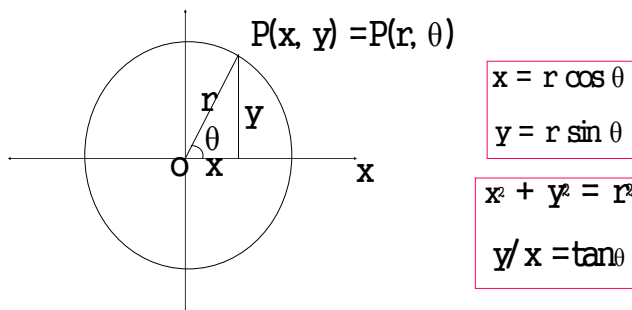
$(-r, \theta)$  and  $(r, \theta + 180^\circ)$  to be polar coordinates for the same point .

With  $r = 3$  and  $\theta = 45^\circ$  in (2) it follows that  $(-3, 45^\circ)$  and  $(3, 225^\circ)$  represent the same point.

### RELATION BETWEEN POLAR AND RECTANGULAR COORDINATES



### CONVERSION FORMULA FROM POLAR TO CARTESIAN COORDINATES AND VICE VERSA



#### **Example :**

Find the rectangular coordinates of the point P whose polar coordinates are  $(6, 135^\circ)$

**Solution:**

Substituting the polar coordinates,

$r = 6$  and  $\theta = 135^\circ$  in  $x = r \cos \theta$  and  $y = r \sin \theta$  yields

$$x = 6 \cos 135^\circ = 6 \left(-\frac{\sqrt{2}}{2}\right) = -3\sqrt{2}$$

$$y = 6 \sin 135^\circ = 6 \left(\frac{\sqrt{2}}{2}\right) = 3\sqrt{2}$$

Thus, the rectangular coordinates of the point P are  $(-3\sqrt{2}, 3\sqrt{2})$

**Example:** Find polar coordinates of the point P whose rectangular coordinates are  $(-2, 2\sqrt{3})$

**Solution:** We will find polar coordinates  $(r, \theta)$  of P such that  $r > 0$  and  $0 \leq \theta \leq 2\pi$ .

$$r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4 + 12} = \sqrt{16} = 4$$

$$\tan \theta = \frac{y}{x} = \frac{2\sqrt{3}}{-2} = -\sqrt{3} \Rightarrow \theta = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$$

From this we have  $(-2, 2\sqrt{3})$  lies in the second quadrant of P. All other Polar coordinates of P have the form

$$\left(4, \frac{2\pi}{3} + 2n\pi\right) \text{ or } \left(-4, \frac{5\pi}{3} + 2n\pi\right), \text{ where } n \text{ is integer}$$

**LINES IN POLAR COORDINATES**

A line perpendicular to the x-axis and passing through the point with xy coordinates with  $(a, 0)$  has the equation  $x = a$ . To express this equation in polar coordinates we substitute  $x = r \cos \theta \Rightarrow a = r \cos \theta$  -----(1)

This result makes sense geometrically since each point P  $(r, \theta)$  on this line will yield the value  $a$  for  $r \cos \theta$ .

A line parallel to the x-axis that meets the y-axis at the point with xy-coordinates  $(0, b)$  has the equation  $y = b$ .

Substituting  $y = r \sin \theta$  yields.

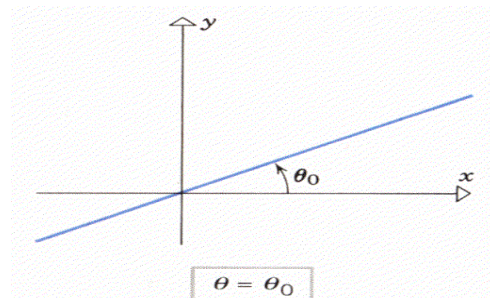
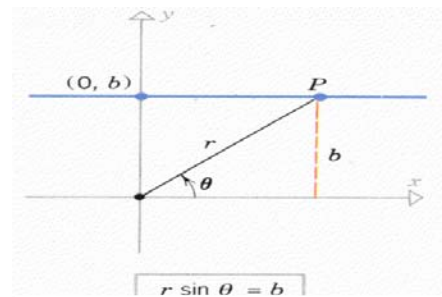
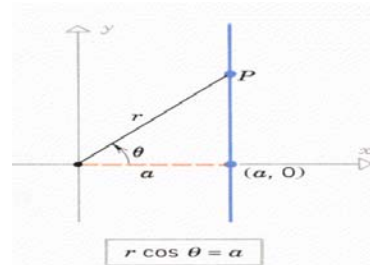
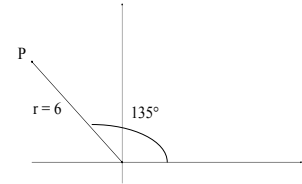
$$r \sin \theta = b \quad (2)$$

as the polar equation of this line. This makes sense geometrically since each point P  $(r, \theta)$  on this line will yield the value  $b$  for  $r \sin \theta$

For Any constant  $\theta_0$ , the equation

$$\theta = \theta_0 \quad (3)$$

is satisfied by the coordinates of all points of the form P  $(r, \theta_0)$ , regardless of the value of  $r$ . Thus, the equation represents the line through the origin making an angle of  $\theta_0$  (radians) with the polar axis.



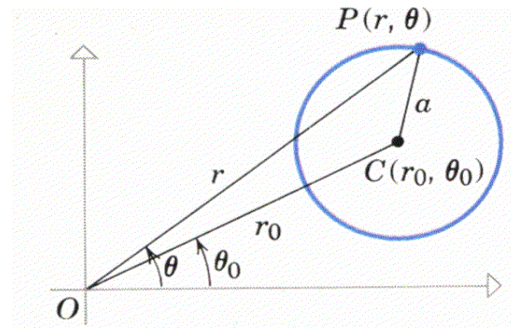
By substitution  $x = r \cos\theta$  and  $y = r \sin\theta$  in the equation  $Ax + By + C = 0$ . We obtain the general polar form of the line,

$$r (A \cos\theta + B \sin\theta) + C = 0$$

### CIRCLES IN POLAR COORDINATES

Let us try to find the polar equation of a circle whose radius is  $a$  and whose center has polar coordinates  $(r_0, \theta_0)$ . If we let  $P(r, \theta)$  be an arbitrary point on the circle, and if we apply the law of cosines to the triangle  $OCP$  we obtain

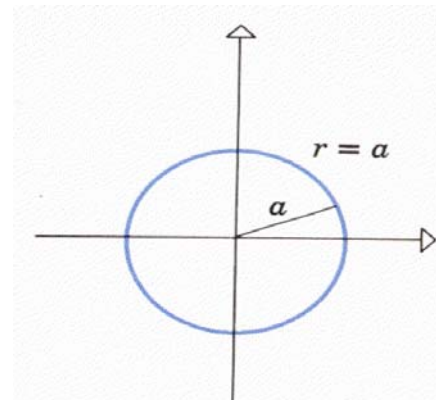
$$r^2 - 2rr_0 \cos(\theta - \theta_0) + r_0^2 = a^2 \quad (1)$$



### SOME SPECIAL CASES OF EQUATION OF CIRCLE IN POLAR COORDINATES

A circle of radius  $a$ , centered at the origin, has an especially simple polar equation. If we let  $r_0 = 0$  in (1), we obtain  $r^2 = a^2$  or, since  $a \geq 0$ ,  $r = a$

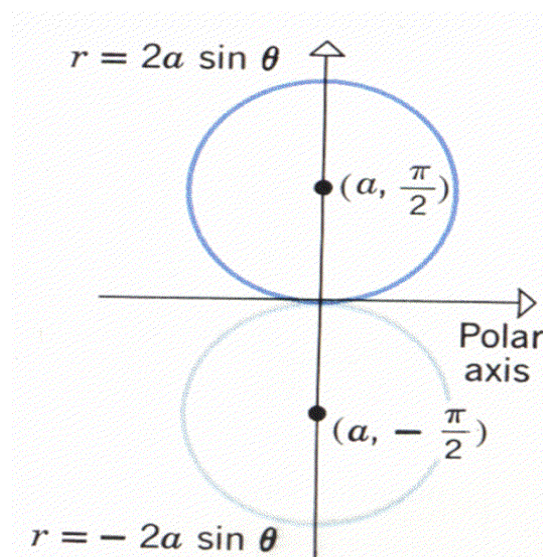
This equation makes sense geometrically since the circle of radius  $a$ , centered at the origin, consists of all points  $P(r, \theta)$  for which  $r = a$ , regardless of the value of  $\theta$



If a circle of radius  $a$  has its center on the  $y$ -axis and passes through the origin, then the polar coordinates of the center are either

$(a, 0)$  or  $(a, \pi)$

depending on whether the center is to the right or left of the origin



## LECTURE No. 24

## SKETCHING

**Draw graph of the curve having the equation  $r = \sin \theta$** 

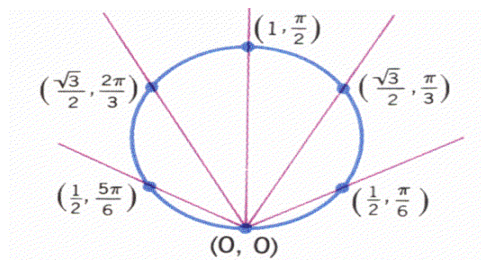
By substituting values for  $\theta$  at increments of  $\frac{\pi}{6}$  ( $30^\circ$ ) and calculating  $r$ , we can construct

The following table:

$\theta$ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$
$r = \sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$

$\theta$ (radians)	$\pi$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	$2\pi$
$r = \sin \theta$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0

Note that there are 13 pairs listed in Table, but only 6 points plotted in This is because the pairs from  $\theta = \pi$  on yield duplicates of the preceding points. For example,  $(-\frac{1}{2}, 7\pi/6)$  and  $(1/2, \pi/6)$  represent the same point. The points appear to lie on a circle.



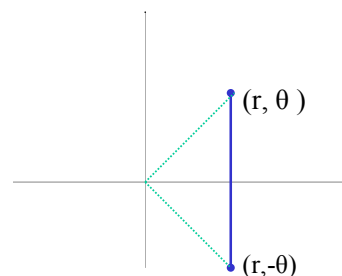
This is indeed the case may be seen by expressing the given equation in terms of  $x$  and  $y$ . We first multiply the given equation through by  $r$  to obtain  $r^2 = r \sin \theta$  which can be rewritten as

$$x^2 + y^2 = y \quad \text{or} \quad x^2 + y^2 - y = 0$$

or on completing the square  $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$ . This is a circle of radius  $\frac{1}{2}$  centered at the point  $(0, \frac{1}{2})$  in the  $xy$ -plane.

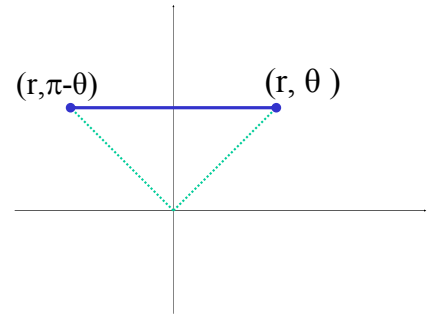
**Sketching of Curves in Polar Coordinates****1. SYMMETRY****(i) Symmetry about the Initial Line**

If the equation of a curve remains unchanged when  $(r, \theta)$  is replaced by either  $(r, -\theta)$  in its equation, then the curve is symmetric about initial line.

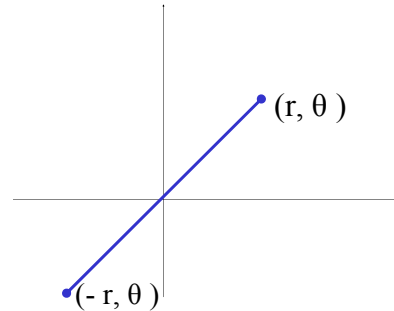


**(ii) Symmetry about the y-axis**

If when  $(r, \theta)$  is replaced by either  $(r, \pi - \theta)$  in The equation of a curve and an equivalent equation is obtained, then the curve is symmetric about the line perpendicular to the initial i.e, the y-axis

**(ii) Symmetry about the Pole**

If the equation of a curve remains unchanged when either  $(-r, \theta)$  or is substituted for  $(r, \theta)$  in its equation, then the curve is symmetric about the pole. In such a case, the center of the curve.

**2. Position of the Pole Relative to the Curve**

See whether the pole on the curve by putting  $r = 0$  in the equation of the curve and solving for  $\theta$ .

**3. Table Of Values**

Construct a sufficiently complete table of values. This can be of great help in sketching the graph of a curve.

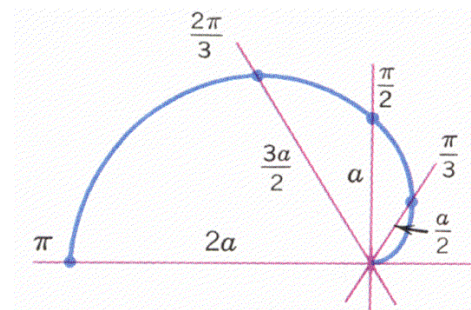
**II Position Of The Pole Relative To The Curve.**

When  $r = 0$ ,  $\theta = 0$ . Hence the curve passes through the pole.

**III. Table of Values**

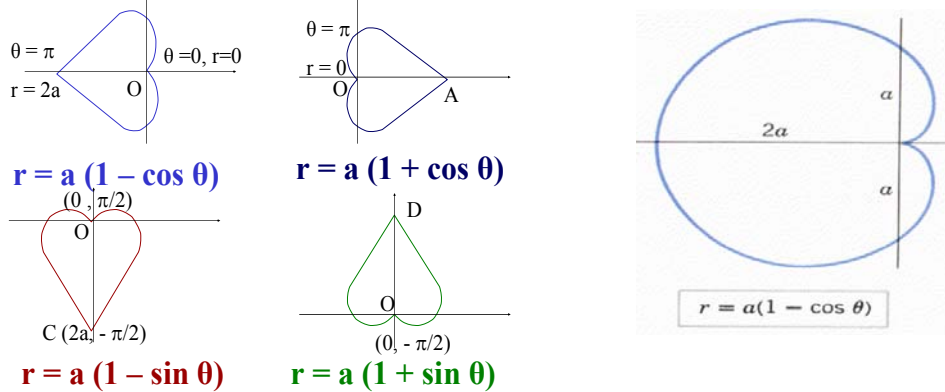
$\theta$	0	$\pi/3$	$\pi/2$	$2\pi/3$	$\pi$
$r = a(1 - \cos\theta)$	0	$a/2$	$a$	$3a/2$	$2a$

As  $\theta$  varies from 0 to  $\pi$ ,  $\cos \theta$  decreases steadily from 1 to  $-1$ , and  $1 - \cos \theta$  increases steadily from 0 to 2. Thus, as  $\theta$  varies from 0 to  $\pi$ , the value of  $r = a(1 - \cos \theta)$  will increase steadily from an initial value of  $r = 0$  to a final value of  $r = 2a$ .



On reflecting the curve in about the x-axis, we obtain the curve.

### CARDIOIDS

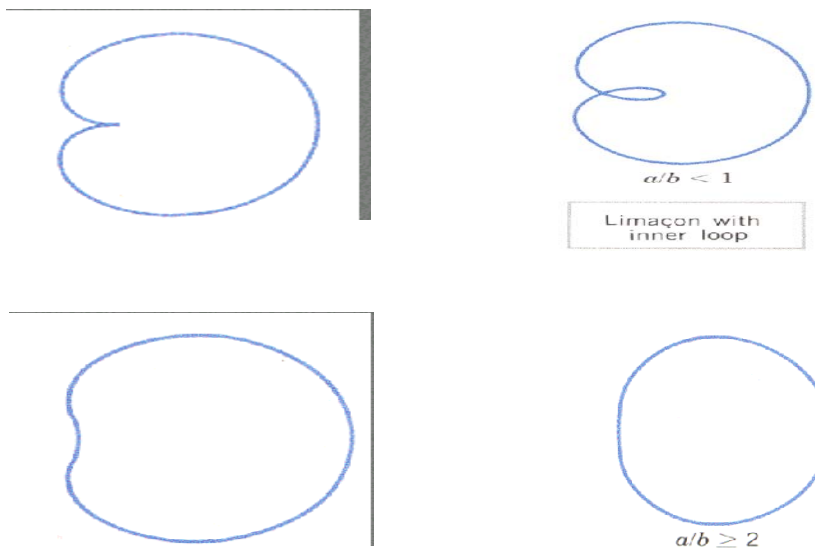


### CARDIOIDS AND LIMAÇONS

$$r = a + b \sin \theta, \quad r = a - b \sin \theta$$

$$r = a + b \cos \theta, \quad r = a - b \cos \theta$$

The equations of above form produce polar curves called limaçons. Because of the heart-shaped appearance of the curve in the case  $a = b$ , limaçons of this type are called cardioids. The position of the limaçon relative to the polar axis depends on whether  $\sin \theta$  or  $\cos \theta$  appears in the equation and whether the  $+$  or  $-$  occurs.



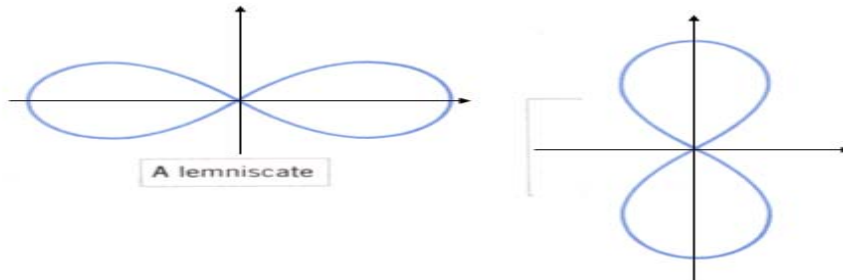
### LEMNISCATE

If  $a > 0$ , then equation of the form

$$r^2 = a^2 \cos 2\theta, \quad r^2 = -a^2 \cos 2\theta$$

$$r^2 = a^2 \sin 2\theta, \quad r^2 = -a^2 \sin 2\theta$$

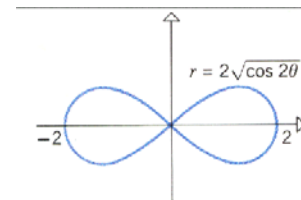
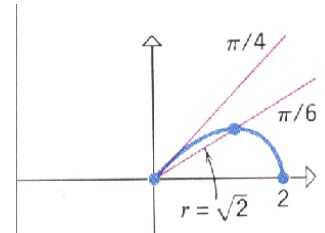
represent propeller-shaped curves, called lemniscates (from the Greek word “lemniscos” for a looped ribbon resembling the Fig 8. The lemniscates are centered at the origin, but the position relative to the polar axis depends on the sign preceding the  $a^2$  and whether  $\sin 2\theta$  or  $\cos 2\theta$  appears in the equation.



### Example

$$r^2 = 4 \cos 2\theta$$

The equation represents a lemniscate. The graph is **symmetric** about the **x-axis** and the **y-axis**. Therefore, we can obtain each graph by first sketching the portion of the graph in the range  $0 \leq \theta < \pi/2$  and then reflecting that portion about the x- and y-axes. The curve passes through the origin when  $\theta = \pi/4$ , so the line  $\theta = \pi/4$  is tangent to the curve at the origin. As  $\theta$  varies from 0 to  $\pi/4$ , the value of  $\cos 2\theta$  decreases steadily from 1 to 0, so that  $r$  decreases steadily from 2 to 0. For  $\theta$  in the range  $\pi/4 < \theta < \pi/2$ , the quantity  $\cos 2\theta$  is negative, so there are no real values of  $r$  satisfying first equation. Thus, there are no points on the graph for such  $\theta$ . The entire graph is obtained by reflecting the curve about the x-axis and then reflecting the resulting curve about the y-axis.

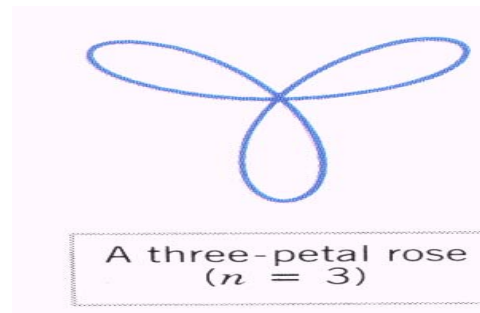
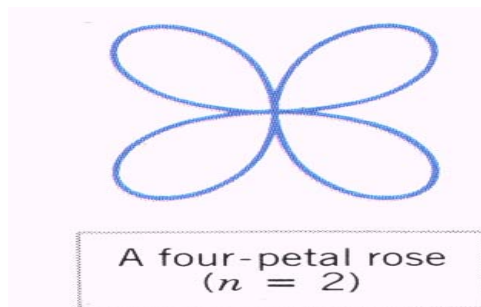


### ROSE CURVES

Equations of the form

$$r = a \sin n\theta \text{ and } r = a \cos n\theta$$

represent flower-shaped curves called roses. The rose has  **$n$  equally spaced petals or loops** if  **$n$  is odd** and  **$2n$  equally spaced petals** if  **$n$  is even**



The orientation of the rose relative to the polar axis depends on the sign of the constant  $a$  and whether  $\sin\theta$  or  $\cos\theta$  appears in the equation.

### **SPIRAL**

A curve that “winds around the origin” infinitely many times in such a way that  $r$  increases (or decreases) steadily as  $\theta$  increases is called a spiral. The most common example is the spiral of Archimedes, which has an equation of the form.

$$r = a\theta \quad (\theta \geq 0) \quad \text{or} \quad r = -a\theta \quad (\theta \leq 0)$$

In these equations,  $\theta$  is in radians and  $a$  is positive.

### **EXAMPLE**

Sketch the curve  $r = \theta$  ( $\theta \geq 0$ ) in polar coordinates.

This is an equation of spiral with  $a = 1$ ; thus, it represents an Archimedean spiral.

Since  $r = 0$  when  $\theta = 0$ , the origin is on the curve and the polar axis is tangent to the spiral.

A reasonably accurate sketch may be obtained by plotting the intersections of the spiral with the  $x$  and  $y$  axes and noting that  $r$  increases steadily as  $\theta$  increases. The intersections with the  $x$ -axis occur when

$$\theta = 0, \pi, 2\pi, 3\pi, \dots$$

at which points  $r$  has the values

$$r = 0, \pi, 2\pi, 3\pi, \dots$$

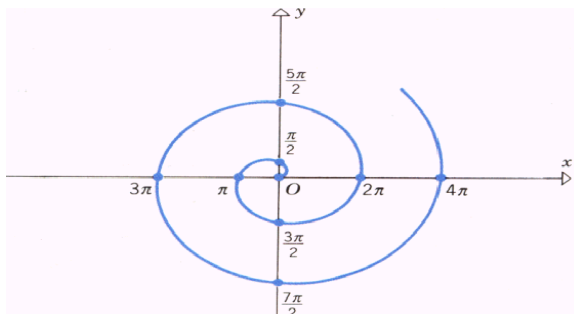
and the intersections with the  $y$ -axis occur when

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \dots$$

at which points  $r$  has the values

$$r = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \dots$$

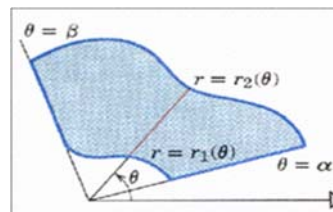
Starting from the origin, the Archimedean spirals  $r = \theta$  ( $\theta \geq 0$ ) loops counterclockwise around the origin.



## LECTURE No. 25

## DOUBLE INTEGRALS IN POLAR COORDINATES

Double integrals in which the integrand and the region of integration are expressed in polar coordinates are important for two reasons: First, they arise naturally in many applications, and second, many double integrals in rectangular coordinates are more easily evaluated if they are converted to polar coordinates. The function  $z = f(r, \theta)$  to be integrated over the region  $R$  as shown in the figure.

INTEGRALS IN POLAR COORDINATES

When we define the double integral of a function over a region  $R$  in the  $xy$ -plane, we begin by cutting  $R$  into rectangles whose sides are parallel to the coordinate axes. These are the natural shapes to use because their sides have either constant  $x$ -values or constant  $y$ -values. In polar coordinates, the natural shape is a “polar rectangle” whose sides have constant  $r$  and  $\theta$  - values.

Suppose that a function  $f(r, \theta)$  is defined over a region  $R$  that is bounded by the ray  $\theta = \alpha$  and  $\theta = \beta$  and by the continuous curves  $r = r_1(\theta)$  and  $r = r_2(\theta)$ . Suppose also that  $0 \leq r_1(\theta) \leq r_2(\theta) \leq a$  for every value of  $\theta$  between  $\alpha$  and  $\beta$ . Then  $R$  lies in a fan-shaped region  $Q$  defined by the inequalities  $0 \leq r \leq a$  and  $\alpha \leq \theta \leq \beta$ .

Then the double integral in polar coordinates is given as

$$\iint_R f(r, \theta) dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=r_1(\theta)}^{r=r_2(\theta)} f(r, \theta) dr d\theta$$

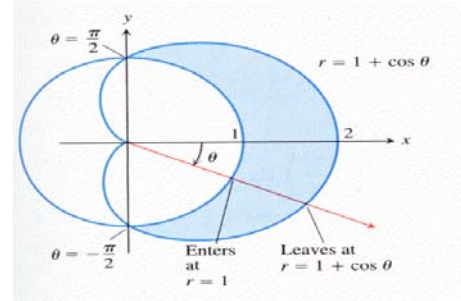
How to find limits of integration from sketch

**Step 1.** Since  $\theta$  is held fixed for the first integration, draw a radial line from the origin through the region  $R$  at a fixed angle  $\theta$ . This line crosses the boundary of  $R$  at most twice. The innermost point of intersection is on the curve  $r = r_1(\theta)$  and the outermost point is on the curve  $r = r_2(\theta)$ . These intersections determine the  $r$ -limits of integration.

**Step 2.** Imagine rotating a ray along the positive  $x$ -axis one revolution counterclockwise about the origin. The smallest angle at which this ray intersects the region  $R$  is  $\theta = \alpha$  and the largest angle is  $\theta = \beta$ . This yields the  $\theta$ -limits of the integration.

**EXAMPLE**

Find the limits of integration for integrating  $f(r, \theta)$  over the region  $R$  that lies inside the cardioids  $r = 1 + \cos \theta$  and outside the circle  $r = 1$ .

**Solution:**

**Step 1.** We sketch the region and label the bounding curves.

**Step 2.** The  $r$ -limits of integration. A typical ray from the origin enters  $R$  where  $r = 1$  and leaves where  $r = 1 + \cos \theta$ .

**Step 3.** The  $\theta$ -limits of integration. The rays from the

origin that intersect  $R$  run from  $\theta = -\frac{\pi}{2}$  to  $\theta = \frac{\pi}{2}$ .

$$\text{The integral is } \int_{-\pi/2}^{\pi/2} \int_1^{1+\cos\theta} f(r, \theta) r \, dr \, d\theta = 2 \int_0^{\pi/2} \int_1^{1+\cos\theta} f(r, \theta) r \, dr \, d\theta$$

**EXAMPLE :** Evaluate  $\iint_R \sin \theta \, dA$  where  $R$  is the region in the first quadrant that is outside the circle  $r = 2$  and inside the cardioids  $r = 2(1 + \cos \theta)$ .

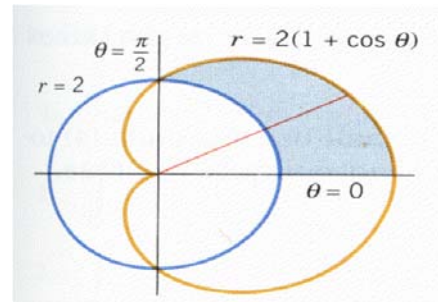
**Solution:**

$$\iint_R \sin \theta \, dA = \int_0^{\pi/2} \int_2^{2(1+\cos\theta)} (\sin \theta) r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \left[ \frac{1}{2} r^2 \sin \theta \right]_{r=2}^{2(1+\cos\theta)} d\theta$$

$$= 2 \int_0^{\pi/2} [(1+\cos\theta)^2 \sin \theta - \sin \theta] d\theta$$

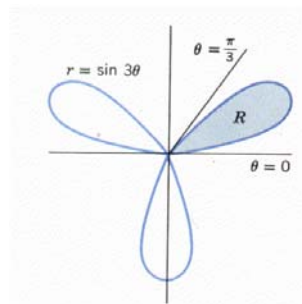
$$= 2 \left[ -\frac{1}{3} (1 + \cos\theta)^3 + \cos\theta \right]_0^{\pi/2} = 2 \left[ -\frac{1}{3} - \left( -\frac{5}{3} \right) \right] = \frac{8}{3}$$



**EXAMPLE :** Use a double polar integral to find the area enclosed by the three-petaled rose  $r = \sin 3\theta$ .

**Solution:** We calculate the area of the petal  $R$  in the first quadrant and multiply by three.

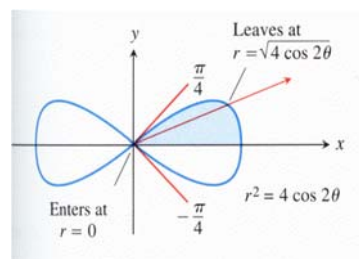
$$\begin{aligned}
 A &= 3 \iint_R dA = 3 \int_0^{\pi/3} \int_0^{\sin 3\theta} r \, dr \, d\theta = 3 \int_0^{\pi/3} \left. \frac{r^2}{2} \right|_0^{\sin 3\theta} d\theta \\
 &= \frac{3}{2} \int_0^{\pi/3} \sin^2 3\theta \, d\theta = \frac{3}{4} \int_0^{\pi/3} (1 - \cos 6\theta) \, d\theta \\
 &= \frac{3}{4} \left[ \theta - \frac{\sin 6\theta}{6} \right]_0^{\pi/3} = \left[ \frac{3}{4} \theta - \frac{3}{24} \sin 6\theta \right]_0^{\pi/3} = \frac{1}{4} \pi
 \end{aligned}$$



**EXAMPLE :** Find the area enclosed by the lemniscate  $r^2 = 4 \cos 2\theta$ . The total area is four times the first-quadrant portion.

**Solution:**

$$\begin{aligned}
 A &= 4 \int_0^{\pi/4} \int_0^{\sqrt{4 \cos 2\theta}} r \, dr \, d\theta = 4 \int_0^{\pi/4} \left[ \frac{r^2}{2} \right]_0^{\sqrt{4 \cos 2\theta}} d\theta \\
 &= 4 \int_0^{\pi/4} 2 \cos 2\theta \, d\theta = 4 \left[ \sin 2\theta \right]_0^{\pi/4} = 4.
 \end{aligned}$$



### CHANGING CARTESIAN INTEGRALS INTO POLAR INTEGRALS

The procedure for changing a Cartesian integral  $\iint_R f(x, y) \, dx \, dy$  into a polar integral has two steps.

**Step 1.** Substitute  $x = r \cos \theta$  and  $y = r \sin \theta$ , and replace  $dx \, dy$  by  $r \, dr \, d\theta$  in the Cartesian integral.

**Step 2.** Supply polar limits of integration for the boundary of  $R$ . The Cartesian integral then becomes

$$\iint_R f(x, y) \, dx \, dy = \iint_G f(r \cos \theta, r \sin \theta) r \, dr \, d\theta$$

where  $G$  denotes the region of integration in polar coordinates.

**Notice** that  $dx \, dy$  is not replaced by  $dr \, d\theta$  but by  $r \, dr \, d\theta$ .

**EXAMPLE:** Evaluate the double integral  $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) \, dy \, dx$  by changing to polar coordinates.

**Solution:** The region of integration is bounded by

$$0 \leq y \leq \sqrt{1-x^2} \text{ and } 0 \leq x \leq 1$$

$y = \sqrt{1-x^2}$  is the circle, which gives  $x^2 + y^2 = 1$ ,  $r = 1$

On changing into the polar coordinates, the given integral is

$$\int_0^{\pi/2} \int_0^1 r^3 dr d\theta = \int_0^{\pi/2} \left[ \frac{r^4}{4} \right]_0^1 d\theta = \int_0^{\pi/2} \frac{1}{4} d\theta = \left[ \frac{\theta}{4} \right]_0^{\pi/2} = \frac{\pi}{8}$$

### EXAMPLE

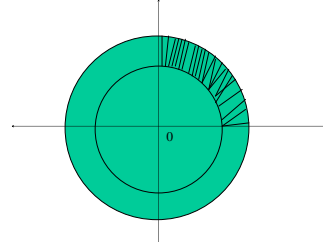
Evaluate  $I = \int_D \frac{dx dy}{x^2 + y^2}$  by changing to polar coordinates,

where D is the region in the first quadrant between the circles.

**Solution:**

Two circles are  $x^2 + y^2 = a^2$  and  $x^2 + y^2 = b^2$ ,  $0 < a < b$

$$\begin{aligned} I &= \int_0^{\pi/2} \int_a^b \frac{r dr d\theta}{r^2} = \int_0^{\pi/2} [\ln r]_a^b d\theta \\ &= \int_0^{\pi/2} \ln \left( \frac{b}{a} \right) d\theta = \left[ \theta \ln \left( \frac{b}{a} \right) \right]_0^{\pi/2} = \frac{\pi}{2} \ln \left( \frac{b}{a} \right). \end{aligned}$$



### EXAMPLE

Evaluate the double integral  $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx$  by changing to polar coordinates.

The region of integration is bounded by  $0 < y < \sqrt{1-x^2}$  and  $0 \leq x \leq 1$

$y = \sqrt{1-x^2}$  is the circle  $x^2 + y^2 = 1$ ,  $r = 1$

On changing into the polar coordinates, the given integral is

$$\int_0^{\pi/2} \int_0^1 r^3 dr d\theta = \int_0^{\pi/2} \left[ \frac{r^4}{4} \right]_0^1 d\theta = \int_0^{\pi/2} \frac{1}{4} d\theta = \frac{1}{4} \left[ \theta \right]_0^{\pi/2} = \frac{1}{4} (\pi/2) = \pi/8$$

## LECTURE No. 26

## EXAMPLES

**Example 1 :** Evaluate  $I = \int_0^4 \int_0^{\sqrt{4y-y^2}} (x^2+y^2) dx dy$  by changing into polar coordinates.

**Solution:** The region of integration is bounded by  $0 \leq x \leq \sqrt{4y-y^2}$  and  $0 \leq y \leq 4$

Now  $x = \sqrt{4y-y^2}$  is the circle  $x^2+y^2-4y=0 \Rightarrow x^2+y^2=4y$ . In polar coordinates this takes the form  $r^2 = 4r \sin \theta$ ,  $r = 4 \sin \theta$

On changing the integral into polar coordinates, we have

$$I = \int_0^{\pi/2} \int_0^{4\sin\theta} r^2 \cdot r dr d\theta = \int_0^{\pi/2} 64 \sin^4 \theta d\theta = 64 \cdot \frac{3.1}{4.2} \cdot \frac{\pi}{2} = 12\pi \quad (\text{using Walli's formula})$$

**Example 2 :** Evaluate  $\iint_R e^{x^2+y^2} dy dx$ , where R is the semicircular region bounded by the x-axis and the curve  $y = \sqrt{1-x^2}$

**Solution:** In Cartesian coordinates, the integral in question is a non-elementary integral and there is no direct way to integrate  $e^{x^2+y^2}$  with respect to either x or y.

Substituting  $x = r \cos \theta$ ,  $y = r \sin \theta$ , and replacing  $dy dx$  by  $r dr d\theta$  enables us to evaluate the integral as

$$\iint_R e^{x^2+y^2} dy dx = \int_0^{\pi} \int_0^1 e^{r^2} r dr d\theta = \int_0^{\pi} \left[ \frac{1}{2} e^{r^2} \right]_0^1 d\theta = \int_0^{\pi} \frac{1}{2} (e-1) d\theta = \frac{\pi}{2} (e-1).$$

**Example 3 :** Let  $R_a$  be the region bounded by the circle  $x^2 + y^2 = a^2$ . Define

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy = \lim_{a \rightarrow \infty} \iint_{R_a} e^{-(x^2+y^2)} dx dy$$

**Solution:** To evaluate this improper integral,

$$\begin{aligned} I &= \lim_{a \rightarrow \infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy = \lim_{a \rightarrow \infty} \iint_{D_a} e^{-(x^2+y^2)} dx dy \\ &= \lim_{a \rightarrow \infty} \int_0^{2\pi} \int_0^a e^{-r^2} r dr d\theta = \lim_{a \rightarrow \infty} \int_0^{2\pi} \frac{1}{2} (1 - e^{-a^2}) d\theta = \lim_{a \rightarrow \infty} \frac{1}{2} (1 - e^{-a^2}) \theta \Big|_0^{2\pi} \\ &= \lim_{a \rightarrow \infty} \pi (1 - e^{-a^2}) = \lim_{a \rightarrow \infty} \left( \pi - \frac{\pi}{e^{a^2}} \right) = \pi - \lim_{a \rightarrow \infty} \frac{\pi}{e^{a^2}} = \pi - \frac{\pi}{e^{\infty}} = \pi \end{aligned}$$

**Example 4 : Prove that**  $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

**Solution :** To prove it, we take

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dx dy = \int_{-\infty}^{\infty} e^{-y^2} \int_{-\infty}^{\infty} e^{-x^2} dx dy \\ &= \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-x^2} dx \quad y \text{ is a dummy variable, so we can change } y \text{ to } x. \\ &= \left( \int_{-\infty}^{\infty} e^{-x^2} dx \right)^2 = \left( 2 \int_0^{\infty} e^{-x^2} dx \right)^2 \end{aligned}$$

Therefore,  $4 \left( \int_0^{\infty} e^{-x^2} dx \right)^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy = \pi$  by Example 3

$$\left( \int_0^{\infty} e^{-x^2} dx \right)^2 = \frac{\pi}{4}$$

By taking square root on both sides,  $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

**THEOREM :** Let G be the rectangular box defined by the inequalities

$$a \leq x \leq b, \quad c \leq y \leq d, \quad k \leq z \leq \ell$$

If f is continuous on the region G, then  $\iiint_G f(x, y, z) dv = \int_a^b \int_c^d \int_k^{\ell} f(x, y, z) dz dy dx$

Moreover, the iterated integral on the right can be replaced with any of the five other iterated integrals that result by altering the order of integration.

$$\begin{aligned} &= \int_a^b \int_k^{\ell} \int_c^d f(x, y, z) dy dz dx = \int_k^{\ell} \int_a^b \int_c^d f(x, y, z) dy dx dz = \int_c^d \int_k^{\ell} \int_a^b f(x, y, z) dx dy dz \\ &= \int_c^d \int_a^b \int_k^{\ell} f(x, y, z) dx dz dy = \int_a^b \int_c^d \int_k^{\ell} f(x, y, z) dz dx dy \end{aligned}$$

**Example 5:** Evaluate the triple integral  $\iiint_G 12xy^2z^3 dV$  over the rectangular box G

defined by the inequalities  $-1 \leq x \leq 2, 0 \leq y \leq 3, 0 \leq z \leq 2$ .

**Solution:** We first integrate with respect to z, holding x and y fixed, then with respect to y holding x fixed, and finally with respect to x.

$$\begin{aligned} \iiint_G 12xy^2z^3 dV &= \int_{-1}^2 \int_0^3 \int_0^2 12xy^2z^3 dz dy dx = \int_{-1}^2 \int_0^3 [3xy^2z^4]_{z=0}^2 dy dx = \int_{-1}^2 \int_0^3 48xy^2 dy dx \\ &= \int_{-1}^2 [16xy^3]_{y=0}^3 dx = \int_{-1}^2 432x dx = [216x^2]_{-1}^2 = 648 \end{aligned}$$

**Example 6 :** Evaluate  $\iiint_R (x - 2y + z) dx dy dz$  Region R :  $0 \leq x \leq 1, 0 \leq y \leq x^2, 0 \leq z \leq x + y$

$$\begin{aligned} \text{Solution: } &= \int_0^1 \int_0^{x^2} \int_0^{x+y} (x - 2y + z) dz dy dx = \int_0^1 \int_0^{x^2} \left| \frac{(x - 2y + z)^2}{2} \right|_0^{x+y} dy dx \\ &= \int_0^1 \int_0^{x^2} \left[ \frac{(x - 2y + x + y)^2}{2} - \frac{(x - 2y)^2}{2} \right] dy dx = \frac{1}{2} \int_0^1 \int_0^{x^2} (3x^2 - 3y^2) dy dx = \frac{3}{2} \int_0^1 \left| x^2 y - \frac{y^3}{3} \right|_0^{x^2} dx \\ &= \frac{3}{2} \int_0^1 \left( x^4 - \frac{x^6}{3} \right) dx = \frac{3}{2} \left| \frac{x^5}{5} - \frac{x^7}{21} \right|_0^1 = \frac{3}{2} \left[ \frac{1}{5} - \frac{1}{21} \right] = \frac{8}{35} \end{aligned}$$

**Example 7 :** Evaluate  $\iiint_S x y z dx dy dz$  Where  $S = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1, x \geq 0, y \geq 0, z \geq 0\}$

**Solution:** S is the sphere  $x^2 + y^2 + z^2 = 1$ . Since x, y, z are all +ve so we have to consider only the +ve octant of the sphere. Now  $x^2 + y^2 + z^2 = 1$ . So that  $z = \sqrt{1 - x^2 - y^2}$   
The Projection of the sphere on xy plan is the circle  $x^2 + y^2 = 1$ .

This circle is covered as y-varies from 0 to  $\sqrt{1 - x^2}$  and x varies from 0 to 1.

$$\begin{aligned} \iiint_R x y z dx dy dz &= \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} x y z dz dy dx = \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left| \frac{z^2}{2} \right|_0^{\sqrt{1-x^2-y^2}} dy dx \\ &= \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left( \frac{1 - x^2 - y^2}{2} \right) dy dx = \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} x (y - x^2 y - y^3) dy dx \\ &= \frac{1}{2} \int_0^1 x \left( \frac{y^2}{2} - \frac{x^2 y^2}{2} - \frac{y^4}{4} \right) \Big|_0^{\sqrt{1-x^2}} dx = \frac{1}{4} \int_0^1 x \left[ 1 - x^2 - x^2 (1 - x^2) - \frac{(1 - x^2)^2}{2} \right] dx \\ &= \frac{1}{8} \int_0^1 (x - 2x^3 + x^5) dx = \frac{1}{8} \left| \frac{x^2}{2} - \frac{x^4}{2} + \frac{x^6}{6} \right|_0^1 = \frac{1}{8} \left( \frac{1}{2} - \frac{1}{2} + \frac{1}{6} \right) = \frac{1}{48} \end{aligned}$$

## LECTURE No. 27

## VECTOR VALUED FUNCTIONS

Recall that a function is a rule that assigns to each element in its domain one and only one element in its range. Thus far, we have considered only functions for which the domain and range are sets of real numbers; such functions are called real-valued functions of a real variable or sometimes simply real-valued functions. In this section we shall consider functions for which the domain consists of real numbers and the range consists of vectors in 2-space or 3-space; such functions are called vector-valued functions of a real variable or more simply vector-valued functions. In 2-space such functions can be expressed in the form.

$$\mathbf{r}(t) = (x(t), y(t)) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

and in 3-space in the form

$$\mathbf{r}(t) = (x(t), y(t), z(t)) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

where  $x(t)$ ,  $y(t)$ , and  $z(t)$  are real-valued functions of the real variable  $t$ . These real-valued functions are called the component functions or components of  $\mathbf{r}$ . As a matter of notation, we shall denote vector-valued functions with boldface type  $\mathbf{f}(t)$ ,  $\mathbf{g}(t)$  and  $\mathbf{r}(t)$  and real-valued functions, as usual, with lightface italic type  $f(t)$ ,  $g(t)$  and  $r(t)$ .

**EXAMPLE:**  $\mathbf{r}(t) = (\ln t)\mathbf{i} + \sqrt{t^2 + 2}\mathbf{j} + (\cos t\pi)\mathbf{k}$

Then the component functions are  $x(t) = \ln t$ ,  $y(t) = \sqrt{t^2 + 2}$ , and  $z(t) = \cos t\pi$

The vector that  $\mathbf{r}(t)$  associates with  $t = 1$  is  $\mathbf{r}(1) = (\ln 1)\mathbf{i} + \sqrt{3}\mathbf{j} + (\cos \pi)\mathbf{k} = \sqrt{3}\mathbf{j} - \mathbf{k}$

The function  $\mathbf{r}$  is undefined if  $t \leq 0$  because  $\ln t$  is undefined for such  $t$ .

If the domain of a vector-valued function is not stated explicitly, then it is understood to consist of all real numbers for which every component is defined and yields a real value. This is called the natural domain of the function. Thus the natural domain of a vector-valued function is the intersection of the natural domains of its components.

**PARAMETRIC EQUATIONS IN VECTOR FORM**

Vector-valued functions can be used to express parametric equations in 2-space or 3-space in a compact form.

For example, consider the parametric equations  $x = x(t)$ ,  $y = y(t)$

Because two vectors are equivalent if and only if their corresponding components are equal, this pair of equations can be replaced by the single vector equation.

$$x = x(t), y = y(t) \quad \text{-----}(1)$$

$$x\mathbf{i} + y\mathbf{j} = x(t)\mathbf{i} + y(t)\mathbf{j} \quad \text{-----}(2)$$

Similarly, in 3-space the three parametric equations

$$x = x(t), y = y(t), z = z(t) \quad \text{-----}(3)$$

can be replaced by the single vector equation

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \quad \text{-----}(4)$$

if we let  $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$  and  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$  in 2-space and let  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$  and  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$  in 3-space, then both (2) and (4) can be written as  $\mathbf{r} = \mathbf{r}(t)$  -----(5) which is the vector form of the parametric equations in (1) and (3). Conversely, every vector equation of form (5) can be rewritten as parametric equations by equating components on the two sides.

**EXAMPLE: Express the given parametric equations as a single vector equation.**

(a)  $x = t^2, y = 3t$

(b)  $x = \cos t, y = \sin t, z = t$

**Solution:** (a) Using the two sides of the equations as components of a vector yields.

$$x\mathbf{i} + y\mathbf{j} = t^2\mathbf{i} + 3t\mathbf{j}$$

(b) Proceeding as in part (a) yields

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$

**EXAMPLE: Find parametric equations that correspond to the vector equation**

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (t^3 + 1)\mathbf{i} + 3\mathbf{j} + e^t\mathbf{k}$$

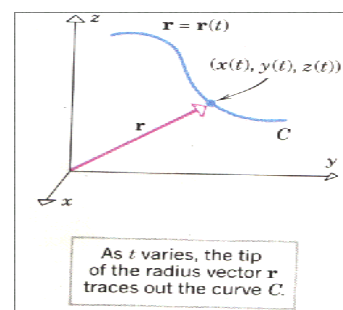
Equating corresponding components yields.

$$x = t^3 + 1, y = 3, z = e^t$$

### **GRAPHS OF VECTOR-VALUED FUNCTIONS**

One method for interpreting a vector-valued function  $\mathbf{r}(t)$  in 2-space or 3-space geometrically is to position the vector  $\mathbf{r} = \mathbf{r}(t)$  with its initial point at the origin, and let  $C$  be the curve generated by the tip of the vector  $\mathbf{r}$  as the parameter  $t$  varies.

The vector  $\mathbf{r}$ , when positioned in this way, is called the radius vector or position vector of  $C$ , and  $C$  is called the graph of the function  $\mathbf{r}(t)$  or, equivalently, the graph of the equation  $\mathbf{r} = \mathbf{r}(t)$ . The vector equation  $\mathbf{r} = \mathbf{r}(t)$  is equivalent to a set of parametric equations, so  $C$  is also called the graph of these parametric equations.



**EXAMPLE: Sketch the graph of the vector-valued function  $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$ ,  $0 \leq t \leq 2\pi$**

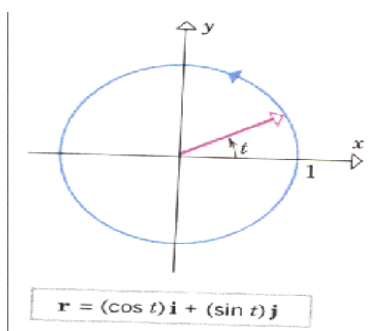
The graph of  $\mathbf{r}(t)$  is the graph of the vector equation

$$x\mathbf{i} + y\mathbf{j} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad 0 \leq t \leq 2\pi$$

or equivalently, it is the graph of the parametric equations

$$x = \cos t, y = \sin t \quad (0 \leq t \leq 2\pi)$$

This is a circle of radius 1 that is centered at the origin with the direction of increasing  $t$  counterclockwise. The graph and a radius vector are shown in Fig.



**EXAMPLE:** Sketch the graph of the vector-valued function  $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + 2\mathbf{k}$ ,  $0 \leq t \leq 2\pi$

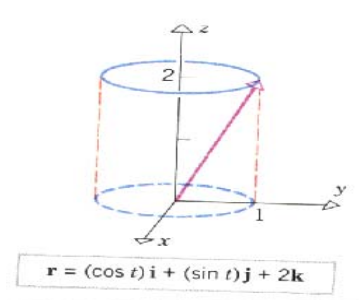
The graph of  $\mathbf{r}(t)$  is the graph of the vector equation

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + 2\mathbf{k}, \quad 0 \leq t \leq 2\pi$$

or, equivalently, it is the graph of the parametric equations

$$x = \cos t, \quad y = \sin t, \quad z = 2 \quad (0 \leq t \leq 2\pi)$$

From the last equation, the tip of the radius vector traces a curve in the plane  $z = 2$ , and from the first two equations and the preceding example, the curve is a circle of radius 1 centered on the  $z$ -axis and traced counterclockwise looking down the  $z$ -axis. The graph and a radius vector are shown in Fig.



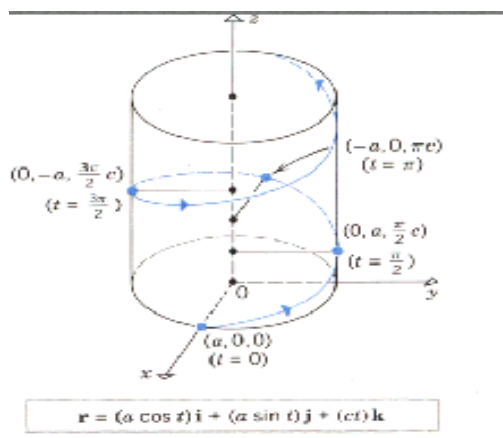
**EXAMPLE:** Sketch the graph of the vector-valued function  $\mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j} + (ct)\mathbf{k}$ , where  $a$  and  $c$  are positive constant.

The graph of  $\mathbf{r}(t)$  is the graph of the parametric equations.

$$x = a \cos t, \quad y = a \sin t, \quad z = ct$$

As the parameter  $t$  increases, the value of  $z = ct$  also increases, so the point  $(x, y, z)$  moves upward. However, as  $t$  increases, the point  $(x, y, z)$  also moves in a path directly over the circle  $x = a \cos t$ ,  $y = a \sin t$  in the  $xy$ -plane. The combination of these upward and circular motions produces a corkscrew-shaped curve that wraps around a right-circular cylinder of radius  $a$  centered on the  $z$ -axis.

**This curve is called a circular helix.**



**EXAMPLE:** Describe the graph of the vector equation  $\mathbf{r} = (-2 + t)\mathbf{i} + 3t\mathbf{j} + (5 - 4t)\mathbf{k}$

The corresponding parametric equations are  $x = -2 + t$ ,  $y = 3t$ ,  $z = 5 - 4t$

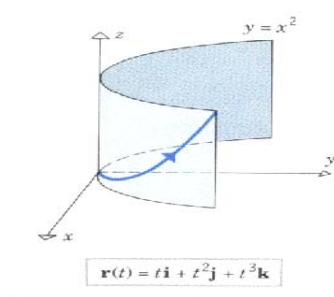
The graph is the line in 3-space that passes through the point  $(-2, 0, 5)$  and is parallel to the vector  $\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ .

### EXAMPLE

The graph of the vector-valued function  $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$  is called a twisted cubic. Show that this curve lies on the parabolic cylinder  $y = x^2$ , and sketch the graph for  $t \geq 0$

The corresponding parametric equations are  $x = t$ ,  $y = t^2$ ,  $z = t^3$

Eliminating the parameter  $t$  in the equations for  $x$  and  $y$  yields  $y = x^2$ , so the curve lies on the parabolic cylinder with this equation. The curve starts at the origin for  $t = 0$ ; as  $t$  increases, so do  $x$ ,  $y$ , and  $z$ , so the curve is traced in the upward direction, moving away from the origin along the cylinder.



### GRAPHS OF CONSTANT VECTOR-VALUED FUNCTIONS

If  $\mathbf{c}$  is a constant vector in the sense that it does not depend on a parameter, then the graph of  $\mathbf{r} = \mathbf{c}$  is a single point since the radius vector remains fixed with its tip at  $\mathbf{c}$ .

If  $\mathbf{c} = x_0\mathbf{i} + y_0\mathbf{j}$  (in 2-space), then the graph is the point  $(x_0, y_0)$ , and if  $\mathbf{c} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$  (in 3-space), then the graph is the point  $(x_0, y_0, z_0)$ .

**EXAMPLE:** The graph of the equation  $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$  is the point  $(2, 3, -1)$  in 3-space.

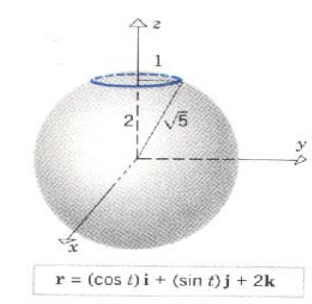
**Remark:** If  $\mathbf{r}(t)$  is a vector-valued function, then for each value of the parameter  $t$ , the expression  $\|\mathbf{r}(t)\|$  is a real-valued function of  $t$  because the norm (or length of  $\mathbf{r}(t)$ ) is a real number.

For example, if  $\mathbf{r}(t) = t\mathbf{i} + (t-1)\mathbf{j}$  Then  $\|\mathbf{r}(t)\| = \sqrt{t^2 + (t-1)^2}$  which is a real-valued function of  $t$ .

**EXAMPLE:** The graph of  $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + 2\mathbf{k}$ ,  $0 \leq t \leq 2\pi$  is a circle of radius 1 centered on the  $z$ -axis and lying in the plane  $z = 2$ . This circle lies on the surface of a sphere of radius  $\sqrt{5}$  because for each value of  $t$

$$\|\mathbf{r}(t)\| = \sqrt{\cos^2 t + \sin^2 t + 2^2} = \sqrt{1 + 4} = \sqrt{5}$$

which shows that each point on the circle is a distance of  $\sqrt{5}$  units from the origin.



## LECTURE No. 28

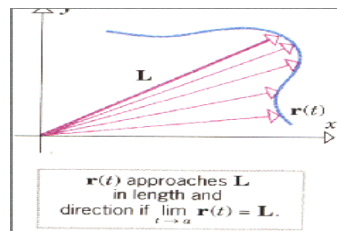
## LIMITS OF VECTOR VALUED FUNCTIONS

The limit of a vector-valued function is defined to be the vector that results by taking the limit of each component. Thus, for a function  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$  in 2-space we define.

$$\lim_{t \rightarrow \alpha} \mathbf{r}(t) = (\lim_{t \rightarrow \alpha} x(t))\mathbf{i} + (\lim_{t \rightarrow \alpha} y(t))\mathbf{j}$$

and for a function  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$  in 3-space we define.

$$\lim_{t \rightarrow \alpha} \mathbf{r}(t) = (\lim_{t \rightarrow \alpha} x(t))\mathbf{i} + (\lim_{t \rightarrow \alpha} y(t))\mathbf{j} + (\lim_{t \rightarrow \alpha} z(t))\mathbf{k}$$



If the limit of any component does not exist, then we shall agree that the limit of  $\mathbf{r}(t)$  does not exist.

These definitions are also applicable to the one-sided limits  $\lim_{t \rightarrow \alpha^+}$ ,  $\lim_{t \rightarrow \alpha^-}$  and infinite limits,  $\lim_{t \rightarrow +\infty}$ , and  $\lim_{t \rightarrow -\infty}$ . It follows from (1) and (2) that

$$\lim_{t \rightarrow \alpha} \mathbf{r}(t) = \mathbf{L}$$

if and only if the components of  $\mathbf{r}(t)$  approach the components of  $\mathbf{L}$  as  $t \rightarrow \alpha$ . Geometrically, this is equivalent to stating that the length and direction of  $\mathbf{r}(t)$  approach the length and direction of  $\mathbf{L}$  as  $t \rightarrow \alpha$ .

**CONTINUITY OF VECTOR-VALUED FUNCTIONS**

The definition of continuity for vector-valued functions is similar to that for real-valued functions. We shall say that  $\mathbf{r}$  is continuous at  $t_0$  if

1.  $\mathbf{r}(t_0)$  is defined;
2.  $\lim_{t \rightarrow t_0} \mathbf{r}(t)$  exists;
3.  $\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$ .

It can be shown that  $\mathbf{r}$  is continuous at  $t_0$  if and only if each component of  $\mathbf{r}$  is continuous. As with real-valued functions, we shall call  $\mathbf{r}$  continuous everywhere or simply continuous if  $\mathbf{r}$  is continuous at all real values of  $t$ . Geometrically, the graph of a continuous vector-valued function is an unbroken curve.

**DERIVATIVES OF VECTOR-VALUED FUNCTIONS**

The definition of a derivative for vector-valued functions is analogous to the definition for real-valued functions.

**DEFINITION**

The derivative  $\mathbf{r}'(t)$  of a vector-valued function  $\mathbf{r}(t)$  is defined by

$$\mathbf{r}'(t) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

Provided this limit exists.

For computational purposes the following theorem is extremely useful; it states that the derivative of a vector-valued function can be computed by differentiating each component.

### **THEOREM**

(a) If  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$  is a vector-valued function in 2-space, and if  $x(t)$  and  $y(t)$  are differentiable, then  $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$

(b) If  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$  is a vector-valued function in 3-space, and if  $x(t)$ ,  $y(t)$ , and  $z(t)$  are differentiable, then

$$\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$

We shall prove part (a). The proof of (b) is similar.

**Proof (a):**

$$\begin{aligned}\mathbf{r}'(t) &= \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \lim_{h \rightarrow 0} \frac{[x(t+h) - x(t)]}{h} \mathbf{i} + \lim_{h \rightarrow 0} \frac{[y(t+h) - y(t)]}{h} \mathbf{j} \\ &= x'(t)\mathbf{i} + y'(t)\mathbf{j}\end{aligned}$$

As with real-valued functions, there are various notations for the derivative of a vector-valued function. If  $\mathbf{r} = \mathbf{r}(t)$ , then some possibilities are  $\frac{d}{dt}[\mathbf{r}(t)]$ ,  $\frac{d\mathbf{r}}{dt}$ ,  $\mathbf{r}'(t)$ , and  $\mathbf{r}'$

### **EXAMPLE**

Let  $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$ . Find  $\mathbf{r}'(t)$  and  $\mathbf{r}'(1)$

$$\begin{aligned}\mathbf{r}'(t) &= \frac{d}{dt}[t^2]\mathbf{i} + \frac{d}{dt}[t^3]\mathbf{j} \\ &= 2t\mathbf{i} + 3t^2\mathbf{j}\end{aligned}$$

Substituting  $t=1$  yields

$$\mathbf{r}'(1) = 2\mathbf{i} + 3\mathbf{j}.$$

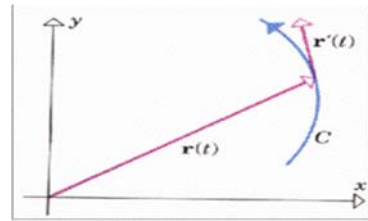
### **TAGENT VECTORS AND TANGENT LINES**

#### **GEOMETRIC INTERPRETATIONS OF THE DERIVATIVE.**

Suppose that  $C$  is the graph of a vector-valued

function  $\mathbf{r}(t)$  and that  $\mathbf{r}'(t)$  exists and is nonzero

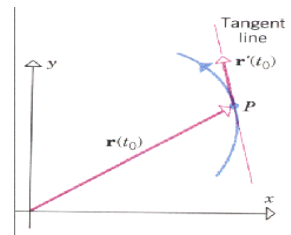
for a given value of  $t$ . If the vector  $\mathbf{r}'(t)$  is positioned with its initial point at the terminal point of the radius vector



### **DEFINITION**

Let  $P$  be a point on the graph of a vector-valued function  $\mathbf{r}(t)$ , and let  $\mathbf{r}(t_0)$  be the radius vector from the origin to  $P$

If  $\mathbf{r}'(t_0)$  exists and  $\mathbf{r}'(t_0) \neq \mathbf{0}$ , then we call  $\mathbf{r}'(t_0)$  the tangent vector to the graph of  $\mathbf{r}$  at  $\mathbf{r}(t_0)$



**REMARKS**

Observe that the graph of a vector-valued function can fail to have a tangent vector at a point either because the derivative in (4) does not exist or because the derivative is zero at the point. If a vector-valued function  $\mathbf{r}(t)$  has a tangent vector  $\mathbf{r}'(t_0)$  at a point on its graph, then the line that is parallel to  $\mathbf{r}'(t_0)$  and passes through the tip of the radius vector  $\mathbf{r}(t_0)$  is called the tangent line of the graph of  $\mathbf{r}(t)$  at  $\mathbf{r}(t_0)$ . Vector equation of the tangent line is

$$\mathbf{r} = \mathbf{r}(t_0) + t \mathbf{r}'(t_0)$$

**EXAMPLE**

**Find parametric equation of the tangent line to the circular helix**

$$x = \cos t, \quad y = \sin t, \quad z = 1 \quad \text{at the point where } t = \pi/6$$

**Solution:**

To find a vector equation of the tangent line, then we shall equate components to obtain the parametric equations. A vector equation  $\mathbf{r} = \mathbf{r}(t)$  of the helix is

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$

$$\text{Thus, } \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$

$$\Rightarrow \mathbf{r}'(t) = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}$$

At the point where  $t = \pi/6$ , these vectors are

$$\mathbf{r}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{\pi}{6}\mathbf{k} \quad \text{and}$$

$$\mathbf{r}'\left(\frac{\pi}{6}\right) = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} + \mathbf{k}$$

so from (5) with  $t_0 = \pi/6$  a vector equation of the tangent line is

$$\mathbf{r}\left(\frac{\pi}{6}\right) + t \mathbf{r}'\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{\pi}{6}\mathbf{k}\right) + t\left(-\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} + \mathbf{k}\right)$$

Simplifying, then equating the resulting components with the corresponding components of  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$  yields the parametric equation.

$$x = \frac{\sqrt{3}}{2} - \frac{1}{2}t, \quad y = \frac{1}{2} + \frac{\sqrt{3}}{2}t, \quad z = \frac{\pi}{6} + t$$

**EXAMPLE**

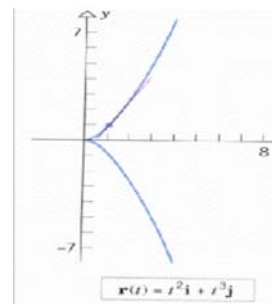
The graph of  $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$  is called a semi-cubical parabola

Find a vector equation of the tangent line to the graph of  $\mathbf{r}(t)$  at

- (a) the point (0,0)      (b) the point (1,1)

The derivative of  $\mathbf{r}(t)$  is

$$\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$$



- (a) The point (0, 0) on the graph of  $\mathbf{r}$  corresponds to  $t = 0$ . As this point we have  $\mathbf{r}'(0) = 0$ , so there is no tangent vector at the point and consequently a tangent line does not exist at this point.

(b) The point (1, 1) on the graph of  $\mathbf{r}$  corresponds to  $t = 1$ , so from (5) a vector equation of the tangent line at this point is  $\mathbf{r} = \mathbf{r}(1) + t \mathbf{r}'(1)$

From the formulas for  $\mathbf{r}(t)$  and  $\mathbf{r}'(t)$  with  $t = 1$ , this equation becomes

$$\mathbf{r} = (\mathbf{i} + \mathbf{j}) + t(2\mathbf{i} + 3\mathbf{j})$$

If  $\mathbf{r}$  is a vector-valued function in 2-space or 3-space, then we say that  $\mathbf{r}(t)$  is smoothly parameterized or that  $\mathbf{r}$  is a smooth function of  $t$  if the components of  $\mathbf{r}$  have continuous derivatives with respect to  $t$  and  $\mathbf{r}'(t) \neq \mathbf{0}$  for any value of  $t$ . Thus, in 3-space  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$  is a smooth function of  $t$  if  $x'(t)$ ,  $y'(t)$ , and  $z'(t)$  are continuous and there is no value of  $t$  at which all three derivatives are zero. A parametric curve  $C$  in 2-space or 3-space will be called smooth if it is the graph of some smooth vector-valued function.

It can be shown that a smooth vector-valued function has a tangent line at every point on its graph.

### **PROPERTIES OF DERIVATIVES**

#### **(Rules of Differentiation).**

In either 2-space or 3-space let  $\mathbf{r}(t)$ ,  $\mathbf{r}_1(t)$ , and  $\mathbf{r}_2(t)$  be vector-valued functions,  $f(t)$  a real-valued function,  $k$  a scalar, and  $\mathbf{c}$  a fixed (constant) vector. Then the following rules of differentiation hold:

1.  $\frac{d}{dt} [\mathbf{c}] = \mathbf{0}$
2.  $\frac{d}{dt} [k \mathbf{r}(t)] = k \frac{d}{dt} [\mathbf{r}(t)]$
3.  $\frac{d}{dt} [\mathbf{r}_1(t) + \mathbf{r}_2(t)] = \frac{d}{dt} [\mathbf{r}_1(t)] + \frac{d}{dt} [\mathbf{r}_2(t)]$
4.  $\frac{d}{dt} [\mathbf{r}_1(t) - \mathbf{r}_2(t)] = \frac{d}{dt} [\mathbf{r}_1(t)] - \frac{d}{dt} [\mathbf{r}_2(t)]$
5.  $\frac{d}{dt} [f(t)\mathbf{r}(t)] = f(t)\frac{d}{dt} [\mathbf{r}(t)] + \mathbf{r}(t)\frac{d}{dt} [f(t)]$

In addition to the rules listed in the foregoing theorem, we have the following rules for differentiating dot products in 2-space or 3-space and cross products in 3-space:

6.  $\frac{d}{dt} [\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)] = \mathbf{r}_1 \cdot \frac{d\mathbf{r}_2}{dt} + \frac{d\mathbf{r}_1}{dt} \cdot \mathbf{r}_2$
7.  $\frac{d}{dt} [\mathbf{r}_1(t) \times \mathbf{r}_2(t)] = \mathbf{r}_1 \times \frac{d\mathbf{r}_2}{dt} + \frac{d\mathbf{r}_1}{dt} \times \mathbf{r}_2$

### **REMARKS:**

In (6), the order of the factors in each term on the right does not matter, but in (7) it does.

In plane geometry one learns that a tangent line to a circle is perpendicular to the radius at the point of tangency. Consequently, if a point moves along a circular arc in 2-space,

one would expect the radius vector and the tangent vector at any point on the arc to be perpendicular. This is the motivation for the following useful theorem, which is applicable in both 2-space and 3-space.

### **THEOREM:**

If  $\mathbf{r}(t)$  is a vector-valued function in 2-space or 3-space and  $\|\mathbf{r}(t)\|$  is constant for all  $t$ , then  $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$

**Proof:** That is,  $\mathbf{r}(t)$  and  $\mathbf{r}'(t)$  are orthogonal vectors for all  $t$ . It follows from (6) with  $\mathbf{r}_1(t) = \mathbf{r}_2(t) = \mathbf{r}(t)$  that

$$\frac{d}{dt} [\mathbf{r}(t) \cdot \mathbf{r}(t)] = \mathbf{r}(t) \cdot \frac{d\mathbf{r}}{dt} + \frac{d\mathbf{r}}{dt} \cdot \mathbf{r}(t)$$

$$\text{or, equivalently, } \frac{d}{dt} [\|\mathbf{r}(t)\|^2] = 2\mathbf{r}(t) \cdot \frac{d\mathbf{r}}{dt}$$

But  $\|\mathbf{r}(t)\|^2$  is constant, so its derivative is zero. Thus  $2\mathbf{r}(t) \cdot \frac{d\mathbf{r}}{dt} = 0$  that is  $\mathbf{r}(t) \cdot \frac{d\mathbf{r}}{dt} = 0$

That is the  $\mathbf{r}(t)$  is perpendicular  $\frac{d\mathbf{r}}{dt}$

### **EXAMPLE**

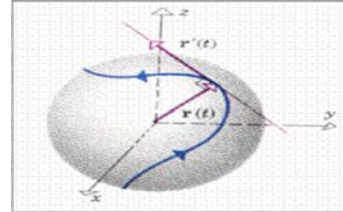
Just as a tangent line to a circular arc in 2-space is perpendicular to the radius at the point of tangency, so a tangent line to a curve on the surface of a sphere in 3-space is perpendicular to the radius at the point of tangency.

To see that this is so, suppose that the graph of  $\mathbf{r}(t)$  lies on the surface of the sphere of radius  $k > 0$  centered at the origin. For each value of  $t$  we have  $\|\mathbf{r}(t)\| = k$ ,

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$$

which implies that the radius vector  $\mathbf{r}(t)$  and the

tangent vector  $\mathbf{r}'(t)$  are perpendicular. This completes the argument because the tangent line, where it exists, is parallel to the tangent vector.



### **INTEGRALS OF VECTOR VALUED FUNCTION**

(a) If  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$  is a vector-valued function in 2-space, then we define.

$$\int \mathbf{r}(t) dt = \left( \int x(t) dt \right) \mathbf{i} + \left( \int y(t) dt \right) \mathbf{j} \quad (1a)$$

$$\int_a^b \mathbf{r}(t) dt = \left( \int_a^b x(t) dt \right) \mathbf{i} + \left( \int_a^b y(t) dt \right) \mathbf{j} \quad (1b)$$

(a) If  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$  is a vector-valued function in 3-space, then we define.

$$\int \mathbf{r}(t)dt = \left(\int x(t)dt\right)\mathbf{i} + \left(\int y(t)dt\right)\mathbf{j} + \left(\int z(t)dt\right)\mathbf{k} \quad (2a)$$

$$\int_a^b \mathbf{r}(t)dt = \left(\int_a^b x(t)dt\right)\mathbf{i} + \left(\int_a^b y(t)dt\right)\mathbf{j} + \left(\int_a^b z(t)dt\right)\mathbf{k} \quad (2b)$$

**Example:**

Let  $\mathbf{r}(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$ , find

$$\begin{aligned} (a) \int \mathbf{r}(t)dt & \quad (b) \int_0^2 \mathbf{r}(t)dt \\ \int \mathbf{r}(t)dt &= \int (2t\mathbf{i} + 3t^2\mathbf{j})dt = \left(\int 2tdt\right)\mathbf{i} + \left(\int 3t^2dt\right)\mathbf{j} \\ (t^2 + C_1)\mathbf{i} + (t^3 + C_2)\mathbf{j} &= t^2\mathbf{i} + C_1\mathbf{i} + t^3\mathbf{j} + C_2\mathbf{j} \\ &= t^2\mathbf{i} + t^3\mathbf{j} + C_1\mathbf{i} + C_2\mathbf{j} \\ &= t^2\mathbf{i} + t^3\mathbf{j} + \mathbf{C} \end{aligned}$$

Where  $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j}$  is an arbitrary vector constant of integration

$$(b) \int_0^2 \mathbf{r}(t)dt = \int_0^2 (2t\mathbf{i} + 3t^2\mathbf{j})dt = \left(\int_0^2 2tdt\right)\mathbf{i} + \left(\int_0^2 3t^2dt\right)\mathbf{j} = \left[t^2\right]_0^2\mathbf{i} + \left[t^3\right]_0^2\mathbf{j} = (2^2 - 0)\mathbf{i} + (2^3 - 0)\mathbf{j} = 4\mathbf{i} + 8\mathbf{j}$$

**PROPERTIES OF INTEGRALS**

$$\int c\mathbf{r}(t) dt = c \int \mathbf{r}(t) dt \quad (3)$$

$$\int [\mathbf{r}_1(t) + \mathbf{r}_2(t)] dt = \int \mathbf{r}_1(t)dt + \int \mathbf{r}_2(t) dt \quad (4)$$

$$\int [\mathbf{r}_1(t) - \mathbf{r}_2(t)]dt = \int \mathbf{r}_1(t) dt - \int \mathbf{r}_2(t) dt \quad (5)$$

These properties also hold for definite integrals of vector-valued functions. In addition, we leave it for the reader to show that if  $\mathbf{r}$  is a vector-valued function in 2-space or 3-space, then

$$\frac{d}{dt} \left[ \int \mathbf{r}(t) dt \right] = \mathbf{r}(t) \quad (6)$$

This shows that an indefinite integrals of  $\mathbf{r}(t)$  is, in fact, the set of anti-derivatives of  $\mathbf{r}(t)$ , just as for real-valued functions.

If  $\mathbf{r}(t)$  is any anti-derivative of  $\mathbf{r}(t)$  in the sense that  $\mathbf{r}'(t) = \mathbf{r}(t)$ , then

$$\int \mathbf{r}(t) dt = \mathbf{R}(t) + \mathbf{C} \quad (7)$$

where  $\mathbf{C}$  is an arbitrary vector constant of integration.

Moreover,

$$\int_a^b \mathbf{r}(t) dt = \mathbf{R}(t) \Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

## LECTURE No.29

## CHANGE OF PARAMETER

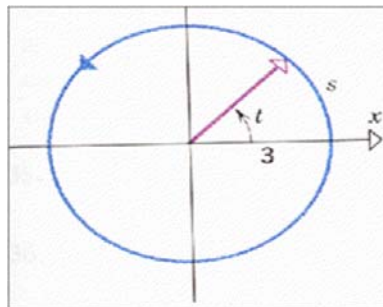
It is possible for different vector-valued functions to have the same graph.

For example, the graph of the function

$$\mathbf{r} = (3 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j}, \quad 0 \leq t \leq 2\pi \quad \text{-----(1)}$$

is the circle of radius 3 centered at the origin with counterclockwise orientation. The parameter  $t$  can be interpreted geometrically as the positive angle in radians from the  $x$ -axis to the radius vector.

For each value of  $t$ , let  $s$  be the length of the arc subtended by this angle on the circle



The parameters  $s$  and  $t$  are related by

$$t = \frac{s}{3}, \quad 0 < s < 6\pi$$

If we substitute this in equation (1), we obtain a vector-valued function of the parameter  $s$ , namely

$$\mathbf{r} = 3 \cos (s/3)\mathbf{i} + 3 \sin (s/3)\mathbf{j}, \quad 0 \leq s \leq 6\pi$$

whose graph is also the circle of radius 3 centered at the origin with counterclockwise orientation. In various problems it is helpful to change the parameter in a vector-valued function by making an appropriate substitution. For example, we changed the parameter above from  $t$  to  $s$  by substituting  $t = \frac{s}{3}$  in equation(1).

In general, if  $g$  is a real-valued function, then substituting  $t = g(u)$  in  $\mathbf{r}(t)$  changes the parameter from  $t$  to  $u$ .

**SMOOTH FUNCTION**

When making such a change of parameter, it is important to ensure that the new vector-valued function of  $u$  is smooth if the original vector-valued function of  $t$  is smooth. It can be proved that this will be so if  $g$  satisfies the following conditions:

1.  $g$  is differentiable.
2.  $g'$  is continuous.
3.  $g'(u) \neq 0$  for any  $u$  in the domain of  $g$ .
4. The range of  $g$  is the domain of  $\mathbf{r}$ .

If  $g$  satisfies these conditions, then we call  $t = g(u)$  a smooth change of parameter. Henceforth, we shall assume that all changes of parameter are smooth, even if it is not stated explicitly.

**ARC LENGTH**

Because derivatives of vector-valued functions are calculated by differentiating components, it is natural to define integrals of vector-functions in terms of components.

**EXAMPLE:** If  $x'(t)$  and  $y'(t)$  are continuous for  $a \leq t \leq b$ , then the curve given by the parametric equations

$$x = x(t), \quad y = y(t) \quad (a \leq t \leq b) \quad (9)$$

has arc length

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad (10)$$

This result generalizes to curves in 3-spaces exactly as one would expect:

If  $x'(t)$ ,  $y'(t)$ , and  $z'(t)$  are continuous for  $a \leq t \leq b$ , then the curve given by the parametric equations

$$x = x(t), \quad y = y(t), \quad z = z(t) \quad (a \leq t \leq b)$$

has arc length

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \quad (12)$$

**EXAMPLE :** Find the arc length of that portion of the circular helix

$$x = \cos t, \quad y = \sin t, \quad z = t$$

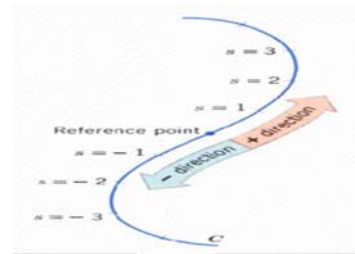
From  $t = 0$  to  $t = \pi$

The arc length is

$$\begin{aligned} L &= \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \int_0^\pi \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} dt \\ &= \int_0^\pi \sqrt{2} dt = \sqrt{2} \pi \end{aligned}$$

### ARC LENGTH AS A PARAMETER

For many purposes the best parameter to use for representing a curve in 2-space or 3-space parametrically is the length of arc measured along the curve from some fixed reference point. This can be done as follows:



Step 1: Select an arbitrary point on the curve  $C$  to serve as a reference point.

Step 2: Starting from the reference point, choose one direction along the curve to be the positive direction and the other to be the negative direction.

Step 3: If  $P$  is a point on the curve, let  $s$  be the “signed” arc length along  $C$  from the reference point to  $P$ , where  $s$  is positive if  $P$  is in the positive direction from the reference point, and  $s$  is negative if  $P$  is in the negative direction.

By this procedure, a unique point  $P$  on the curve is determined when a value for  $s$  is given. For example,  $s = 2$  determines the point that is 2 units along the curve in the positive direction from the reference point, and  $s = -\frac{3}{2}$  determines the point that is  $\frac{3}{2}$  units along the curve in the negative direction from the reference point.

Let us now treat  $s$  as a variable. As the value of  $s$  changes, the corresponding point  $P$  moves along  $C$  and the coordinates of  $P$  become functions of  $s$ . Thus, in 2-space the coordinates of  $P$  are  $(x(s), y(s))$ , and in 3-space they are  $(x(s), y(s), z(s))$ . Therefore, in 2-space the curve  $C$  is given by the parametric equations  $x = x(s)$ ,  $y = y(s)$  and in 3-space by  $x = x(s)$ ,  $y = y(s)$ ,  $z = z(s)$

### **REMARKS**

When defining the parameter  $s$ , the choice of positive and negative directions is arbitrary. However, it may be that the curve  $C$  is already specified in terms of some other parameter  $t$ , in which case we shall agree always to take the direction of increasing  $t$  as the positive direction for the parameter  $s$ . By so doing,  $s$  will increase as  $t$  increases and vice versa. The following theorem gives a formula for computing an arc-length parameter  $s$  when the curve  $C$  is expressed in terms of some other parameter  $t$ . This result will be used when we want to change the parameterization for  $C$  from  $t$  to  $s$ .

### **THEOREM**

(a) Let  $C$  be a curve in 2-space given parametrically by

$$x = x(t), \quad y = y(t)$$

where  $x'(t)$  and  $y'(t)$  are continuous functions. If an arc-length parameter  $s$  is introduced with its reference point at  $(x(t_0), y(t_0))$ , then the parameters  $s$  and  $t$  are related by

$$s = \int_{t_0}^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2} du \quad (13a)$$

(b) Let  $C$  be a curve in 3-space given parametrically by

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

where  $x'(t)$ ,  $y'(t)$ , and  $z'(t)$  are continuous functions. If an arc-length parameter  $s$  is introduced with its reference point at  $(x(t_0), y(t_0), z(t_0))$ , then the parameters  $s$  and  $t$  are related by

$$s = \int_{t_0}^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} du \quad (13b)$$

### **Proof**

If  $t > t_0$ , then from (10) (with  $u$  as the variable of integration rather than  $t$ ) it follows that

$$\int_{t_0}^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2} du \quad (14)$$

represents the arc length of that portion of the curve  $C$  that lies between  $(x(t_0), y(t_0))$  and  $(x(t), y(t))$ . If  $t < t_0$ , then (14) is the negative of this arc length. In either case, integral (14) represents the “signed” arc length  $s$  between these points, which proves (13a).

It follows from Formulas (13a) and (13b) and the Second Fundamental Theorem of Calculus (Theorem 5.9.3) that in 2-space,

$$\frac{ds}{dt} = \frac{d}{dt} \text{Error!} = \text{Error!}$$

and in 3-space

$$\frac{ds}{dt} = \frac{d}{dt} \text{Error!} = \text{Error!}$$

Thus, in 2-space and 3-space, respectively,

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \quad (15a)$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \quad (15b)$$

**REMARKS:**

Formulas (15a) and (15b) reveal two facts worth noting. First,  $\frac{ds}{dt}$  does not depend on  $t_0$ ;

that is, the value of  $\frac{ds}{dt}$  is independent of where the reference point for the parameter  $s$  is

located. This is to be expected since changing the position of the reference point shifts each value of  $s$  by a constant (the arc length between the reference points), and this constant drops out when we differentiate. The second fact to be noted from (15a) and

(15b) is that  $\frac{ds}{dt} \geq 0$  for all  $t$ . This is also to be expected since  $s$  increases with  $t$  by the remark preceding Theorem 15.3.2. If the curve  $C$  is smooth, then it follows from (15a) and (15b) that  $\frac{ds}{dt} \geq 0$  for all  $t$ .

**EXAMPLE**

$$x = 2t + 1, \quad y = 3t - 2 \quad (16)$$

using arc length  $s$  as a parameter, where the reference point for  $s$  is the point  $(1, -2)$ .

In formula (13a) we used  $u$  as the variable of integration because  $t$  was needed as a limit of integration. To apply (13a), we first rewrite the given parametric equations with  $u$  in place of  $t$ ; this gives

from which we obtain

$$\begin{aligned} x &= 2u + 1, & y &= 3u - 2 \\ \frac{dx}{du} &= 2, & \frac{dy}{du} &= 3 \end{aligned}$$

we see that the reference point  $(1, -2)$  corresponds to  $t = t_0 = 0$

$$s = \int_{t_0}^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2} du = \int_0^t \sqrt{13} du = \left[ \sqrt{13}u \right]_{u=0}^{u=t} = \sqrt{13}t$$

Therefore,  $t = \frac{1}{\sqrt{13}} s$

Substituting this expression in the given parametric equations yields.

$$\begin{aligned} x &= 2 \left( \frac{1}{\sqrt{13}} s \right) + 1 = \frac{2}{\sqrt{13}} s + 1 \\ y &= 3 \left( \frac{1}{\sqrt{13}} s \right) - 2 = \frac{3}{\sqrt{13}} s - 2 \end{aligned}$$

**EXAMPLE:** Find parametric equations for the circle  $x = a \cos t$ ,  $y = a \sin t$  ( $0 \leq t \leq 2\pi$ ) using arc length  $s$  as a parameter, with the reference point for  $s$  being  $(a, 0)$ , where  $a > 0$ .

**Solution:** We first replace  $t$  by  $u$  in the given equations so that  $x = a \cos u$ ,  $y = a \sin u$

$$\text{And } \frac{dx}{du} = -a \sin u, \quad \frac{dy}{du} = a \cos u$$

Since the reference point  $(a, 0)$  corresponds to  $t = 0$ , we obtain

$$s = \int_0^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2} du = \int_0^t \sqrt{(-a \sin u)^2 + (a \cos u)^2} du = \int_0^t a du = au \Big|_{u=0}^{u=t} = at$$

Solving for  $t$  in terms of  $s$  yields  $t = \frac{s}{a}$

Substituting this in the given parametric equations and using the fact that  $s = at$  ranges from 0 to  $2\pi a$  as  $t$  ranges from 0 to  $2\pi$ , we obtain

$$x = a \cos\left(\frac{s}{a}\right), \quad y = a \sin\left(\frac{s}{a}\right) \quad (0 \leq s \leq 2\pi a)$$

### EXAMPLE

Find Arc length of the curve  $\mathbf{r}(t) = t^3\mathbf{i} + t\mathbf{j} + \frac{1}{2}\sqrt{6}t^2\mathbf{k}$ ,  $1 \leq t \leq 3$

Here  $x = t^3$ ,  $y = t$ ,  $z = \frac{1}{2}\sqrt{6}t^2$

$$\frac{dx}{dt} = 3t^2, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = \sqrt{6}t$$

$$\begin{aligned} \text{Arc length} &= \int_1^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \int_1^3 \sqrt{9t^4 + 1 + 6t^2} dt = \int_1^3 \sqrt{(3t^2 + 1)^2} dt \\ &= \left| t^3 + t \right|_1^3 = (3)^3 + 3 - (1)^3 - 1 = 27 + 3 - 1 - 1 = 28 \end{aligned}$$

**EXAMPLE:** Calculate  $\frac{d\mathbf{r}}{du}$  by chain Rule, where  $\mathbf{r} = e^u\mathbf{i} + 4e^{-u}\mathbf{j}$  and  $t = u^2$

**Solution:**

$$\frac{d\mathbf{r}}{dt} = e^u\mathbf{i} - 4e^{-u}\mathbf{j}$$

$$\frac{dt}{du} = 2u$$

$$\frac{d\mathbf{r}}{du} = \frac{d\mathbf{r}}{dt} \cdot \frac{dt}{du} = (e^u\mathbf{i} - 4e^{-u}\mathbf{j}) \cdot (2u) = 2u e^{u^2}\mathbf{i} - 8u e^{-u^2}\mathbf{j}$$

By expressing  $\mathbf{r}$  in terms of  $u$

$$\mathbf{r} = e^u\mathbf{i} + 4e^{-u}\mathbf{j}$$

$$\frac{d\mathbf{r}}{du} = 2u e^{u^2}\mathbf{i} - 8u e^{-u^2}\mathbf{j}$$

**LECTURE No. 30****EXACT DIFFERENTIAL**

If  $z = f(x, y)$ , then  $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

The result can be extended to functions of more than two independent variables.

If  $z = f(x, y, w)$ ,  $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy + \frac{\partial z}{\partial w} dw$

Make a note of these results in differential form as shown.

**Exercise**

Determine the differential  $dz$  for each of the following functions.

1.  $z = x^2 + y^2$
2.  $z = x^3 \sin 2y$
3.  $z = (2x - 1) e^{3y}$
4.  $z = x^2 + 2y^2 + 3w^2$
5.  $z = x^3 y^2 w$ .

Finish all five and then check the result.

1.  $dz = 2(x dx + y dy)$
2.  $dz = x^2 (3 \sin 2y dx + 2x \cos 2y dy)$
3.  $dz = e^{3y} \{2dx + (6x - 3) dy\}$
4.  $dz = 2(xdx + 2ydy + 3wdw)$
5.  $dz = x^2 y (3ywdx + 2xw dy + xydw)$

**Exact Differential**

We have just established that if  $z = f(x, y)$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

We now work in reverse.

Any expression  $dz = Pdx + Qdy$ , where  $P$  and  $Q$  are functions of  $x$  and  $y$ , is an exact differential if it can be integrated to determine  $z$ .

$$\therefore P = \frac{\partial z}{\partial x} \text{ and } Q = \frac{\partial z}{\partial y}$$

$$\text{Now } \frac{\partial P}{\partial y} = \frac{\partial^2 z}{\partial y \partial x} \text{ and } \frac{\partial Q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y} \text{ and we know that } \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$$

Therefore, for  $dz$  to be an exact differential  $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$  and this is the test we apply.

**EXAMPLE**

$$dz = (3x^2 + 4y^2) dx + 8xy dy.$$

If we compare the right-hand side with  $Pdx + Qdy$ , then

$$P = 3x^2 + 4y^2 \therefore \frac{\partial P}{\partial y} = 8y$$

$$Q = 8xy \quad \therefore \frac{\partial Q}{\partial x} = 8y$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \therefore dz \text{ is an exact differential}$$

Similarly, we can test this one.

### EXAMPLE

$$dz = (1 + 8xy) dx + 5x^2 dy.$$

From this we find  $dz$  is not an exact differential

$$\text{for } dz = (1 + 8xy) dx + 5x^2 dy$$

$$\therefore P = 1 + 8xy \quad \therefore \frac{\partial P}{\partial y} = 8x$$

$$Q = 5x^2 \quad \therefore \frac{\partial Q}{\partial x} = 10x$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \quad \therefore dz \text{ is not an exact differential}$$

### EXERCISE

**Determine whether each of the following is an exact differential.**

1.  $dz = 4x^3y^3dx + 3x^4y^2 dy$
2.  $dz = (4x^3y+2xy^3) dx+(x^4+3x^2y^2) dy$
3.  $dz=(15y^2e^{3x}+2xy^2)dx+(10ye^{3x}+x^2y)dy$
4.  $dz=(3x^2e^{2y}-2y^2e^{3x})dx+(2x^3e^{2y}-2ye^{3x})dy$
5.  $dz=(4y^3\cos 4x+3x^2\cos 2y)dx+(3y^2\sin 4x-2x^3\sin 2y) dy.$

**1. Yes 2. Yes 3. No 4. No 5. Yes**

We have just tested whether certain expressions are, in fact, exact differentials—and we said previously that, by definition, an exact differential can be integrated. But how exactly do we go about it? The following examples will show.

### Integration Of Exact Differentials

$$dz = Pdx + Qdy \quad \text{where } P = \frac{\partial z}{\partial x} \quad \text{and } Q = \frac{\partial z}{\partial y}$$

$$\therefore z = \int Pdx \quad \text{and also} \quad z = \int Qdy$$

### Example

$$dz = (2xy + 6x) dx + (x^2 + 2y^3) dy.$$

$$P = \frac{\partial z}{\partial x} = 2xy + 6x \quad \therefore z = \int (2xy + 6x) dx$$

$\therefore z = x^2y + 3x^2 + f(y)$  where  $f(y)$  is an arbitrary function of  $y$  only, and is akin to the constant of integration in a normal integral.

Also  $Q = \frac{\partial z}{\partial y} = x^2 + 2y^3 \quad \therefore z = \int (x^2 + 2y^3) dy$

$\therefore z = x^2y + \frac{y^4}{2} + F(x)$  where  $F(x)$  is an arbitrary function of  $x$  only.

$z = x^2y + 3x^2 + f(y) \quad (i)$

and  $z = x^2y + \frac{y^4}{2} + F(x) \quad (ii)$

For these two expressions to represent the same function, then

$f(y)$  in (i) must be  $\frac{y^4}{2}$  already in (i)

and  $F(x)$  in (ii) must be  $3x^2$  already in (i)

$\therefore z = x^2y + 3x^2 + \frac{y^4}{2}$

### EXAMPLE

**Integrate  $dz = (8e^{4x} + 2xy^2) dx + (4 \cos 4y + 2x^2y) dy$ .**

$dz = (8e^{4x} + 2xy^2) dx + (4 \cos 4y + 2x^2y) dy$

$P = \frac{\partial z}{\partial x} = 8e^{4x} + 2xy^2$

$\therefore z = \int (8e^{4x} + 2xy^2) dx$

$\therefore z = 2e^{4x} + x^2y^2 + f(y) \quad (i)$

$Q = \frac{\partial z}{\partial y} = 4 \cos 4y + 2x^2y$

$\therefore z = \int (4 \cos 4y + 2x^2y) dy$

$\therefore z = \sin 4y + x^2y^2 + F(x) \quad (ii)$

For (i) and (ii) to agree,  $f(y) = \sin 4y$  and  $F(x) = 2e^{4x}$

$\therefore z = 2e^{4x} + x^2y^2 + \sin 4y$

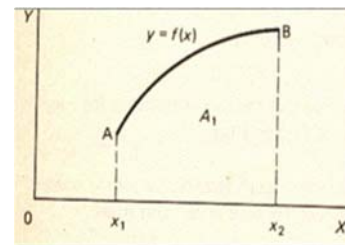
### Area enclosed by the closed curve

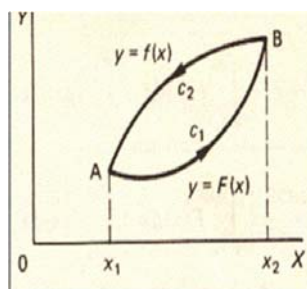
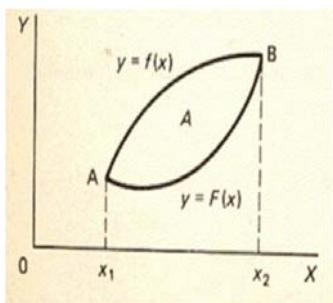
One of the earliest applications of integration is finding the area of a plane figure bounded by the  $x$ -axis, the curve  $y = f(x)$  and ordinates at  $x=x_1$  and  $x=x_2$ .

$$A_1 = \int_{x_1}^{x_2} y dx = \int_{x_1}^{x_2} f(x) dx$$

If points A and B are joined by another curve,  $y = F(x)$

$$A_2 = \int_{x_1}^{x_2} f(x) dx$$





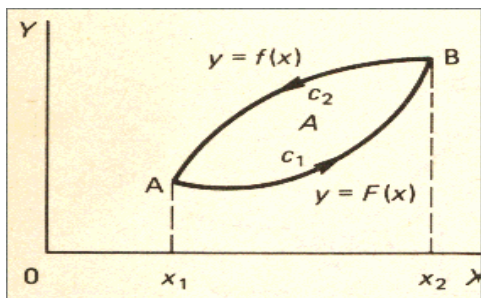
Combining the two figures, we have

$$A = A_1 - A_2 \quad \therefore A = \int_{x_1}^{x_2} F(x) dx - \int_{x_1}^{x_2} f(x) dx$$

The final result above can be written in the form

$$A = -\oint y dx$$

Where the symbol  $\oint$  indicates that the integral is to be evaluated round the closed boundary in the positive



### EXAMPLE

Determine the area enclosed by the graph of  $y = x^3$  and  $y = 4x$  for  $x \geq 0$ .

First we need to know the points of intersection. These are  $x = 0$  and  $x = 2$

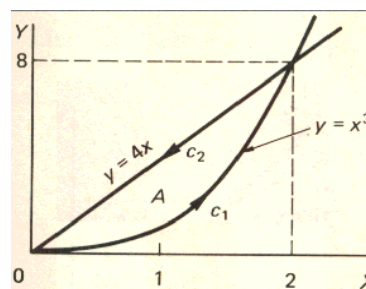
We integrate in an anticlockwise manner

$c_1$ :  $y = x^3$ , limits  $x = 0$  to  $x = 2$

$c_2$ :  $y = 4x$ , limits  $x = 2$  to  $x = 0$ .

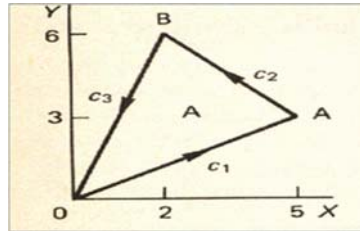
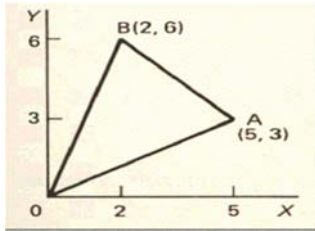
$$A = -\oint y dx = A = 4 \text{ square units}$$

$$\text{For } A = -\oint y dx = -\text{Error!} = -\text{Error!} = 4$$



**EXAMPLE**

Find the area of the triangle with vertices  $(0, 0)$ ,  $(5, 3)$  and  $(2, 6)$ .



The equation of OA is  $y = \frac{3}{5}x$ , BA is  $y = 8 - x$ , OB is  $y = 3x$

Then  $A = - \oint y \, dx$

Write down the component integrals with appropriate limits.

-----

$A = - \oint y \, dx =$  **Error!**

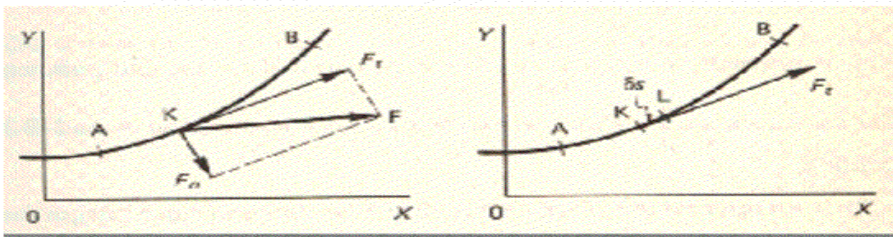
The limits chosen must progress the integration round the boundary of the figure in an anticlockwise manner. Finishing off the integration, we have

$A = 12$  square units

The actual integration is easy enough. The work we have just done leads us on to consider line integrals, so let us make a fresh start in the next frame.

**Line Integrals**

If a field exists in the  $xy$ -plane, producing a force  $F$  on a particle at  $K$ , then  $F$  can be resolved into two components.  $F_1$  along the tangent to the curve  $AB$  at  $K$ .  $F_2$  along the normal to the curve  $AB$  at  $K$ .

**Line Integrals**

The work done in moving the particle through a small distance  $\delta s$  from  $K$  to  $L$  along the curve is then approximately  $F_1 \delta s$ . So the total work done in moving a particle along the curve from  $A$  to  $B$  is given by

$$\lim_{\delta \rightarrow 0} \sum F_t \delta s = \int F_t \, ds \text{ from } A \text{ to } B$$

This is normally written  $\int_{AB} F_t \, ds$  where  $A$  and  $B$  are the end points of the curve,

or as  $\int_C F_t ds$  where the curve  $c$  connecting  $A$  and  $B$  is defined.

Such an integral thus formed, is called a line integral since integration is carried out along the path of the particular curve  $c$  joining  $A$  and  $B$ .

$$\therefore I = \int_{AB} F_t dx = \int_C F_t ds$$

where  $c$  is the curve  $y = f(x)$  between  $A(x_1, y_2)$  and  $B(x_2, y_2)$ .

There is in fact an alternative form of the integral which is often useful, so let us also consider that.

### **Alternative form of a line integral**

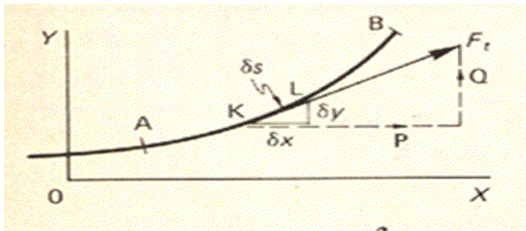
It is often more convenient to integrate with respect to  $x$  or  $y$  than to take arc length as the variable.

If  $F_t$  has a component

$P$  in the  $x$ -direction

$Q$  in the  $y$ -direction

then the work done from  $K$  to  $L$  can be stated as  $P\delta x + Q\delta y$



$$\therefore \int_{AB} F_t ds = \int_{AB} (P dx + Q dy)$$

where  $P$  and  $Q$  are functions of  $x$  and  $y$ .

In general then, the line integral can be expressed as

$$I = \int_C F_t ds = \int_C (P dx + Q dy)$$

where  $c$  is the prescribed curve and  $F$ , or  $P$  and  $Q$ , are functions of  $x$  and  $y$ .

Make a note of these results –then we will apply them to one or two examples.

## LECTURE No.31

## LINE INTEGRAL

The work done in moving the particle through a small distance  $\delta s$  from K to L along the curve is then approximately  $F_t \delta s$ . So the total work done in moving a particle along the curve from A to B is given by

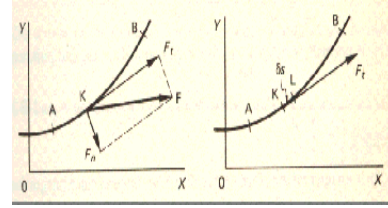
$$\lim_{\delta \rightarrow 0} \sum F_t \delta s = \int_C F_t ds \text{ from A to B}$$

This is normally written  $\int_{AB} F_t ds$  where A and B are the end points of the curve, or as  $\int_C F_t ds$  where the curve  $c$  connecting A and B is defined. Such an integral thus formed, is called a line integral since integration is carried out along the path of the particular curve  $c$  joining A and B.

$$\therefore I = \int_{AB} F_t dx = \int_C F_t ds$$

where  $c$  is the curve  $y = f(x)$  between  $A(x_1, y_1)$  and  $B(x_2, y_2)$ .

There is in fact an alternative form of the integral which is often useful, so let us also consider that.

**Alternative form of a line integral**

It is often more convenient to integrate with respect to  $x$  or  $y$  than to take arc length as the variable.

If  $F_t$  has a component  $P$  in the  $x$ -direction,  $Q$  in the  $y$ -direction then the work done from K to L can be stated as  $P\delta x + Q\delta y$

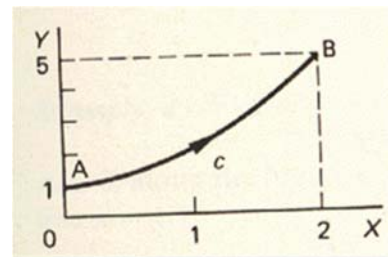
**Example 1:**

Evaluate  $\int_C (x + 3y) dx$  from A (0, 1) to B (2, 5)

along the curve  $y = 1 + x^2$ .

**Solution:** The line integral is of the form

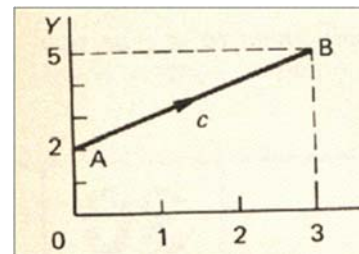
$\int_C (P dx + Q dy)$  where, in this case,  $Q = 0$  and  $c$  is the curve  $y = 1 + x^2$ .



It can be converted at once into an ordinary integral by substituting for  $y$  and applying the appropriate limits of  $x$ .

$$I = \int_C (P dx + Q dy) = \int_C (x + 3y) dx = \int_0^2 (x + 3(1 + x^2)) dx$$

$$= \left[ \frac{x^2}{2} + 3x + x^3 \right]_0^2 = 16$$



**Example 2**

Evaluate  $I = \int_C (x^2 + y) dx + (x - y^2) dy$  from A (0, 2) to B (3, 5) along the curve  $y = 2 + x$ .

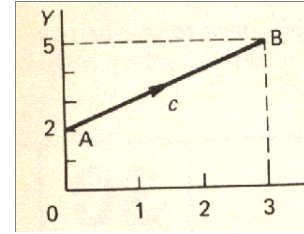
**Solution:**  $I = \int_C (Pdx + Qdy)$

$$P = x^2 + y = x^2 + 2 + x = x^2 + x + 2$$

$$Q = x - y^2 = x - (4 + 4x + x^2) = -(x^2 + 3x + 4)$$

Also  $y = 2 + x$

$\therefore dy = dx$  and the limits are  $x=0$  to  $x=3$



$$I = \int_0^3 \{ (x^2 + x + 2) dx - (x^2 + 3x + 4) dx \} = \int_0^3 -(2x + 2) dx = \left[ -x^2 - 2x \right]_0^3 = -9 - 6 = -15$$

**Example 3**

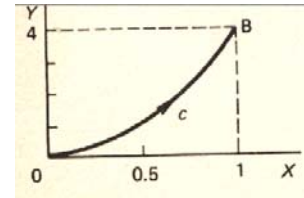
Evaluate  $I = \int_C \{ (x^2 + 2y) dx + xy dy \}$  from O(0, 0) to B(1, 4) along the curve  $y = 4x^2$ .

**Solution:** In this case,  $c$  is the curve  $y = 4x^2$ .

$\therefore dy = 8x dx$

Substitute for  $y$  in the integral and apply the limits.

$$I = \int_C \{ (x^2 + 2y) dx + xy dy \}$$



also  $x^2 + 2y = x^2 + 8x^2 = 9x^2$ ;  $xy = 4x^3$

$$\therefore I = \int_0^1 (9x^2 dx + x(4x^2)(8x dx)) = \int_0^1 \{ 9x^2 dx + 32x^4 dx \} = \frac{47}{5} = 9.4$$

They are all done in very much the same way.

**Example 4**

Evaluate  $I = \int_C \{ (x^2 + 2y) dx + xy dy \}$  from O(0, 0) to A (1, 0) along the line  $y = 0$  and then from A (1, 0) to B (1, 4) along the line  $x = 1$ .

**Solution:** (i) OA :  $c_1$  is the line  $y = 0 \therefore dy = 0$ .

Substituting  $y = 0$  and  $dy = 0$  in the given integral gives.

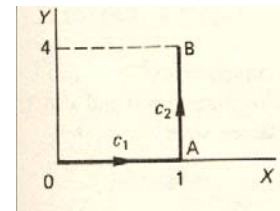
$$I_{OA} = \int_0^1 x^2 dx = \left[ \frac{x^3}{3} \right]_0^1 = \frac{1}{3}$$

(ii) AB: Here  $c_2$  is the line  $x = 1 \therefore dx = 0$

$\therefore I_{AB} = 8$

$$\text{For } I_{AB} = \int_0^4 \{ (1 + 2y) (0) + y dy \} = \int_0^4 y dy = \left[ \frac{y^2}{2} \right]_0^4 = 8$$

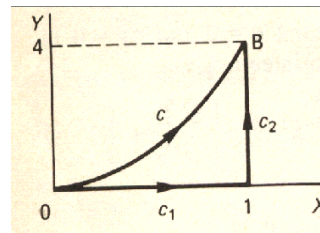
$$\text{Then } I = I_{OA} + I_{AB} = \frac{1}{3} + 8 = 8\frac{1}{3} \therefore I = \frac{25}{3} = 8\frac{1}{3}$$



If we now look back to Example 3 and 4 just completed, we find that we have evaluated the same integral between the same two end points, but along different paths of integration. If we combine the two diagrams, we have where  $c$  is the curve  $y = 4x^2$  and  $c_1 + c_2$  are the lines  $y = 0$  and  $x = 1$ . The result obtained were

$$I_c = 9\frac{2}{3} \text{ and } I_{c_1+c_2} = 8\frac{1}{3}$$

**Remark:** The integration along two distinct paths joining the same two end points does not necessarily give the same results.



### Properties of line integrals

$$1. \quad \int_C F \, ds = \int_C \{P \, dx + Q \, dy\}$$

$$2. \quad \int_{AB} F \, ds = - \int_{BA} F \, ds \quad \text{and} \quad \int_{AB} \{P \, dx + Q \, dy\} = \int_{BA} \{P \, dx + Q \, dy\}$$

i.e. the sign of a line integral is reversed when the direction of the integration along the path is reversed.

3. (a) For a path of integration parallel to the y-axis, i.e.  $x = k$ ,  $dx = 0$

$$\therefore \int_C P \, dx = 0 \quad \therefore I_C = \int_C Q \, dy.$$

(b) For a path of integration parallel to the x-axis, i.e.  $y = k$ ,  $dy = 0$ .

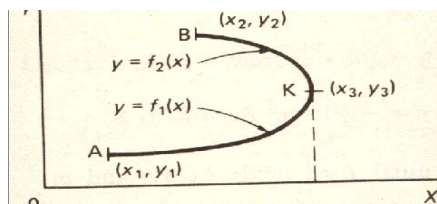
$$\therefore \int_C Q \, dy = 0 \quad \therefore I_C = \int_C P \, dx.$$

4. If the path of integration  $c$  joining  $A$  to  $B$  is divided into two parts  $AK$  and  $KB$ , then

$$I_c = I_{AB} = I_{AK} + I_{KB}.$$

5. If the path of integration  $c$  is not single valued for part of its extent, the path is divided into two sections.

$y = f_1(x)$  from  $A$  to  $K$ ,  $y = f_2(x)$  from  $K$  to  $B$ .



6. In all cases, the actual path of integration involved must be continuous and single-valued.

### Example 5

Evaluate  $I = \int_C (x + y) \, dx$  from  $A(0, 1)$  to  $B(0, -1)$  along the semi-circle  $x^2 + y^2 = 1$

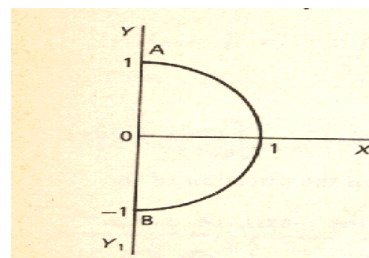
for  $x \geq 0$ .

**Solution:** The first thing we notice is that the path of integration  $c$  is not single-valued

For any value of  $x$ ,  $y = \pm \sqrt{1 - x^2}$ . Therefore, we divided  $c$  into two parts

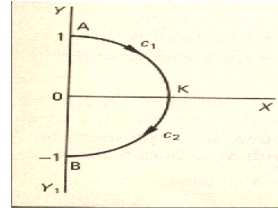
$$(i) \quad y = \sqrt{1 - x^2} \text{ from } A \text{ to } K \text{ ( } x=0 \text{ to } x=1)$$

$$(ii) \quad y = -\sqrt{1 - x^2} \text{ from } K \text{ to } B \text{ ( } x=1 \text{ to } x=0)$$



As usual,  $I = \int_C (Pdx + Qdy)$  and in this particular case,  $Q = 0$

$$\begin{aligned}\therefore I &= \int_C Pdx = \int_0^1 (x + \sqrt{1-x^2}) dx + \int_1^0 (x - \sqrt{1-x^2}) dx \\ &= \int_0^1 (x + \sqrt{1-x^2} - x + \sqrt{1-x^2}) dx = 2 \int_0^1 \sqrt{1-x^2} dx\end{aligned}$$



Now by trigonometric substitution, put  $x = \sin \theta$

$$\therefore dx = \cos \theta d\theta \quad \text{and} \quad \sqrt{1-x^2} = \sqrt{1-\sin^2 \theta} = \sqrt{\cos^2 \theta} = \cos \theta$$

Limits :  $x = 0, \theta = 0; x = 1, \theta = \frac{\pi}{2}$

$$\begin{aligned}I &= 2 \int_0^1 \sqrt{1-x^2} dx \\ &= 2 \int_0^{\pi/2} \cos \theta \cos \theta d\theta = 2 \int_0^{\pi/2} \cos^2 \theta d\theta = \int_0^{\pi/2} (1 + \cos 2\theta) d\theta = \left[ \theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2} = \frac{\pi}{2}\end{aligned}$$

Now let us extend this line of development a stage further.

### Example 6

Evaluate the line integral

$I = \oint (x^2 dx - 2xy dy)$  where  $c$  comprises the three sides of the triangle joining  $O(0, 0)$ ,  $A(1, 0)$  and  $B(0, 1)$ .

**Solution:** First draw the diagram and mark in  $c_1, c_2$  and  $c_3$ , the proposed directions of integration. Do just that. The three sides of the path of integration must be arranged in an anticlockwise manner round the figure.

Now we deal with each part separately.

(a)  $OA : c_1$  is the line  $y = 0$

Therefore,  $dy = 0$ .

Then  $I = \oint (x^2 dx - 2xy dy)$  for this part becomes

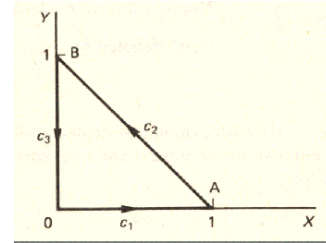
$$I_1 = \int_0^1 x^2 dx = \left[ \frac{x^3}{3} \right]_0^1 = \frac{1}{3}$$

(b)  $AB : c_2$  is the line  $y = 1 - x$

$\therefore dy = -dx$ .

$$I_2 = \int_1^0 \{x^2 dx + 2x(1-x)dx\} = \int_1^0 (x^2 + 2x - 2x^2) dx = \int_1^0 (2x - x^2) dx = \left[ x^2 - \frac{x^3}{3} \right]_1^0 = -\frac{2}{3}$$

$$\therefore I_2 = -\frac{2}{3}$$



Note that anticlockwise progression is obtained by arranging the limits in the appropriate order. Now we have to determine  $I_3$  for BO.

(c) BO:  $c_3$  is the line  $x = 0$

$$\therefore dx = 0 \quad \therefore I_3 = \int 0 dy = 0 \quad \therefore I_3 = 0$$

$$\text{Finally, } I = I_1 + I_2 + I_3 = \frac{1}{3} - \frac{2}{3} + 0 = -\frac{1}{3} \quad \therefore I = -\frac{1}{3}$$

### Example 7

Evaluate  $\oint_C y \, dx$  when  $c$  is the circle  $x^2 + y^2 = 4$ .

**Solution:**  $x^2 + y^2 = 4 \quad \therefore y = \pm \sqrt{4 - x^2}$

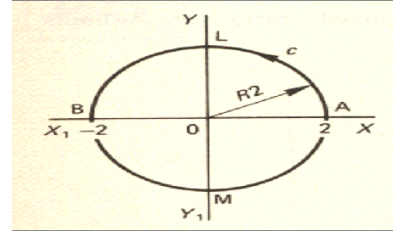
$y$  is thus not single-valued. Therefore use

$y = \sqrt{4 - x^2}$  for ALB between

$x = 2$  and  $x = -2$  and

$y = -\sqrt{4 - x^2}$  for BMA between

$x = -2$  and  $x = 2$ .



$$\begin{aligned} \therefore I &= \int_{-2}^2 \sqrt{4 - x^2} \, dx + \int_{-2}^2 \{-\sqrt{4 - x^2}\} \, dx \\ &= -2 \int_{-2}^2 \sqrt{4 - x^2} \, dx = -4 \int_0^2 \sqrt{4 - x^2} \, dx. \end{aligned}$$

$$I = -4 \int_0^2 \sqrt{4 - x^2} \, dx$$

$$\text{Put } x = 2 \sin \theta \Rightarrow \frac{dx}{d\theta} = 2 \cos \theta \Rightarrow dx = 2 \cos \theta \, d\theta$$

$$\text{When } x=0, \quad 0 = 2 \sin \theta \Rightarrow 0 = \sin \theta \Rightarrow \theta = 0$$

$$\text{When } x=2, \quad 2 = 2 \sin \theta \Rightarrow 1 = \sin \theta \Rightarrow \theta = \frac{\pi}{2}$$

$$\begin{aligned} I &= -4 \int_0^{\frac{\pi}{2}} \sqrt{4 - 4 \sin^2 \theta} \cdot 2 \cos \theta \, d\theta = -16 \int_0^{\frac{\pi}{2}} \sqrt{\cos^2 \theta} \cos \theta \, d\theta \\ &= -16 \int_0^{\frac{\pi}{2}} \cos \theta \cos \theta \, d\theta = -16 \int_0^{\frac{\pi}{2}} \cos^2 \theta \, d\theta = -8 \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2\theta}{2} \, d\theta \\ &= -8 \int_0^{\frac{\pi}{2}} (1 + \cos 2\theta) \, d\theta = -8 \left[ \theta + \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{2}} = -8 \left( \left( \frac{\pi}{2} - 0 \right) + \frac{1}{2} (0 - 0) \right) \\ &= -4\pi \end{aligned}$$

## LECTURE No. 32

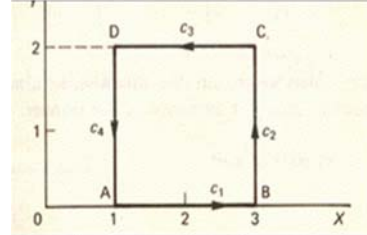
## EXAMPLES

**Example 1: Evaluate  $I = \oint \{xydx + (1+y^2)dy\}$  where  $c$  is the boundary of the rectangle joining  $A(1,0)$ ,  $B(3,0)$ ,  $C(3,2)$ ,  $D(1,2)$ .**

**Solution:** First draw the diagram and insert  $c_1, c_2, c_3, c_4$ .

That give

Now evaluate  $I_1$  for  $AB$ ;  $I_2$  for  $BC$ ;  $I_3$  for  $CD$ ;  $I_4$  for  $DA$ ; and finally  $I$ .



$$I = \oint \{xydx + (1+y^2)dy\}$$

(a)  $AB: c_1$  is  $y = 0 \therefore dy = 0 \therefore I_1 = 0$

(b)  $BC: c_2$  is  $x = 3 \therefore dx = 0$

$$\therefore I_2 = \int_0^2 (1+y^2)dy = \left[ y + \frac{y^3}{3} \right]_0^2 = 4\frac{2}{3} \therefore I_2 = 4\frac{2}{3}$$

(c)  $CD: c_3$  is  $y = 2 \therefore dy = 0$

$$\therefore I_3 = \int_3^1 2xdx = \left[ x^2 \right]_3^1 = -8 \therefore I_3 = -8$$

(d)  $DA: c_4$  is  $x = 1 \therefore dx = 0$

$$\therefore I_4 = \int_2^0 (1+y^2)dy = \left[ y + \frac{y^3}{3} \right]_2^0 = -4\frac{2}{3}$$

$$\text{Finally } I = I_1 + I_2 + I_3 + I_4 = 0 + 4\frac{2}{3} - 8 - 4\frac{2}{3} = -8 \therefore I = -8$$

**Remember** that, unless we are directed otherwise, we always proceed round the closed boundary in an anticlockwise manner.

### Line integral with respect to arc length

We have already established that

$$I = \int_{AB} F_t ds = \int_{AB} \{Pdx + Qdy\}$$

where  $F_t$  denoted the tangential force along the curve  $c$  at the sample point  $K(x,y)$ .

The same kind of integral can, of course, relate to any function  $f(x,y)$  which is a function of the position of a point on the stated curve, so that

$$I = \int_C f(x, y) ds.$$

This can readily be converted into an integral in terms of  $x$ :

$$I = \int_C f(x,y)dx = \int_C f(x,y) \frac{ds}{dx} dx$$

$$\text{where } \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\therefore \int_C f(x,y) dx = \int_{x_1}^{x_2} f(x,y) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \text{----- (1)}$$

**Example 2**

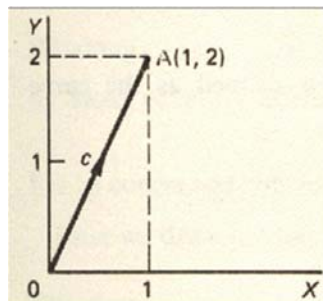
**Evaluate  $I = \int_C (4x+3xy)ds$  where  $c$  is the straight line joining  $O(0,0)$  to  $A(1,2)$ .**

**Solution:**  $c$  is the line  $y = 2x \therefore \frac{dy}{dx} = 2$

$$\therefore \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{5}$$

$$\therefore I = \int_{x=0}^{x=1} (4x+3xy)ds = \int_0^1 (4x+3xy)(\sqrt{5}) dx. \quad \text{But } y = 2x$$

$$\text{for } I = \int_0^1 (4x+6x^2)(\sqrt{5}) dx = 2\sqrt{5} \int_0^1 (2x+3x^2) dx = 4\sqrt{5}$$

**Parametric Equations**

When  $x$  and  $y$  are expressed in parametric form, e.g.  $x = x(t)$ ,  $y = y(t)$ , then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \quad \therefore ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$I = \int_C f(x,y) ds = \int_{t_1}^{t_2} f(x,y) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \text{----- (2)}$$

**Example 3 : Evaluate  $I = \oint 4xyds$  where  $c$  is defined as the curve  $x = \sin t$ ,  $y = \cos t$  between  $t=0$  and  $t=\frac{\pi}{4}$ .**

**Solution:** We have  $x = \sin t \therefore \frac{dx}{dt} = \cos t$ ,

$$y = \cos t \therefore \frac{dy}{dt} = -\sin t$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\cos^2 t + \sin^2 t} = 1 \quad \therefore ds = dt$$

$$\begin{aligned} \therefore I &= \int_C f(x,y) ds = \int_{t_1}^{t_2} f(x,y) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^{\pi/4} 4 \sin t \cos t dt = 2 \int_0^{\pi/4} \sin 2t dt = -2 \left[ \frac{\cos 2t}{2} \right]_0^{\pi/4} = 1 \end{aligned}$$

### Dependence of the line integral on the path of integration

We know that integration along two separate paths joining the same two end points does not necessarily give identical results. With this in mind, let us investigate the following problem.

**Example 4 :** Evaluate  $I = \oint_C \{3x^2 y^2 dx + 2x^3 y dy\}$  between O (0, 0) and A (2, 4)

(a) along  $c_1$  i.e.  $y = x^2$

(b) along  $c_2$  i.e.  $y = 2x$

(c) along  $c_3$  i.e.  $x = 0$  from (0,0) to (0,4) and  $y = 4$  from (0,4) to (2,4).

**Solution:**

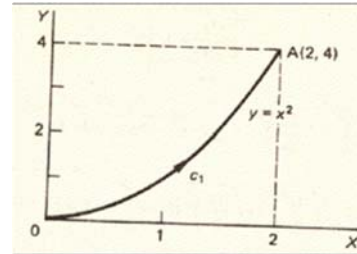
(a). First we draw the figure and insert relevant information.

$$I = \int_C \{3x^2 y^2 dx + 2x^3 y dy\}$$

The path  $c_1$  is  $y = x^2 \therefore dy = 2x dx$

$$\therefore I_1 = \int_0^2 \{3x^2 x^4 dx + 2x^3 x^2 2x dx\} = \int_0^2 (3x^6 + 4x^6) dx$$

$$\therefore I_1 = \left[ x^7 \right]_0^2 = 128 \therefore I_1 = 128$$



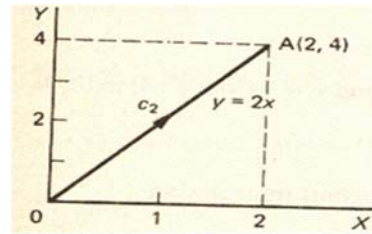
(b) Here the path of integration is  $c_2$ , i.e.  $y = 2x$

So, in this case, for with  $c_2$ ,  $y = 2x \therefore dy = 2dx$

$$\therefore I_2 = \int_0^2 \{3x^2 4x^2 dx + 2x^3 2x 2dx\}$$

$$= \int_0^2 (12x^4 dx + 8x^4 dx)$$

$$= \int_0^2 20x^4 dx = 4 \left[ x^5 \right]_0^2 = 128 \therefore I_2 = 128$$



(c) In the third case, the path  $c_3$  is split

$x = 0$  from (0,0) to (0, 4),

$y = 4$  from (0, 4) to (2, 4)

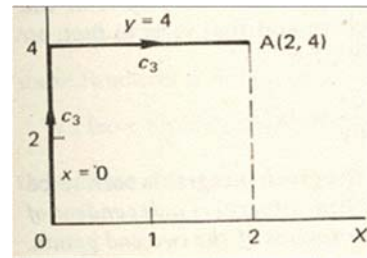
Sketch the diagram and determine  $I_3$ .

from (0,0) to (0,4)  $x=0 \therefore dx=0 \therefore I_{3a}=0$

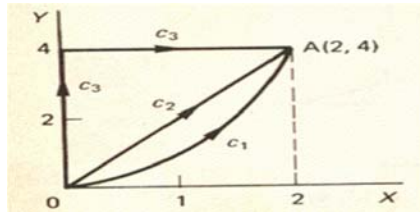
from (0,4) to (2,4)  $y=4 \therefore dy=0$

$$\therefore I_{3b} = \int_0^2 48x^2 dx = 128$$

$$\therefore I_3 = 0 + 128 = 128$$



In the example we have just worked through, we took three different paths and in each case, the line integral produced the same result. It appears, therefore, that in this case, the value of the integral is independent of the path of integration taken.



We have been dealing with  $I = \int_C \{3x^2y^2dx + 2x^3ydy\}$

On reflection, we see that the integrand  $3x^2y^2 dx + 2x^3y dy$  is of the form  $P dx + Q dy$  which we have met before and that it is, in fact, an exact differential of the function

$$z = x^3y^2, \text{ for } \frac{\partial z}{\partial x} = 3x^2y^2 \text{ and } \frac{\partial z}{\partial y} = 2x^3y$$

This always happens. If the integrand of the given integral is seen to be an exact differential, then the value of the line integral is independent of the path taken and depends only on the coordinates of the two end points.

## LECTURE No.33

## EXAMPLES

**Example 1:** Evaluate  $I = \int_C \{3y dx + (3x+2y) dy\}$  from A(1, 2) to B (3, 5).

**Solution:** No path is given, so the integrand is doubtless an exact differential of some function  $z = f(x, y)$ . In fact  $\frac{\partial P}{\partial y} = 3 = \frac{\partial Q}{\partial x}$ . We have already dealt with the integration of

exact differentials, so there is no difficulty. Compare with  $I = \int_C \{P dx + Q dy\}$ .

$$P = \frac{\partial z}{\partial x} = 3y \quad \therefore z = \int 3y dx = 3xy + f(y) \quad \text{----- (i)}$$

$$Q = \frac{\partial z}{\partial y} = 3x + 2y \quad \therefore z = \int (3x + 2y) dy = 3xy + y^2 + F(x) \quad \text{----(ii)}$$

For (i) and (ii) to agree  $f(y) = y^2$ ;  $F(x) = 0$

Hence  $z = 3xy + y^2$

$$\therefore I = \int_C \{3y dx + (3x+2y) dy\} = \int_{(1,2)}^{(3,5)} d(3xy + y^2) = [3xy + y^2]_{(1,2)}^{(3,5)} = (45+25) - (6+4) = 60$$

**Example 2:** Evaluate  $I = \int_C \{(x^2 + ye^x) dx + (e^x + y) dy\}$  between A (0, 1) and B (1, 2).

**Solution:** As before, compare with  $\int_C \{P dx + Q dy\}$ .

$$P = \frac{\partial z}{\partial x} = x^2 + ye^x \quad \therefore z = \frac{x^3}{3} + ye^x + f(y)$$

$$Q = \frac{\partial z}{\partial y} = e^x + y \quad \therefore z = ye^x + \frac{y^2}{2} + F(x)$$

For these expressions to agree,

$$f(y) = \frac{y^2}{2}; F(x) = \frac{x^3}{3} \quad \text{Then } I = \left[ \frac{x^3}{3} + ye^x + \frac{y^2}{2} \right]_{(0,1)}^{(1,2)} = \frac{5}{6} + 2e$$

**REMARKS:** The main points are that, if  $(Pdx + Qdy)$  is an exact differential

(a)  $I = \int_C (Pdx + Qdy)$  is independent of the path of integration

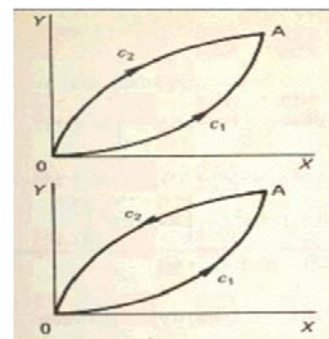
(b)  $I = \oint_C (P dx + Q dy)$  is zero.

If  $I = \int_C \{P dx + Q dy\}$  and  $(Pdx + Qdy)$  is an exact differential,

Then  $I_{c_1} = - I_{c_2}$

$$I_{c_1} + I_{c_2} = 0$$

Hence, the integration taken round a closed curve is zero, provided  $(Pdx + Qdy)$  is an exact differential.



∴ If  $(P dx + Q dy)$  is an exact differential,  $\oint (P dx + Q dy) = 0$

### Exact differentials in three independent variables

A line integral in space naturally involves three independent variables, but the method is very much like that for two independent variables.

$dz = Pdx + Q dy + R dw$  is an exact differential of  $z = f(x, y, w)$

$$\text{if } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}; \frac{\partial P}{\partial w} = \frac{\partial R}{\partial x}; \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial w}$$

If the test is successful, then

(a)  $\int_C (P dx + Q dy + R dw)$  is independent of the path of integration.

(b)  $\oint_C (P dx + Q dy + R dw)$  is zero.

**Example 3:** Verify that  $dz = (3x^2yw + 6x) dx + (x^3w - 8y) dy + (x^3y + 1) dw$  is an exact differential and hence evaluate  $\int_C dz$  from A (1, 2, 4) to B (2, 1, 3).

**Solution:** First check that  $dz$  is an exact differential by finding the partial derivatives above, when

$$P = 3x^2yw + 6x; \quad Q = x^3w - 8y; \quad \text{and} \quad R = x^3y + 1$$

$$\frac{\partial P}{\partial y} = 3x^2w; \quad \frac{\partial Q}{\partial x} = 3x^2w \quad \therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\frac{\partial P}{\partial w} = 3x^2y; \quad \frac{\partial R}{\partial x} = 3x^2y \quad \therefore \frac{\partial P}{\partial w} = \frac{\partial R}{\partial x}$$

$$\frac{\partial R}{\partial y} = x^3; \quad \frac{\partial Q}{\partial w} = x^3 \quad \therefore \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial w}$$

∴  $dz$  is an exact differential

Now to find  $z$ .  $P = \frac{\partial z}{\partial x}; \quad Q = \frac{\partial z}{\partial y}; \quad R = \frac{\partial z}{\partial w}$

$$\therefore \frac{\partial z}{\partial x} = 3x^2yw + 6x \quad \therefore \quad z = \int (3x^2yw + 6x) dx = x^3yw + 3x^2 + f(y) + F(w)$$

$$\therefore \frac{\partial z}{\partial y} = x^3w - 8x \quad \therefore \quad z = \int (x^3w - 8y) dy = x^3yw - 4y^2 + g(x) + F(w)$$

$$\frac{\partial z}{\partial w} = x^3y + 1 \quad \therefore \quad z = \int (x^3y + 1) dw = x^3yw + w + f(y) + g(x)$$

For these three expressions for  $z$  to agree

$$f(y) = -4y^2; \quad F(w) = w; \quad g(x) = 3x^2$$

$$\therefore \quad z = x^3yw + 3x^2 - 4y^2 + w$$

$$\therefore \quad I = \left[ x^3yw + 3x^2 - 4y^2 + w \right]_{(1,2,4)}^{(2,1,3)}$$

$$\text{for} \quad I = \left[ x^3yw + 3x^2 - 4y^2 + w \right]_{(1,2,4)}^{(2,1,3)} = (24 + 12 - 4 + 3) - (8 + 3 - 16 + 4) = 36$$

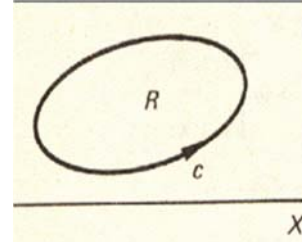
The extension to line integrals in space is thus quite straightforward.

Finally, we have a theorem that can be very helpful on occasions and which links up with the work we have been doing. It is important, so let us start a new section.

### **Green's Theorem**

Let  $P$  and  $Q$  be two function of  $x$  and  $y$  that are finite and continuous inside and the boundary  $c$  of a region  $R$  in the  $xy$ -plane. If the first partial derivatives are continuous within the region and on the boundary, then Green's theorem states that.

$$\iint_R \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) dx dy = - \oint_C (P dx + Q dy)$$



That is, a double integral over the plane region  $R$  can be transformed into a line integral over the boundary  $c$  of the region and the action is reversible.

Let us see how it works.

#### **EXAMPLE 4**

Evaluate  $I = \oint_C \{(2x - y)dx + (2y + x)dy\}$  around the boundary  $c$  of the ellipse  $x^2 + 9y^2 = 16$ .

**Solution:** The integral is of the form

$$I = \oint_C \{P dx + Q dy\} \text{ where } P = 2x - y \therefore \frac{\partial P}{\partial y} = -1 \text{ and } Q = 2y + x \therefore \frac{\partial Q}{\partial x} = 1.$$

$$\therefore I = - \iint_R \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) dx dy = - \iint_R (-1 - 1) dx dy = 2 \iint_R dx dy = 2A$$

But  $\iint_R dx dy$  over any closed region give the area of the figure.

In this case, then,  $I = 2A$  where  $A$  is the area of the ellipse ( $A = \pi ab$ )

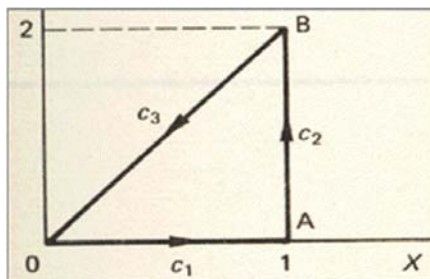
$$x^2 + 9y^2 = 16 \text{ i.e. } \frac{x^2}{16} + \frac{9y^2}{16} = 1$$

$$\therefore a = 4; b = \frac{4}{3} \quad \therefore A = \pi ab = \frac{16\pi}{3} \quad \therefore I = 2A = \frac{32\pi}{3}$$

To demonstrate the advantage of Green's theorem, let us work through the next example (a) by the previous method, and (b) by applying Green's theorem.

**Example 5:** Evaluate  $I = \oint_C \{(2x + y) dx + (3x - 2y) dy\}$  taken in anticlockwise manner round the triangle with vertices at  $O(0,0)$   $A(1,0)$   $B(1,2)$ .

**Solution:**  $I = \oint_C \{(2x + y) dx + (3x - 2y) dy\}$



**(a) By the previous method**

There are clearly three stages with  $c_1, c_2, c_3$ . Work through the complete evaluation to determine the value of  $I$ . It will be good revision. When you have finished, check the result with the solution in the next frame.  $I = 2$

(a) (i)  $c_1$  is  $y = 0 \quad \therefore dy = 0$

$$\therefore I_1 = \int_0^1 2x \, dx = \left[ x^2 \right]_0^1 = 1 \quad \therefore I_1 = 1$$

(ii)  $c_2$  is  $x = 1 \quad \therefore dx = 0$

$$\therefore I_2 = \int_0^2 (3-2y) \, dy = \left[ 3y - y^2 \right]_0^2 = 2 \quad \therefore I_2 = 2$$

(iii)  $c_3$  is  $y = 2x \quad \therefore dy = 2 \, dx$

$$\begin{aligned} \therefore I_3 &= \int_1^0 \{ 4x \, dx + (3x - 4x) 2 \, dx \} \\ &= \int_1^0 2x \, dx = \left[ x^2 \right]_1^0 = -1 \quad \therefore I_3 = -1 \end{aligned}$$

$$I = I_1 + I_2 + I_3 = 1 + 2 + (-1) = 2 \quad \therefore I = 2$$

Now we will do the same problem by applying Green's theorem, so more

**(b) By Green's theorem**

$$I = \oint_C \{ (2x + y) \, dx + (3x - 2y) \, dy \}$$

$$P = 2x + y \quad \therefore \quad \frac{\partial P}{\partial y} = 1;$$

$$Q = 3x - 2y \quad \therefore \quad \frac{\partial Q}{\partial x} = 3$$

$$I = - \iint_R \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) \, dx \, dy$$

$$= - \iint_R (1-3) \, dx \, dy = 2 \iint_R \, dx \, dy = 2A$$

$$= 2 \times \text{the area of the triangle} = 2 \times \left( \frac{1}{2} \times 1 \times 2 \right) = 2$$

$$\therefore I = 2$$

**Remark:** Application of Green's theorem is not always the quickest method. It is useful, however, to have both methods available.

If you have not already done so, make a note of Green's theorem.

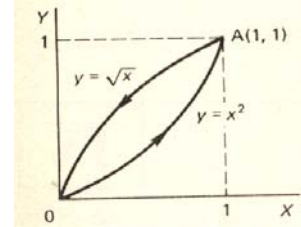
$$\iint_R \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) \, dx \, dy = - \oint_C (P \, dx + Q \, dy)$$

**Note:** Green's theorem can, in fact, be applied to a region that is not simply connected by arranging a link between outer and inner boundaries, provided the path of integration is such that the region is kept on the left-hand side.

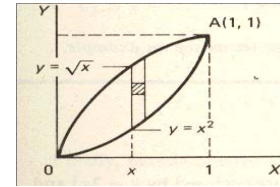
## LECTURE No.34

## EXAMPLES

**Example 1:** Evaluate the line integral  $I = \oint_C \{xy \, dx + (2x - y) \, dy\}$  round the region bounded by the curves  $y = x^2$  and  $x = y^2$  by Green's theorem  
**Solution:** Points of intersection are  $O(0, 0)$  and  $A(1, 1)$ .



$$\begin{aligned}
 I &= \oint_C \{xy \, dx + (2x - y) \, dy\} \\
 &= \oint_C \{Pdx + Qdy\} = - \iint_R \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) dx \, dy \\
 P &= xy \quad \therefore \frac{\partial P}{\partial y} = x; \quad Q = 2x - y \quad \therefore \frac{\partial Q}{\partial x} = 2 \\
 I &= - \iint_R (x - 2) \, dx \, dy = - \int_0^1 \int_{y=x^2}^{y=\sqrt{x}} (x - 2) \, dy \, dx \\
 &= - \int_0^1 (x - 2) \left[ y \right]_{x^2}^{\sqrt{x}} dx \\
 \therefore I &= - \int_0^1 (x - 2) (\sqrt{x} - x^2) \, dx = - \int_0^1 (x^{3/2} - x^3 - 2x^{1/2} + 2x^2) \, dx \\
 &= - \left[ \frac{2}{5} x^{5/2} - \frac{1}{4} x^4 - \frac{4}{3} x^{3/2} + \frac{2}{3} x^3 \right]_0^1 = \frac{31}{60}
 \end{aligned}$$



In this special case when  $P=y$  and  $Q=-x$  so  $\frac{\partial P}{\partial y} = 1$  and  $\frac{\partial Q}{\partial x} = -1$

Green's theorem then states  $\iint_R \{1 - (-1)\} \, dx \, dy = - \oint_C (P \, dx + Q \, dy)$

$$\text{i.e. } 2 \iint_R dx \, dy = - \oint_C (y \, dx - x \, dy) = \oint_C (x \, dy - y \, dx)$$

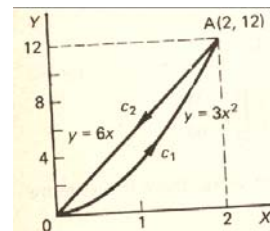
Therefore, the area of the closed region  $A = \iint_R dx \, dy = \frac{1}{2} \oint_C (x \, dy - y \, dx)$

**Example 2:** Determine the area of the figure enclosed by  $y = 3x^2$  and  $y = 6x$ .

**Solution:** Points of intersection:  $3x^2 = 6x \quad \therefore x = 0 \text{ or } 2$

$$\text{Area } A = \frac{1}{2} \oint_C (x \, dy - y \, dx)$$

We evaluate the integral in two parts, i.e.  
 OA along  $c_1$  and AO along  $c_2$



$$2A = \int_{c_1 \text{ (along OA)}} (x dy - y dx) + \int_{c_2 \text{ (along OA)}} (x dy - y dx) = I_1 + I_2$$

$$I_1: c_1 \text{ is } y = 3x^2 \quad \therefore dy = 6x dx$$

$$\therefore I_1 = \int_0^2 (6x^2 dx - 3x^2 dx) = \int_0^2 3x^2 dx = \left[ x^3 \right]_0^2 = 8 \quad \therefore I_1 = 8$$

$$\text{Similarly, for } c_2 \text{ is } y = 6x \quad \therefore dy = 6 dx$$

$$\therefore I_2 = \int_2^0 (6x dx - 6x dx) = 0$$

$$\therefore I_2 = 0$$

$$\therefore I = I_1 + I_2 = 8 + 0 = 8$$

$$\therefore A = 4 \text{ square units}$$

**Example 3:** Determine the area bounded by the curves  $y = 2x^3$ ,  $y = x^3 + 1$  and the axis  $x = 0$  for  $x \geq 0$ .

**Solution:** Here it is  $y = 2x^3$ ;  $y = x^3 + 1$ ;  $x = 0$

$$\text{Point of intersection } 2x^3 = x^3 + 1 \quad \therefore x^3 = 1 \quad \therefore x = 1$$

$$\text{Area } A = \frac{1}{2} \oint_C (x dy - y dx) \quad \therefore 2A = \oint_C (x dy - y dx)$$

$$(a) \text{ OA : } c_1 \text{ is } y = 2x^3 \quad \therefore dy = 6x^2 dx$$

$$\therefore I_1 = \int_{c_1} (x dy - y dx) = \int_0^1 (6x^3 dx - 2x^3 dx) = \int_0^1 4x^3 dx = \left[ x^4 \right]_0^1 = 1$$

$$\therefore I_1 = 1$$

$$(b) \text{ AB: } c_2 \text{ is } y = x^3 + 1 \quad \therefore dy = 3x^2 dx$$

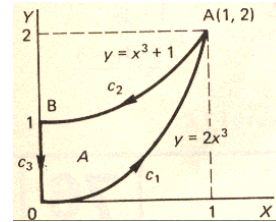
$$\therefore I_2 = \int_1^0 \{ 3x^3 dx - (x^3 + 1) dx \} = \int_1^0 (2x^3 - 1) dx = \left[ \frac{x^4}{2} - x \right]_1^0 = -\left( \frac{1}{2} - 1 \right) = \frac{1}{2}$$

$$\therefore I_2 = \frac{1}{2}$$

$$(c) \text{ BO: } c_3 \text{ is } x = 0 \quad \therefore dx = 0$$

$$I_3 = \int_{y=1}^{y=0} (x dy - y dx) = 0 \quad \therefore I_3 = 0$$

$$\therefore 2A = I = I_1 + I_2 + I_3 = 1 + \frac{1}{2} + 0 = 1\frac{1}{2} \quad \therefore A = \frac{3}{4} \text{ square units}$$



### Revision Summary

#### Properties of line integrals

- Sign of line integral is reversed when the direction of integration along the path is reversed.
- Path of integration parallel to y-axis,  $dx = 0 \quad \therefore I_c = \int_c Q dy$ .

- Path of integration parallel to x-axis,  $dy = 0 \therefore I_c = \int_c P dx$ .
- Path of integration must be continuous and single-valued.
- Dependence of line integral on path of integration.
- In general, the value of the line integral depends on the particular path of integration.

- **Exact differential**

If  $P dx + Q dy$  is an exact differential, then

$$(a) \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$(b) \quad I = \int_c (P dx + Q dy) \text{ is independent of the path of integration}$$

$$(c) \quad I = \oint_c (P dx + Q dy) \text{ is zero.}$$

- **Exact differential in three variables.**

If  $P dx + Q dy + R dw$  is an exact differential

$$(a) \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}; \quad \frac{\partial P}{\partial w} = \frac{\partial R}{\partial x}; \quad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial w}$$

$$(b) \quad \int_c (P dx + Q dy + R dw) \text{ is independent of the path of integration.}$$

$$(c) \quad \oint_c (P dx + Q dy + R dw) \text{ is zero.}$$

- **Green's theorem**

$$\oint_c (P dx + Q dy) = - \iint_R \left\{ \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right\} dx dy \quad \text{and, for a simple closed curve,}$$

$$\oint_c (x dy - y dx) = 2 \iint_R dx dy = 2A$$

where A is the area of the enclosed figure.

### Gradient of a scalar function

Del operator is given by  $\nabla = \left( \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right)$

$$\nabla \phi = \text{grad } \phi = \left( \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \phi = \mathbf{i} \frac{\partial \phi}{\partial x} + \mathbf{j} \frac{\partial \phi}{\partial y} + \mathbf{k} \frac{\partial \phi}{\partial z}$$

$$\text{grad } \phi = \nabla \phi = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k}$$

### Div (Divergence of a vector function)

If  $\mathbf{A} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$ , then

$$\text{div } \mathbf{A} = \nabla \cdot \mathbf{A} = \left( \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot (a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k})$$

$$\therefore \text{div } \mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$$

**Note that**

- (a) the grad operator  $\nabla$  acts on a scalar and gives a vector
- (b) the div operator  $\nabla$  acts on a vector and gives a scalar.

**Example 4:** If  $\mathbf{A} = x^2y\mathbf{i} - xyz\mathbf{j} + yz^2\mathbf{k}$ , then find Div  $\mathbf{A}$ .

**Solution:**

$$\text{Div } \mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\partial}{\partial x}(x^2y) - \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(yz^2) = 2xy - xz + 2yz$$

**Example 5:** If  $\mathbf{A} = 2x^2y\mathbf{i} - 2(xy^2 + y^3z)\mathbf{j} + 3y^2z^2\mathbf{k}$ , determine  $\nabla \cdot \mathbf{A}$  i.e. div  $\mathbf{A}$ .

**Solution:**  $\mathbf{A} = 2x^2y\mathbf{i} - 2(xy^2 + y^3z)\mathbf{j} + 3y^2z^2\mathbf{k}$

$$\nabla \cdot \mathbf{A} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = 4xy - 2(2xy + 3y^2z) + 6y^2z = 4xy - 4xy - 6y^2z + 6y^2z = 0$$

Such a vector  $\mathbf{A}$  for which  $\nabla \cdot \mathbf{A} = 0$  at all points, i.e. for all values of  $x, y, z$ , is called a solenoid vector. It is rather a special case.

**Curl (Curl of a Vector Function)**

The curl operator denoted by  $\nabla \times \mathbf{A}$ , acts on a vector and gives another vector as a result.

If  $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$  then  $\text{curl } \mathbf{A} = \nabla \times \mathbf{A}$ .

$$\begin{aligned} \text{i.e. } \text{curl } \mathbf{A} &= \nabla \times \mathbf{A} = \left( \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \times (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} \\ \therefore \nabla \times \mathbf{A} &= \mathbf{i} \left( \frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right) + \mathbf{j} \left( \frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x} \right) + \mathbf{k} \left( \frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y} \right) \end{aligned}$$

Curl  $\mathbf{A}$  is thus a vector function.

**Example 6:** If  $\mathbf{A} = (y^4 - x^2z^2)\mathbf{i} + (x^2 + y^2)\mathbf{j} - x^2yz\mathbf{k}$ , determine curl  $\mathbf{A}$  at the point  $(1, 3, -2)$ .

$$\text{Solution: } \text{Curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^4 - x^2z^2 & x^2 + y^2 & -x^2yz \end{vmatrix}$$

Now we expand the determinant

$$\begin{aligned} \nabla \times \mathbf{A} &= \mathbf{i} \left\{ \frac{\partial}{\partial y} (-x^2yz) - \frac{\partial}{\partial z} (x^2 + y^2) \right\} - \mathbf{j} \left\{ \frac{\partial}{\partial x} (-x^2yz) - \frac{\partial}{\partial z} (y^4 - x^2z^2) \right\} \\ &\quad + \mathbf{k} \left\{ \frac{\partial}{\partial x} (x^2 + y^2) - \frac{\partial}{\partial y} (y^4 - x^2z^2) \right\} \end{aligned}$$

$$\nabla \times \mathbf{A} = \mathbf{i} \{-x^2z\} - \mathbf{j} \{-2xyz + 2x^2z\} + \mathbf{k} \{2x - 4y^3\}. \quad \therefore \text{At } (1, 3, -2),$$

$$\nabla \times \mathbf{A} = \mathbf{i} (2) - \mathbf{j} (12 - 4) + \mathbf{k} (2 - 108) = 2\mathbf{i} - 8\mathbf{j} - 106\mathbf{k}$$

**Example 7:**

**Determine curl F at the point (2,0,3) given that  $\mathbf{F} = z e^{2xy}\mathbf{i} + 2xz \cos y\mathbf{j} + (x+2y)\mathbf{k}$ .**

**Solution:** In determinant form,  $\text{curl } \mathbf{F} = \nabla \times \mathbf{F}$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ze^{2xy} & 2xz \cos y & x+2y \end{vmatrix}$$

Now expand the determinant and substitute the values for x, y and z, finally obtaining curl

$$\begin{aligned} \nabla \times \mathbf{F} &= \mathbf{i}\{2 - 2x \cos y\} - \mathbf{j}\{1 - e^{2xy}\} + \mathbf{k}\{2z \cos y - 2xe^{2xy}\} \\ \therefore \text{At } (2, 0, 3) \quad \nabla \times \mathbf{F} &= \mathbf{i}(2-4) - \mathbf{j}(1-1) + \mathbf{k}(6-12) = -2\mathbf{i} - 6\mathbf{k} = -2(\mathbf{i} + 3\mathbf{k}) \end{aligned}$$

### Summary of grad, div and curl

- (a) Grad operator acts on a scalar field to give a vector field.
- (b) Div operator acts on a vector field to give a scalar field.
- (c) Curl operator acts on a vector field to give a vector field.
- (d) With a scalar function  $\phi(x, y, z)$

$$\text{Grad } \phi = \nabla \phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$$

- (e) With a vector function  $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$

$$(i) \text{div } \mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

$$(ii) \text{Curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

### Multiple Operations

We can combine the operators grad, div and curl in multiple operations, as in the examples that follow.

**EXAMPLE 8:** If  $\mathbf{A} = x^2y\mathbf{i} + yz^3\mathbf{j} - zx^3\mathbf{k}$ , then find grad div A.

$$\begin{aligned} \text{Solution: } \text{div } \mathbf{A} &= \nabla \cdot \mathbf{A} = \left( \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot (x^2y\mathbf{i} + yz^3\mathbf{j} - zx^3\mathbf{k}) \\ &= 2xy + z^3 - x^3 = \phi \quad (\text{say}) \end{aligned}$$

$$\text{Now grad (div } \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k} = (2y - 3x^2)\mathbf{i} + (2x)\mathbf{j} + (3z^2)\mathbf{k}$$

$$\text{i.e., grad (div } \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) = (2y - 3x^2)\mathbf{i} + 2x\mathbf{j} + 3z^2\mathbf{k}$$

**Example 9:** If  $\phi = xyz - 2y^2z + x^2z^2$  determine **div grad  $\phi$**  at the point (2, 4, 1).

**Solution:** First find grad  $\phi$  and then the div of the result.

$$\text{div grad } \phi = \nabla \cdot (\nabla \phi)$$

We have  $\phi = xyz - 2y^2z + x^2z^2$

$$\text{grad } \phi = \nabla \phi = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k} = (yz + 2xz^2)\mathbf{i} + (xz - 4yz)\mathbf{j} + (xy - 2y^2 + 2x^2z)\mathbf{k}$$

$$\therefore \text{div grad } \phi = \nabla \cdot (\nabla \phi) = 2z^2 - 4z + 2x^2$$

$$\therefore \text{At (2,4,1), } \text{div grad } \phi = \nabla \cdot (\nabla \phi) = 2 - 4 + 8 = 6$$

**REMARK:** Let  $\text{grad } \phi = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k}$

$$\text{Then div grad } \phi = \nabla \cdot (\nabla \phi) = \left( \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot \left( \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k} \right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$\therefore \text{div grad } \phi = \nabla \cdot (\nabla \phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

**Example 10:** If  $\mathbf{F} = x^2yzi + xyz^2j + y^2zk$  determine **curl  $\mathbf{F}$**  at the point (2, 1, 1). Determine an expression for curl  $\mathbf{F}$  in the usual way, which will be a vector, and then the curl of the result. Finally substitute values.

**Solution:**  $\text{Curl curl } \mathbf{F} = \nabla \times (\nabla \times \mathbf{F}) = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xyz^2 & y^2z \end{vmatrix} = (2yz - 2xyz)\mathbf{i} + x^2y\mathbf{j} + (yz^2 - x^2z)\mathbf{k}$$

$$\text{Curl Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz - 2xyz & x^2y & yz^2 - x^2z \end{vmatrix} = z^2\mathbf{i} - (-2xz - 2y + 2xy)\mathbf{j} + (2xy - 2z + 2xz)\mathbf{k}$$

$$\therefore \text{At (2, 1, 1), } \text{curl curl } \mathbf{F} = \nabla \times (\nabla \times \mathbf{F}) = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

### Two interesting general results

(a) Curl grad  $\phi$  where  $\phi$  is a scalar

$$\text{grad } \phi = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k}$$

$$\therefore \text{curl grad } \phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$\begin{aligned}
&= \mathbf{i} \left\{ \frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} \right\} - \mathbf{j} \left\{ \frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial^2 \phi}{\partial x \partial z} \right\} + \mathbf{k} \left\{ \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} \right\} \\
&= \mathbf{i} 0 - \mathbf{j} 0 + \mathbf{k} 0 = \mathbf{0}
\end{aligned}$$

$$\therefore \text{curl grad } \phi = \nabla \times (\nabla \phi) = \mathbf{0}$$

(b) Div curl  $\mathbf{A}$  where  $\mathbf{A}$  is a vector.

$$\mathbf{A} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}$$

$$\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} = \mathbf{i} \left( \frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right) - \mathbf{j} \left( \frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z} \right) + \mathbf{k} \left( \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right)$$

$$\begin{aligned}
\text{Then div curl } \mathbf{A} &= \nabla \cdot (\nabla \times \mathbf{A}) = \left( \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot (\nabla \times \mathbf{A}) \\
&= \frac{\partial^2 a_z}{\partial x \partial y} - \frac{\partial^2 a_y}{\partial z \partial x} - \frac{\partial^2 a_z}{\partial x \partial y} + \frac{\partial^2 a_x}{\partial y \partial z} + \frac{\partial^2 a_y}{\partial z \partial x} - \frac{\partial^2 a_x}{\partial y \partial z} = 0
\end{aligned}$$

$$\therefore \text{div curl } \mathbf{A} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$$

(c) Div grad  $\phi$  where  $\phi$  is a scalar.

$$\text{grad } \phi = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k}$$

$$\begin{aligned}
\text{Then div grad } \phi &= \nabla \cdot (\nabla \phi) = \left( \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot \left( \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k} \right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \\
\therefore \text{div grad } \phi &= \nabla \cdot (\nabla \phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}
\end{aligned}$$

This result is sometimes denoted by  $\nabla^2 \phi$ .

So these general results are

$$(a) \text{curl grad } \phi = \nabla \times (\nabla \phi) = \mathbf{0}$$

$$(b) \text{div curl } \mathbf{A} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$(c) \text{div grad } \phi = \nabla \cdot (\nabla \phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

## LECTURE No. 35

## DEFINITE INTEGRALS

**Definite Integral for  $\sin^n x$  and  $\cos^n x$ ,  $0 \leq x \leq \pi/2$** 

$$(1) \int_0^{\frac{\pi}{2}} \sin^2 x dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - \cos 2x) dx = \frac{1}{2} \left[ x - \frac{\sin 2x}{2} \right]_0^{\frac{\pi}{2}} = \frac{1}{2} \left[ \frac{\pi}{2} - \frac{\sin \pi}{2} \right] = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \sin^2 x dx = \frac{1}{2} \frac{\pi}{2}$$

$$(2) \int_0^{\frac{\pi}{2}} \cos^2 x dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2x) dx = \frac{1}{2} \left[ x + \frac{\sin 2x}{2} \right]_0^{\frac{\pi}{2}} = \frac{1}{2} \left[ \frac{\pi}{2} + \frac{\sin \pi}{2} \right] = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \cos^2 x dx = \frac{1}{2} \frac{\pi}{2}$$

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$$\int_0^{\frac{\pi}{2}} \sin^3 x dx = \int_0^{\frac{\pi}{2}} \sin^2 x \sin x dx = \int_0^{\frac{\pi}{2}} (1 - \cos^2 x) \sin x dx = \int_0^{\frac{\pi}{2}} \sin x dx + \int_0^{\frac{\pi}{2}} \cos^2 x (-\sin x) dx$$

$$= \left[ -\cos x \right]_0^{\frac{\pi}{2}} + \left[ \frac{\cos^3 x}{3} \right]_0^{\frac{\pi}{2}} = -\cos \frac{\pi}{2} + \cos 0 + \frac{1}{3} \left[ \cos^3 \frac{\pi}{2} - \cos^3 0 \right] = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \cos^3 x dx = \int_0^{\frac{\pi}{2}} \cos^2 x \cos x dx = \int_0^{\frac{\pi}{2}} (1 - \sin^2 x) \cos x dx = \int_0^{\frac{\pi}{2}} \cos x dx - \int_0^{\frac{\pi}{2}} \sin^2 x (\cos x) dx$$

$$= \left[ \sin x \right]_0^{\frac{\pi}{2}} - \left[ \frac{\sin^3 x}{3} \right]_0^{\frac{\pi}{2}} = \sin \frac{\pi}{2} - \sin 0 - \frac{1}{3} \left[ \sin^3 \frac{\pi}{2} - \sin^3 0 \right] = 1 - \frac{1}{3} = \frac{2}{3}$$

---


$$\int_0^{\frac{\pi}{2}} \sin^4 x dx = \int_0^{\frac{\pi}{2}} (\sin^2 x)^2 dx = \int_0^{\frac{\pi}{2}} \left[ \frac{1 - \cos 2x}{2} \right]^2 dx = \frac{1}{4} \int_0^{\frac{\pi}{2}} (1 - 2\cos 2x + \cos^2 2x) dx$$

$$= \frac{1}{4} \int_0^{\frac{\pi}{2}} \left( 1 - 2\cos 2x + \frac{1 + \cos 4x}{2} \right) dx = \frac{1}{4} \int_0^{\frac{\pi}{2}} \left( \frac{3}{2} - 2\cos 2x + \frac{\cos 4x}{2} \right) dx$$

$$= \frac{1}{4} \left[ \frac{3}{2} x - \sin 2x + \frac{\sin 4x}{8} \right]_0^{\frac{\pi}{2}} = \frac{1}{4} \left[ \frac{3}{2} \frac{\pi}{2} - \sin \pi + \frac{\sin 2\pi}{8} \right]$$

$$\int_0^{\frac{\pi}{2}} \sin^4 x dx = \frac{1}{4} \left[ \frac{3}{2} \frac{\pi}{2} \right] \quad \text{so} \quad \int_0^{\frac{\pi}{2}} \sin^4 x dx = \frac{3}{4} \frac{1}{2} \frac{\pi}{2}$$

$$\begin{aligned}
 \int_0^{\frac{\pi}{2}} \cos^4 x dx &= \int_0^{\frac{\pi}{2}} (\cos^2 x)^2 dx = \int_0^{\frac{\pi}{2}} \left[ \frac{1 + \cos 2x}{2} \right]^2 dx = \frac{1}{4} \int_0^{\frac{\pi}{2}} (1 + 2 \cos 2x + \cos^2 2x) dx \\
 &= \frac{1}{4} \int_0^{\frac{\pi}{2}} \left( 1 + 2 \cos 2x + \frac{1 + \cos 4x}{2} \right) dx = \frac{1}{4} \int_0^{\frac{\pi}{2}} \left( \frac{3}{2} + 2 \cos 2x + \frac{\cos 4x}{2} \right) dx \\
 &= \frac{1}{4} \left[ \frac{3}{2} x + \sin 2x + \frac{\sin 4x}{4} \right]_0^{\frac{\pi}{2}} = \frac{1}{4} \left[ \frac{3}{2} \cdot \frac{\pi}{2} + \sin \pi + \frac{\sin 2\pi}{4} \right]
 \end{aligned}$$

$$\int_0^{\frac{\pi}{2}} \cos^4 x dx = \frac{1}{4} \left[ \frac{3}{2} \cdot \frac{\pi}{2} \right] \quad \text{So} \quad \int_0^{\frac{\pi}{2}} \cos^4 x dx = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^5 x dx = \frac{4}{5} \cdot \frac{2}{3} \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^5 x dx = \frac{4}{5} \cdot \frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^6 x dx = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^6 x dx = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^7 x dx = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^7 x dx = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^8 x dx = \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^8 x dx = \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^9 x dx = \frac{8}{9} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^9 x dx = \frac{8}{9} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^{10} x dx = \frac{9}{10} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^{10} x dx = \frac{9}{10} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

### **Wallis Sine Formula**

**When n is even**

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \frac{n-7}{n-6} \cdot \dots \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

**When n is odd**

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \frac{n-7}{n-6} \cdot \dots \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^{11} x dx = \frac{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2}{11 \cdot 9 \cdot 7 \cdot 5 \cdot 3} \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^{11} x dx = \frac{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2}{11 \cdot 9 \cdot 7 \cdot 5 \cdot 3}$$

$$\int_0^{\frac{\pi}{2}} \sin^{12} x dx = \frac{11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^{12} x dx = \frac{11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}$$

### Integration By Parts

$$\int UV dx = U \int V dx - \int \left[ \int V dx \cdot \frac{dU}{dx} \right] dx$$

**Example:** Evaluate  $\int x \ln x dx$

$$\begin{aligned} \int x \ln x dx &= \ln x \int x dx - \int \left[ \int x dx \cdot \frac{d}{dx} (\ln x) \right] dx \quad (\text{Using integrating by parts}) \\ &= \ln x \left( \frac{x^2}{2} \right) - \int \left( \frac{x^2}{2} \right) \left( \frac{1}{x} \right) dx = \left( \frac{x^2}{2} \right) \ln x - \int \left( \frac{x}{2} \right) dx = \left( \frac{x^2}{2} \right) \ln x - \frac{1}{2} \left( \frac{x^2}{2} \right) \end{aligned}$$

**Example:** Evaluate  $\int x \sin x dx$

$$\begin{aligned} \int x \sin x dx &= x \int \sin x dx - \int \left[ \int \sin x dx \cdot \frac{d}{dx} (x) \right] dx \quad (\text{We are integrating by parts}) \\ &= x(-\cos x) - \int (-\cos x)(1) dx = -x(\cos x) + \int \cos x dx = -x(\cos x) + \sin x \end{aligned}$$

### Line Integrals

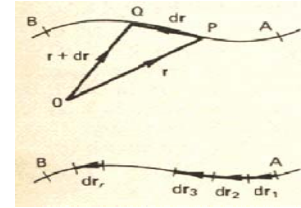
Let a point p on the curve c joining A and B be denoted by the position vector  $\mathbf{r}$  with respect to origin O. If q is a neighboring point on the curve with position vector

$\mathbf{r} + d\mathbf{r}$ , then  $\overrightarrow{PQ} = d\mathbf{r}$

The curve c can be divided up into many n such small arcs, approximating to  $d\mathbf{r}_1, d\mathbf{r}_2, d\mathbf{r}_3, \dots, d\mathbf{r}_p, \dots$

so that  $\overrightarrow{AB} = \sum_{p=1}^n d\mathbf{r}_p$  where  $d\mathbf{r}_p$  is a vector representing the element of the arc in both

magnitude and direction. If  $d\mathbf{r} \rightarrow 0$ , then the length of the curve  $AB = \int_c d\mathbf{r}$ .



### Scalar Field

If a scalar field  $V(\mathbf{r})$  exists for all points on the curve,

the  $\sum_{p=1}^n V(\mathbf{r}) d\mathbf{r}_p$  with  $d\mathbf{r} \rightarrow 0$ , defines the line integral

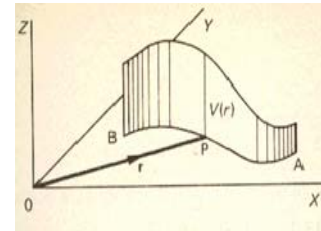
of V i.e line integral  $= \int_c V(\mathbf{r}) d\mathbf{r}$ .

We can illustrate this integral by erecting a continuous

Ordinate to  $V(\mathbf{r})$  at each point of the curve  $\int_c V(\mathbf{r}) d\mathbf{r}$  is then represented by the area of the

curved surface between the ends A and B the curve c. To evaluate a line integral, the integrand is expressed in terms of x, y, z with  $d\mathbf{r} = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k}$

In practice, x, y and z are often expressed in terms of parametric equation of a fourth variable (say u), i.e.  $x = x(u)$ ;  $y = y(u)$ ;  $z = z(u)$ . From these, dx, dy and dz can be written in terms of u and the integral evaluate in terms of this parameter u.



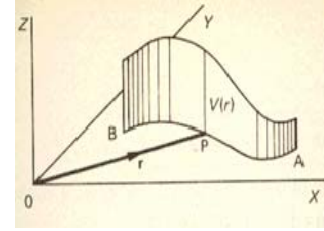
## LECTURE No.36 SCALAR FIELD

### Scalar Field

If a scalar field  $V(r)$  exists for all points on the curve,

then  $\sum_{p=1}^n V(r) dr_p$  with  $dr \rightarrow 0$  defines the line integral

of  $V$  i.e. line integral  $= \int_c V(r) dr$ .



We can illustrate this integral by erecting a continuous

Ordinate to  $V(r)$  at each point of the curve  $\int_c V(r) dr$  is then represented by the area of the

curved surface between the ends A and B the curve c. To evaluate a line integral, the integrand is expressed in terms of  $x, y, z$  with  $d\mathbf{r} = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k}$

In practice,  $x, y$  and  $z$  are often expressed in terms of parametric equation of a fourth variable (say  $u$ ), i.e.  $x = x(u)$ ;  $y = y(u)$ ;  $z = z(u)$ . From these,  $dx, dy$  and  $dz$  can be written in terms of  $u$  and the integral evaluate in terms of this parameter  $u$ .

**Example:** If  $V = xy^2z$ , evaluate  $\int_c V(r) dr$  along the curve  $c$  having parametric equations

$x = 3u$ ;  $y = 2u^2$ ;  $z = u^3$  between  $A(0,0,0)$  and  $B(3,2,1)$

**Solution:**  $V = xy^2z = (3u)(4u^4)(u^3) = 12u^8$

$d\mathbf{r} = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k} \Rightarrow d\mathbf{r} = 3du \mathbf{i} + 4udu \mathbf{j} + 3u^2du \mathbf{k}$

for  $x = 3u$ ;  $\therefore dx = 3du$ ;  $y = 2u^2$   $\therefore dy = 4u du$ ;  $z = u^3$   $\therefore dz = 3u^2du$

Limiting:  $A(0,0,0)$  corresponds to  $B(3,2,1)$  corresponds to  $u$

$A(0,0,0) \equiv u=0$ ;  $B(3,2,1) \equiv u=1$

$$\int_c V(r) dr = \int_0^1 12u^8 (3 \mathbf{i} + 4u \mathbf{j} + 3u^2 \mathbf{k}) du = \left[ 36 \frac{u^9}{9} \mathbf{i} + 48 \frac{u^{10}}{10} \mathbf{j} + 36 \frac{u^{11}}{11} \mathbf{k} \right]_0^1 = 4\mathbf{i} + \frac{24}{5} \mathbf{j} + \frac{36}{11} \mathbf{k}$$

**Example :** If  $V = xy + y^2z$  Evaluate  $\int_c V(r) dr$  along the curve  $c$  defined by  $x = t^2$ ;  $y = 2t$ ;

$z = t+5$  between  $A(0,0,5)$  and  $B(4,4,7)$ . As before, expressing  $V$  and  $d\mathbf{r}$  in term of the parameter  $t$ .

**Solution:**

since  $V = xy + y^2z$

$$= (t^2)(2t) + (4t^2)(t+5)$$

$$= 6t^3 + 20t^2.$$

$$x = t^2 \quad dx = 2t dt$$

$$y = 2t \quad dy = 2 dt$$

$$z = t+5 \quad dz = dt$$

$$\therefore d\mathbf{r} = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k}$$

$$= 2t dt \mathbf{i} + 2 dt \mathbf{j} + dt \mathbf{k}$$

$$\therefore \int_c V dr = \int_c (6t^3 + 20t^2) (2t \mathbf{i} + 2 \mathbf{j} + \mathbf{k}) dt$$

Limits: A (0, 0, 5)  $\equiv t = 0$ ;

B (4, 4, 7)  $\equiv t = 2$

$$\therefore \int_C \mathbf{V} d\mathbf{r} = \int_0^2 (6t^3 + 20t^2)(2t \mathbf{i} + 2 \mathbf{j} + \mathbf{k}) dt$$

$$\int_C \mathbf{V} d\mathbf{r} = 2 \int_0^2 \{6t^4 + 20t^3\} \mathbf{i} + \{6t^3 + 20t^2\} \mathbf{j} + \{3t^3 + 10t^2\} \mathbf{k} \} dt.$$

$$= \frac{8}{15} (444\mathbf{i} + 290\mathbf{j} + 145\mathbf{k})$$

### Vector Field

If a vector field  $\mathbf{F}(\mathbf{r})$  exists for all points of the curve  $c$ , then for each element of arc we can form the scalar product  $\mathbf{F} \cdot d\mathbf{r}$ . Summing these products for all elements of arc, we

have  $\sum_{p=1}^n \mathbf{F} \cdot d\mathbf{r}_p$

The line integral of  $\mathbf{F}(\mathbf{r})$  from A to B along the stated curve  $= \int_C \mathbf{F} \cdot d\mathbf{r}$ .

In this case, since  $\mathbf{F} \cdot d\mathbf{r}$  is a scalar product, then the line integral is a scalar.

To evaluate the line integral,  $\mathbf{F}$  and  $d\mathbf{r}$  are expressed in terms of  $x, y, z$ , and the curve in parametric form. We have

$$\mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}$$

$$\text{And } d\mathbf{r} = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k}$$

$$\text{Then } \mathbf{F} \cdot d\mathbf{r} = (F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}) \cdot (dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k}) = \int_c (F_1 dx + F_2 dy + F_3 dz)$$

Now for an example to show it in operation.

### **Example**

If  $\mathbf{F}(\mathbf{r}) = x^2y \mathbf{i} + xz \mathbf{j} + 2yz \mathbf{k}$ , Evaluate  $\int_c \mathbf{F} \cdot d\mathbf{r}$  between A(0,0,0) and B(4,2,1) along the curve  $c$  having parametric equations  $x=4t$ ;  $y=2t^2$ ;  $z=t^3$

**Solution:** Expressing everything in terms of the parameter  $t$ , we have

$$dx = 4 dt ; dy = 4t dt ; dz = 3t^2 dt$$

$$x^2y = (16t^2)(2t^2) = 32 t^4$$

$$x = 4t \quad \therefore dx = 4 dt$$

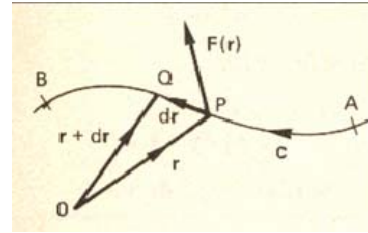
$$xz = (4t)(t^3) = 4 t^4$$

$$y = 2 t^2 \quad \therefore dy = 4t dt$$

$$2yz = (4 t^2)(t^3) = 4 t^5$$

$$z = t^3 \quad \therefore dz = 3t^2 dt$$

$$\mathbf{F} = 32 t^4 \mathbf{i} + 4 t^4 \mathbf{j} - 4 t^5 \mathbf{k}$$



$$d\mathbf{r} = 4dt \mathbf{i} + 4t dt \mathbf{j} + 3t^2 \mathbf{k}$$

$$\begin{aligned} \text{Then } \int \mathbf{F} \cdot d\mathbf{r} &= \int (32t^4 \mathbf{i} + 4t^4 \mathbf{j} - 4t^5 \mathbf{k}) \cdot (4dt \mathbf{i} + 4t dt \mathbf{j} + 3t^2 dt \mathbf{k}) \\ &= \int (128t^4 + 16t^5 + 12t^7) dt \end{aligned}$$

Limits: A(0,0,0)  $\equiv t = 0$ ;

B(4, 2, 1)  $\equiv t = 1$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = (128t^4 + 16t^5 + 12t^7)dt = \frac{128}{5} t^5 + \frac{16}{6} t^6 + \frac{12}{8} t^8 = \frac{128}{5} + \frac{8}{3} + \frac{3}{2} = 29.76$$

### Example

If  $\mathbf{F}(\mathbf{r}) = x^2 y \mathbf{i} + 2yz \mathbf{j} + 3z^2 x \mathbf{k}$

Evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$  between A(0,0,0) and B(1,2,3)

B(1, 2, 3)

(a) along the straight line

$c_1$  from (0, 0, 0) to (1, 0, 0)

then  $c_2$  from (1, 0, 0) to (1, 2, 0)

and  $c_3$  from (1, 2, 0) to (1, 2, 3)

(b) along the straight line  $c_4$  joining (0, 0, 0) to (1, 2, 3).

We first obtain an expression for  $\mathbf{F} \cdot d\mathbf{r}$  which is

$$\mathbf{F} \cdot d\mathbf{r} = (x^2 y \mathbf{i} + 2yz \mathbf{j} + 3z^2 x \mathbf{k}) \cdot (dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k})$$

$$\mathbf{F} \cdot d\mathbf{r} = x^2 y dx + 2yz dy + 3z^2 x dz$$

$$\int \mathbf{F} \cdot d\mathbf{r} = \int x^2 y dx + \int 2yz dy + \int 3z^2 x dz$$

Here the integration is made in three sections, along  $c_1$ ,  $c_2$  and  $c_3$ .

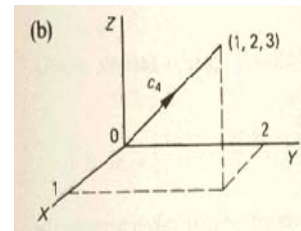
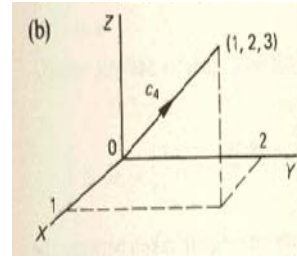
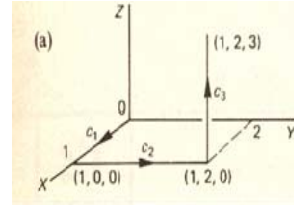
$$\begin{aligned} \text{(i) } c_1: y = 0, z = 0, dy = 0, dz = 0 \\ \therefore \int_{c_1} \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + 0 = 0 \end{aligned}$$

$$\begin{aligned} \text{(ii) } c_2: \text{ The conditions along } c_2 \text{ are} \\ c_2: x = 1, z = 0, dx = 0, dz = 0 \\ \therefore \int_{c_2} \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + 0 = 0 \end{aligned}$$

$$\begin{aligned} \text{(iii) } c_3: x = 1, y = 2, dx = 0, dy = 0 \\ \int_{c_3} \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + \int_0^3 3z^2 dz = 27 \end{aligned}$$

Summing the three partial results

$$\begin{aligned} \int_{(0,0,0)}^{(1,2,3)} \mathbf{F} \cdot d\mathbf{r} &= 0 + 0 + 27 = 27 \\ \therefore \int_{c_1+c_2+c_3} \mathbf{F} \cdot d\mathbf{r} &= 27 \end{aligned}$$



If  $t$  taken as the parameter, the parametric equation of  $c$  are  $x = t; y = 2t; z = 3t$   
 $(0, 0, 0) \Rightarrow t = 0, (1, 2, 3) \Rightarrow t = 1$  and the limits of  $t$  are  $t = 0$  and  $t = 1$

$$\begin{aligned}\mathbf{F} &= 2t^3\mathbf{i} + 12t^2\mathbf{j} + 27t^3\mathbf{k} \\ d\mathbf{r} &= dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} = dt\mathbf{i} + 2dt\mathbf{j} + 3dt\mathbf{k} \\ \int_{c_4} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 (2t^3\mathbf{i} + 12t^2\mathbf{j} + 27t^3\mathbf{k}) \cdot (dt\mathbf{i} + 2dt\mathbf{j} + 3dt\mathbf{k}) = \int_0^1 (2t^3 + 24t^2 + 81t^3) dt \\ &= \int_0^1 (83t^3 + 24t^2) dt = \left[ 83\frac{t^4}{4} + 8t^3 \right]_0^1 = \frac{115}{4} = 28.75\end{aligned}$$

So the value of the line integral depends on the path taken between the two end points A and B

(a)  $\int \mathbf{F} \cdot d\mathbf{r}$  via  $c_1, c_2$  and  $c_3 = 27$

(b)  $\int \mathbf{F} \cdot d\mathbf{r}$  via  $c_4 = 28.75$

### Example

Evaluate  $\int_V F dv$  where  $V$  is the region bounded by the planes  $x = 0, y = 0, z = 0$  and

$2x + y + z = 2$ , and  $F = 2z\mathbf{i} + y\mathbf{k}$ . To sketch the surface  $2x + y + z = 2$ , note that

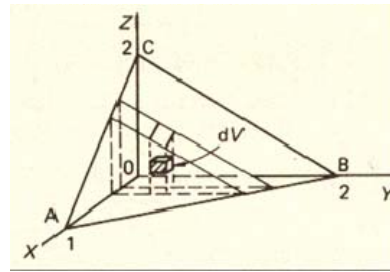
when  $z = 0, 2x + y = 2$  i.e.  $y = 2 - 2x$

when  $y = 0, 2x + z = 2$  i.e.  $z = 2 - 2x$

when  $x = 0, y + z = 2$  i.e.  $z = 2 - y$

Inserting these in the planes  $x = 0, y = 0, z = 0$  will help.

The diagram is therefore.



So  $2x + y + z = 2$  cuts the axes at  $A(1,0,0); B(0,2,0); C(0,0,2)$ .

Also  $F = 2z\mathbf{i} + y\mathbf{k}$ ;

$$z = 2 - 2x - y = 2(1 - x) - y$$

$$\therefore \int_V F dV = \int_0^1 \int_0^{2(1-x)} \int_0^{2(1-x)-y} (2x\mathbf{i} + y\mathbf{k}) dz dy dx$$

$$= \int_0^1 \int_0^{2(1-x)} \left[ z^2\mathbf{i} + yz\mathbf{k} \right]_{z=0}^{z=2(1-x)-y} dy dx$$

$$= \int_0^1 \int_0^{2(1-x)} \{ [4(1-x)^2 - 4(1-x)y + y^2]\mathbf{i} + [2(1-x)y - y^2]\mathbf{k} \} dy dx$$

$$\int_V F dV = \frac{1}{3} (2\mathbf{i} + \mathbf{k})$$

## LECTURE No. 37

## EXAMPLES

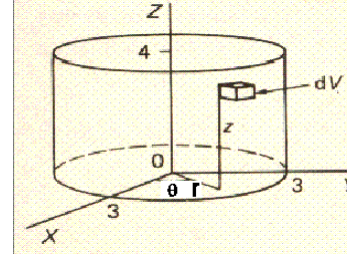
**Example:** Evaluate  $\int_V \mathbf{F} dV$  where  $\mathbf{F} = 2\mathbf{i} + 2z\mathbf{j} + y\mathbf{k}$  and  $V$  is the region bounded by the planes  $z = 0$ ,  $z = 4$  and the surface  $x^2 + y^2 = 9$ .

**Solution:** It will be convenient to use cylindrical polar coordinates  $(r, \theta, z)$  so the relevant transformations are  $x = r \cos \theta$ ;  $y = r \sin \theta$ ;  $z = z$ ;  $dV = r dr d\theta dz$

$$\text{Then } \int_V \mathbf{F} dV = \iiint_V (2\mathbf{i} + 2z\mathbf{j} + y\mathbf{k}) dx dy dz$$

Changing into cylindrical polar coordinates with appropriate change of limits this becomes

$$\begin{aligned} \int_V \mathbf{F} dV &= \int_{\theta=0}^{2\pi} \int_{r=0}^3 \int_{z=0}^4 (2\mathbf{i} + 2z\mathbf{j} + r \sin \theta \mathbf{k}) r dz dr d\theta = \int_{\theta=0}^{2\pi} \int_{r=0}^3 \left[ 2z\mathbf{i} + z^2\mathbf{j} + r \sin \theta z \mathbf{k} \right]_{z=0}^4 r dr d\theta \\ &= \int_0^{2\pi} \int_0^3 (8\mathbf{i} + 16\mathbf{j} + 4r \sin \theta \mathbf{k}) r dr d\theta = 4 \int_0^{2\pi} \int_0^3 (2r\mathbf{i} + 4r\mathbf{j} + r^2 \sin \theta \mathbf{k}) dr d\theta = 4 \int_0^{2\pi} \mathbf{Error!} d\theta \\ &= 4 \int_0^{2\pi} (9\mathbf{i} + 18\mathbf{j} + 9 \sin \theta \mathbf{k}) d\theta = 36 \int_0^{2\pi} (\mathbf{i} + 2\mathbf{j} + \sin \theta \mathbf{k}) d\theta = 36 \left[ \theta \mathbf{i} + 2\theta \mathbf{j} - \cos \theta \mathbf{k} \right]_0^{2\pi} \\ &= 36 \{ (2\pi \mathbf{i} + 4\pi \mathbf{j} - \mathbf{k}) - (-\mathbf{k}) \} = 72\pi (\mathbf{i} + 2\mathbf{j}) \end{aligned}$$

**Scalar Fields**

A scalar field  $F = xyz$  exists over the curved surface  $S$  defined by  $x^2 + y^2 = 4$  between the planes  $z = 0$  and  $z = 3$  in the first octant.

**Example:** Evaluate  $\int_S \mathbf{F} d\mathbf{S}$  over this surface.

**Solution:** We have  $F = xyz$   $S: x^2 + y^2 - 4 = 0$ ,  $z = 0$  to  $z = 3$

$$d\mathbf{S} = \hat{\mathbf{n}} dS \text{ where } \hat{\mathbf{n}} = \frac{\nabla S}{|\nabla S|}$$

$$\text{Now } \nabla S = \frac{\partial S}{\partial x} \mathbf{i} + \frac{\partial S}{\partial y} \mathbf{j} + \frac{\partial S}{\partial z} \mathbf{k} = 2x\mathbf{i} + 2y\mathbf{j}$$

$$|\nabla S| = \sqrt{4x^2 + 4y^2} = 2\sqrt{x^2 + y^2} = 2\sqrt{4} = 4$$

$$\therefore \hat{\mathbf{n}} = \frac{\nabla S}{|\nabla S|} = \frac{x\mathbf{i} + y\mathbf{j}}{2}$$

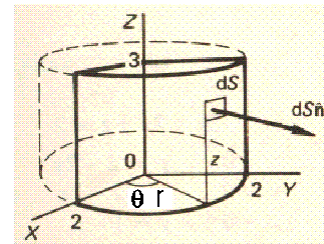
$$\therefore d\mathbf{S} = \hat{\mathbf{n}} dS = \frac{x\mathbf{i} + y\mathbf{j}}{2} dS$$

$$\therefore \int_S \mathbf{F} d\mathbf{S} = \int_S \mathbf{F} \hat{\mathbf{n}} dS = \frac{1}{2} \int_S xyz(x\mathbf{i} + y\mathbf{j}) dS = \frac{1}{2} \int_S (x^2 y z \mathbf{i} + x y^2 z \mathbf{j}) dS \quad (1)$$

We have to evaluate this integral over the prescribed surface.

Changing to cylindrical coordinates with  $r = 2$

$$x = 2 \cos \theta; \quad y = 2 \sin \theta; \quad z = z; \quad dS = 2 d\theta dz$$



$$\therefore x^2yz = (4\cos^2\theta)(2\sin\theta)(z) = 8z \cos^2\theta \sin\theta$$

$$xy^2z = (2\cos\theta)(4\sin^2\theta)(z) = 8z \cos\theta \sin^2\theta$$

Then result  $\int_S \mathbf{F} \cdot d\mathbf{S} = \frac{1}{2} \int_S (x^2yz\mathbf{i} + xy^2z\mathbf{j}) \cdot d\mathbf{S}$  becomes.

$$\begin{aligned} \int_S \mathbf{F} \cdot d\mathbf{S} &= \frac{8}{2} \int_0^{\pi/2} \int_0^3 (\cos^2\theta \sin\theta \mathbf{i} + \cos\theta \sin^2\theta \mathbf{j}) 2z dz d\theta = 4 \int_0^{\pi/2} \int_0^3 (\cos^2\theta \sin\theta \mathbf{i} + \cos\theta \sin^2\theta \mathbf{j}) 2z dz d\theta \\ &= 4 \int_0^{\pi/2} (\cos^2\theta \sin\theta \mathbf{i} + \cos\theta \sin^2\theta \mathbf{j}) z^2 \Big|_0^3 d\theta = 4 \int_0^{\pi/2} (\cos^2\theta \sin\theta \mathbf{i} + \cos\theta \sin^2\theta \mathbf{j}) 9 d\theta \\ &= 36 \left[ -\frac{\cos^3\theta}{3} \mathbf{i} + \frac{\sin^3\theta}{3} \mathbf{j} \right]_0^{\pi/2} = 12 (\mathbf{i} + \mathbf{j}) \end{aligned}$$

### Vector Field

A vector field  $\mathbf{F} = y\mathbf{i} + 2z\mathbf{j} + \mathbf{k}$  exists over a surface  $S$  defined by  $x^2 + y^2 + z^2 = 9$  bounded by  $x = 0, y = 0, z = 0$  in the first octant,

**Example:** Evaluate  $\int_S \mathbf{F} \cdot d\mathbf{S}$  over the surface indicated

$$d\mathbf{S} = \hat{\mathbf{n}} dS; \quad \hat{\mathbf{n}} = \frac{\nabla S}{|\nabla S|}$$

$$S: x^2 + y^2 + z^2 - 9 = 0$$

**Solution:**  $\nabla S = \frac{\partial S}{\partial x} \mathbf{i} + \frac{\partial S}{\partial y} \mathbf{j} + \frac{\partial S}{\partial z} \mathbf{k} = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$$\therefore |\nabla S| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 2\sqrt{9} = 6$$

$$\therefore \hat{\mathbf{n}} = \frac{1}{6} (2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) = \frac{1}{3} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \hat{\mathbf{n}} dS = \int_S (y\mathbf{i} + 2z\mathbf{j} + \mathbf{k}) \cdot \frac{1}{3} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) dS = \frac{1}{3} \int_S (xy + 2yz + z^2) dS$$

Before integrating over the surface, we convert to spherical polar coordinates.

$$x = 3 \sin \phi \cos \theta; \quad y = 3 \sin \phi \sin \theta; \quad z = 3 \cos \phi; \quad dS = 9 \sin \phi d\phi d\theta$$

Limits of  $\phi$  and  $\theta$  are  $\phi = 0$  to  $\frac{\pi}{2}$ ;  $\theta = 0$  to  $\frac{\pi}{2}$

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \frac{1}{3} \int_S (xy + 2yz + z^2) dS$$

$$xy = 3 \sin \phi \cos \theta \cdot 3 \sin \phi \sin \theta = 9 \sin^2 \phi \cos \theta \sin \theta$$

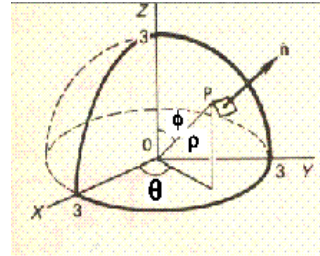
$$2yz = 2 \cdot 3 \sin \phi \sin \theta \cdot 3 \cos \phi = 6 \sin \phi \cos \phi \sin \theta$$

$$z^2 = 3^2 \cos^2 \phi = 9 \cos^2 \phi$$

$$dS = 9 \sin \phi d\phi d\theta$$

Putting these values we get

$$\therefore \int_S \mathbf{F} \cdot d\mathbf{S} = \frac{1}{3} \int_0^{\pi/2} \int_0^{\pi/2} (9 \sin^2 \phi \cos \theta \sin \theta + 6 \sin \phi \cos \phi \sin \theta + 9 \cos^2 \phi) 9 \sin \phi d\phi d\theta$$



$$= 9 \int_0^{\pi/2} \int_0^{\pi/2} (3\sin^3\phi \sin\theta \cos\theta + 2\sin^2\phi \sin\theta + \sin\phi \cos\phi) d\phi d\theta$$

As we know that

$$\int_0^{\pi/2} \sin^3\phi d\phi = \frac{2}{3} \quad \text{by Wallis Formula}$$

$$\text{Also } \int_0^{\pi/2} \sin^2\phi d\phi = \frac{1}{2} \frac{\pi}{2}$$

So we get

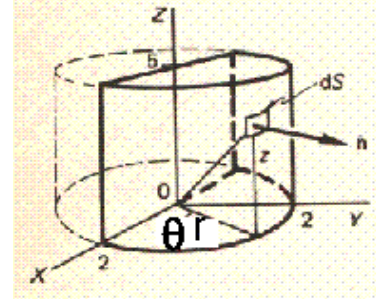
$$\begin{aligned} \int_S \mathbf{F} \cdot d\mathbf{S} &= 9 \int_0^{\pi/2} \left( 2\sin\theta \cos\theta + \frac{\pi}{2} \sin\theta + \frac{1}{2} \right) d\theta \\ &= 9 \left[ \sin 2\theta - \frac{\pi}{2} \cos\theta + \frac{\theta}{2} \right]_0^{\pi/2} = 9 \left[ \left( 1 - 0 + \frac{\pi}{4} \right) - \left( 0 - \frac{\pi}{2} + 0 \right) \right] \\ &= 9 \left[ 1 + \frac{\pi}{4} + \frac{\pi}{2} \right] = 9 \left( 1 + \frac{3\pi}{4} \right) \end{aligned}$$

**Example :** Evaluate  $\int_S \mathbf{F} \cdot d\mathbf{S}$  where  $\mathbf{F} = 2y\mathbf{j} + z\mathbf{k}$  and  $S$  is the surface  $x^2 + y^2 = 4$  in the first two octants bounded by the planes  $z = 0$ ,  $z = 5$  and  $y = 0$ .

**Solution:**  $S : x^2 + y^2 - 4 = 0$

$$\hat{\mathbf{n}} = \frac{\nabla S}{|\nabla S|}$$

$$\nabla S = \frac{\partial S}{\partial x} \mathbf{i} + \frac{\partial S}{\partial y} \mathbf{j} + \frac{\partial S}{\partial z} \mathbf{k} = 2x\mathbf{i} + 2y\mathbf{j}$$



$$\therefore |\nabla S| = \sqrt{4x^2 + 4y^2} = 2\sqrt{x^2 + y^2} = 2\sqrt{4} = 4$$

$$\therefore \hat{\mathbf{n}} = \frac{\nabla S}{|\nabla S|} = \frac{2x\mathbf{i} + 2y\mathbf{j}}{4} = \frac{1}{2} (x\mathbf{i} + y\mathbf{j})$$

$$\therefore \int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \hat{\mathbf{n}} dS$$

$$\int_S \mathbf{F} \cdot \hat{\mathbf{n}} dS = \frac{1}{2} \int_S (2y^2) dS = \int_S y^2 dS$$

This is clearly a case for using cylindrical polar coordinates.

$$x = 2 \cos \theta; \quad y = 2 \sin \theta; \quad z = z; \quad dS = 2d\theta dz$$

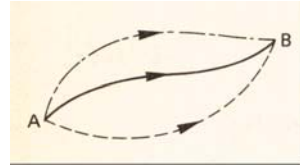
$$\therefore \int_S \mathbf{F} \cdot d\mathbf{S} = \int_S y^2 dS = \iint_S 4\sin^2\theta 2d\theta dz = 8 \iint_S \sin^2\theta d\theta dz$$

Limits:  $\theta = 0$  to  $\theta = \pi$ ;  $z = 0$  to  $z = 5$

$$\int_S \mathbf{F} \cdot d\mathbf{S} = 4 \int_{z=0}^5 \int_{\theta=0}^{\pi} (1 - \cos 2\theta) d\theta dz = 4 \int_0^5 \left[ \theta - \frac{\sin 2\theta}{2} \right]_0^{\pi} dz = 4 \int_0^5 \pi dz = 4\pi [z]_0^5 = 20\pi$$

### Conservative Vector Fields

In general, the value of the integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$  between two stated points A and B depends on the particular path of integration followed. If, however, the line integral between A and B is independent of the path of integration between the two end points, then the vector field  $\mathbf{F}$  is said to be conservative.



It follows that, for a closed path in a conservative

field,  $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ .

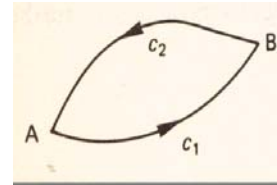
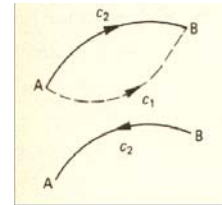
For, if the field is conservative,

$$\int_{C_1(AB)} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2(AB)} \mathbf{F} \cdot d\mathbf{r}$$

But  $\int_{C_2(BA)} \mathbf{F} \cdot d\mathbf{r} = - \int_{C_2(AB)} \mathbf{F} \cdot d\mathbf{r}$

Hence, for the closed path  $AB_{C_1} + BA_{C_2}$ ,

$$\begin{aligned} \oint \mathbf{F} \cdot d\mathbf{r} &= \int_{C_1(AB)} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2(AB)} \mathbf{F} \cdot d\mathbf{r} \\ &= \int_{C_1(AB)} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2(BA)} \mathbf{F} \cdot d\mathbf{r} \\ &= \int_{C_1(AB)} \mathbf{F} \cdot d\mathbf{r} - \int_{C_1(AB)} \mathbf{F} \cdot d\mathbf{r} = 0 \end{aligned}$$

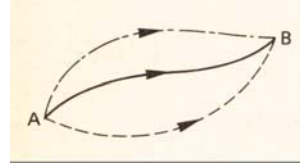


## LECTURE No. 38

## VECTOR FIELD

Conservative Vector Fields

In general, the value of the integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$  between two stated points A and B depends on the particular path of integration followed. If, however, the line integral between A and B is independent of the path of integration between the two end points, then the vector field  $\mathbf{F}$  is said to be conservative.



It follows that, for a closed path in a conservative field,  $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ .

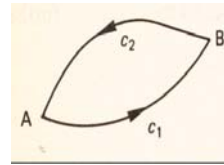
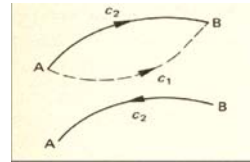
For, if the field is conservative,

$$\int_{C_1(AB)} \mathbf{F} \cdot d\mathbf{r} = \int_{c_2(AB)} \mathbf{F} \cdot d\mathbf{r} \quad \text{But} \quad \int_{c_2(BA)} \mathbf{F} \cdot d\mathbf{r} = - \int_{c_2(AB)} \mathbf{F} \cdot d\mathbf{r}$$

Hence, for the closed path  $AB_{c_1} + BA_{c_2}$ ,  $\oint \mathbf{F} \cdot d\mathbf{r}$

$$= \int_{C_1(AB)} \mathbf{F} \cdot d\mathbf{r} + \int_{c_2(AB)} \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_{C_1(AB)} \mathbf{F} \cdot d\mathbf{r} - \int_{c_2(BA)} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1(AB)} \mathbf{F} \cdot d\mathbf{r} - \int_{c_1(AB)} \mathbf{F} \cdot d\mathbf{r} = 0$$



Note that this result holds good only for a closed curve and when the vector field is a conservative field.

Now for some examples

**Example**

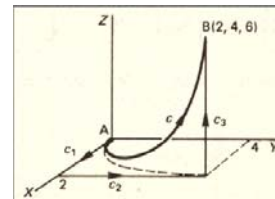
If  $\mathbf{F} = 2xyzi + x^2zj + x^2yk$ , evaluate the line integral  $\int \mathbf{F} \cdot d\mathbf{r}$  between  $A(0,0,0)$  and  $B(2,4,6)$

(a) along the curve  $c$  whose parametric equations are  $x = u$ ,  $y = u^2$ ,  $z = 3u$

(b) along the three straight lines  $c_1: (0,0,0)$  to  $(2, 0, 0)$ ;  $c_2 : (2, 0, 0)$  to  $(2, 4, 0)$ ;  $c_3: (2, 4, 0)$  to  $(2, 4, 6)$ .

Hence determine whether or not  $\mathbf{F}$  is a conservative field.

First draw the diagram.



$$(a) \quad \mathbf{F} = 2xyzi + x^2zj + x^2yk$$

$$x = u; \quad y = u^2; \quad z = 3u$$

$$\therefore \quad dx = du; \quad dy = 2u du; \quad dz = 3 du.$$

$$\mathbf{F} \cdot d\mathbf{r} = (2xyzi + x^2zj + x^2yk) \cdot (dx i + dy j + dz k)$$

$$= 2xyz dx + x^2z dy + x^2y dz$$

Using the transformation shown above, we can now express  $\mathbf{F} \cdot d\mathbf{r}$  in terms of  $u$ .

$$\text{for} \quad 2xyz dx = (2u)(u^2)(3u) du = 6u^4 du$$

$$x^2z dy = (u^2)(3u)(2u) du = 6u^4 du$$

$$x^2y dz = (u^2)(u^2)3 du = 3u^4 du$$

$$\therefore \mathbf{F} \cdot d\mathbf{r} = 15u^4 du$$

The limits of integration in  $u$  are  $u = 0$  to  $u = 2$

$$\therefore \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 15u^4 du = \left[ 3u^5 \right]_0^2 = 96$$

(b) The diagram for (b) is as shown. We consider each straight

$$\int \mathbf{F} \cdot d\mathbf{r} = \int (2xyz dx + x^2z dy + x^2y dz)$$

$c_1$ : (0,0,0) to (2,0,0);  $y = 0, z = 0, dy = 0,$

$$\therefore \int_{c_1} \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + 0 = 0$$

In the same way, we evaluate the line integral along  $c_2$  and  $c_3$ .

$$\int_{c_1} \mathbf{F} \cdot d\mathbf{r} = 0;$$

$$\int \mathbf{F} \cdot d\mathbf{r} = \int (2xyz dx + x^2z dy + x^2y dz)$$

$c_2$ : (2,0,0) to (2,4,0);  $x = 2, z = 0, dx = 0, dz = 0$

$$\therefore \int_{c_2} \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + 0 = 0 \quad \int_{c_2} \mathbf{F} \cdot d\mathbf{r} = 0$$

$c_3$ : (2,4,0) to (2,4,6);  $x = 2, y = 4, dx = 0, dy = 0$

$$\therefore \int_{c_3} \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + \int_0^6 16 dz = \left[ 16z \right]_0^6 = 96 \quad \therefore \int_{c_3} \mathbf{F} \cdot d\mathbf{r} = 96$$

Collecting the three results together

$$\int_{c_1+c_2+c_3} \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + 96 \quad \therefore \int_{c_1+c_2+c_3} \mathbf{F} \cdot d\mathbf{r} = 96$$

In this particular example, the value of the line integral is independent of the two paths we have used joining the same two end points and indicates that  $\mathbf{F}$  is a conservative field.

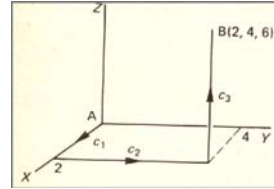
It follows that

$$\begin{aligned} \text{curl } \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz & x^2z & x^2y \end{vmatrix} \\ &= (x^2 - x^2)\mathbf{i} - (2xy - 2xy)\mathbf{j} + (2xz - 2xz)\mathbf{k} = \mathbf{0} \\ \therefore \text{curl } \mathbf{F} &= \mathbf{0} \end{aligned}$$

**REMARK:** So three tests can be applied to determine whether or not a vector field is conservative. They are

- (a)  $\oint \mathbf{F} \cdot d\mathbf{r} = 0$
- (b)  $\text{curl } \mathbf{F} = \mathbf{0}$
- (c)  $\mathbf{F} = \text{grad } V$

Any one of these conditions can be applied as is convenient.

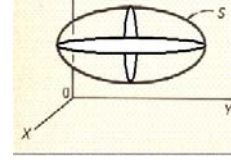


**Divergence Theorem (Gauss' theorem)**

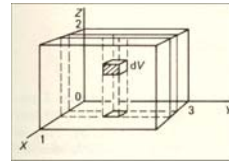
For a closed surface  $S$ , enclosing a region  $V$  in a vector field  $F$

$$\int_V \operatorname{div} \mathbf{F} dV = \int_S \mathbf{F} \cdot d\mathbf{S}$$

In general, this means that the volume integral (triple integral) on the left-hand side can be expressed as a surface integral (double integral) on the right-hand side.

**Example**

Verify the divergence theorem for the vector field  $\mathbf{F} = x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k}$  taken over the region bounded by the planes  $z = 0, z = 2, x = 0, x = 1, y = 0, y = 3$ .  
 $dV = dx dy dz$



we have to show that  $\int_V \operatorname{div} \mathbf{F} dV = \int_S \mathbf{F} \cdot d\mathbf{S}$

(a) To find  $\int_V \operatorname{div} \mathbf{F} dV$

$$\begin{aligned} \operatorname{div} \mathbf{F} &= \nabla \cdot \mathbf{F} = \left( \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) \cdot (x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k}) \\ &= \frac{\partial}{\partial x} (x^2) + \frac{\partial}{\partial y} (z) + \frac{\partial}{\partial z} (y) = 2x + 0 + 0 = 2x \\ \therefore \int_V \operatorname{div} \mathbf{F} dV &= \int_V 2x dV = \iiint_V 2x dz dy dx \\ \int_V \operatorname{div} \mathbf{F} dV &= \int_0^1 \int_0^3 \int_0^2 2x dz dy dx = \int_0^1 \int_0^3 [2xz]_0^2 dy dx = \int_0^1 [4xy]_0^3 dx = \int_0^1 12x dx = [6x^2]_0^1 = 6 \end{aligned}$$

(b) to find  $\int_S \mathbf{F} \cdot d\mathbf{S}$  i.e.  $\int_S \mathbf{F} \cdot \hat{\mathbf{n}} dS$

The enclosing surface  $S$  consists of six separate plane faces denoted as

$S_1, S_2, \dots, S_6$  as shown. We consider each face in turn.

$\mathbf{F} = x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k}$

(i)  $S_1$  (base):  $z = 0; \hat{\mathbf{n}} = -\mathbf{k}$  (outwards and downwards)

$$\therefore \mathbf{F} = x^2\mathbf{i} + y\mathbf{k} \quad dS_1 = dx dy$$

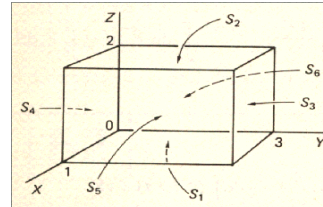
$$\therefore \int_{S_1} \mathbf{F} \cdot \hat{\mathbf{n}} dS = \iint_{S_1} (x^2\mathbf{i} + y\mathbf{k}) \cdot (-\mathbf{k}) dy dx = \int_0^1 \int_0^3 (-y) dy dx = \int_0^1 \left[ -\frac{y^2}{2} \right]_0^3 dx = -\frac{9}{2}$$

(ii)  $S_2$  (top):  $z = 2; \hat{\mathbf{n}} = \mathbf{k}$   $dS_2 = dx dy$

$$\int_{S_2} \mathbf{F} \cdot \hat{\mathbf{n}} dS = \iint_{S_2} (x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k}) \cdot (\mathbf{k}) dy dx = \int_0^1 \int_0^3 y dy dx = \frac{9}{2}$$

(iii)  $S_3$  (right-hand end):  $y = 3;$

$$\begin{aligned} \hat{\mathbf{n}} &= \mathbf{j} \quad dS_3 = dx dz \\ \mathbf{F} &= x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k} \end{aligned}$$



$$\therefore \int_{S_3} \mathbf{F} \cdot \hat{\mathbf{n}} dS = \iint_{S_3} (x^2 \mathbf{i} + z \mathbf{j} + 3 \mathbf{k}) \cdot (\mathbf{j}) dz dx = \int_0^1 \int_0^2 z dz dx = \int_0^1 \left[ \frac{z^2}{2} \right]_0^2 dx = \int_0^1 2 dx = 2$$

(iv)  $S_4$  (left-hand end):  $y = 0$ ,  $\hat{\mathbf{n}} = -\mathbf{j}$ ,  $dS_4 = dx dz$

$$\therefore \int_{S_4} \mathbf{F} \cdot \hat{\mathbf{n}} dS = -2$$

$$\text{for } \int_{S_4} \mathbf{F} \cdot \hat{\mathbf{n}} dS = \iint_{S_4} (x^2 \mathbf{i} + z \mathbf{j} + y \mathbf{k}) \cdot (-\mathbf{j}) dz dx = \int_0^1 \int_0^2 (-z) dz dx = \int_0^1 \left[ -\frac{z^2}{2} \right]_0^2 dx = \int_0^1 (-2) dx = -2$$

Now for the remaining two sides  $S_5$  and  $S_6$ . Evaluate these in the same manner, obtaining

$$\int_{S_5} \mathbf{F} \cdot \hat{\mathbf{n}} dS = 6; \quad \int_{S_6} \mathbf{F} \cdot \hat{\mathbf{n}} dS = 0$$

Check:

(v)  $S_5$  (front):  $x = 1$ ;  $\hat{\mathbf{n}} = \mathbf{i}$   $dS_5 = dy dz$

$$\therefore \int_{S_5} \mathbf{F} \cdot \hat{\mathbf{n}} dS = \iint_{S_5} (\mathbf{i} + z \mathbf{j} + y \mathbf{k}) \cdot (\mathbf{i}) dy dz = \iint_{S_5} 1 dy dz = 6$$

(vi)  $S_6$  (back):  $x = 0$ ;  $\hat{\mathbf{n}} = -\mathbf{i}$   $dS_6 = dy dz$

$$\therefore \int_{S_6} \mathbf{F} \cdot \hat{\mathbf{n}} dS = \iint_{S_6} (z \mathbf{j} + y \mathbf{k}) \cdot (-\mathbf{i}) dy dz = \iint_{S_6} 0 dy dz = 0$$

For the whole surface  $S$  we therefore have  $\int_S \mathbf{F} \cdot d\mathbf{S} = -\frac{9}{2} + \frac{9}{2} + 2 - 2 + 6 + 0 = 6$

and from our previous work in section (a)  $\int_V \text{div } \mathbf{F} dV = 6$

We have therefore verified as required that, in this example  $\int_V \text{div } \mathbf{F} dV = \int_S \mathbf{F} \cdot d\mathbf{S}$

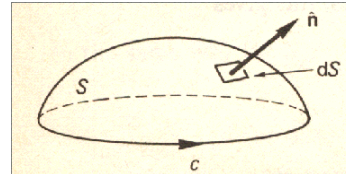
### Stokes Theorem

If  $\mathbf{F}$  is a vector field existing over an open surface  $S$  and around its boundary closed curve

$$C, \text{ then } \int_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

This means that we can express a surface integral in terms of a line integral round the boundary curve.

The proof of this theorem is rather lengthy and is to be found in the Appendix. Let us demonstrate its application in the following examples.



**Example:** A hemisphere  $S$  is defined by  $x^2 + y^2 + z^2 = 4$  ( $z \geq 0$ ). A vector field  $\mathbf{F} = 2y\mathbf{i} - x\mathbf{j} + xz\mathbf{k}$  exists over the surface and around its boundary  $c$ . Verify Stoke's theorem that

$$\int_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}.$$

$$S : x^2 + y^2 + z^2 - 4 = 0$$

$$\mathbf{F} = 2y\mathbf{i} - x\mathbf{j} + xz\mathbf{k} \quad c \text{ is the circle } x^2 + y^2 = 4.$$

$$(a) \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_C (2y\mathbf{i} - x\mathbf{j} + xz\mathbf{k}) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k})$$

$$= \int_C (2ydx - xdy + xzdz)$$

Converting to polar coordinates.  $x = 2 \cos \theta$ ;  $y = 2 \sin \theta$ ;  $z = 0$

$dx = -2 \sin \theta d\theta$ ;  $dy = 2 \cos \theta d\theta$ ; Limits  $\theta = 0$  to  $2\pi$

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (4 \sin \theta [-2 \sin \theta d\theta] - 2 \cos \theta 2 \cos \theta d\theta) = -4 \int_0^{2\pi} (2 \sin^2 \theta + \cos^2 \theta) d\theta \\ &= -4 \int_0^{2\pi} (1 + \sin^2 \theta) d\theta = -2 \int_0^{2\pi} (3 - \cos 2\theta) d\theta = -2 \left[ 3\theta - \frac{\sin 2\theta}{2} \right]_0^{2\pi} = -12\pi \quad (1) \end{aligned}$$

(b) Now we determine  $\int_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$

$$\int_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_S \text{curl } \mathbf{F} \cdot \hat{\mathbf{n}} dS$$

$$\mathbf{F} = 2y\mathbf{i} - x\mathbf{j} + xz\mathbf{k}$$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & -x & xz \end{vmatrix} = \mathbf{i}(0-0) - \mathbf{j}(z-0) + \mathbf{k}(-1-2) = -z\mathbf{j} - 3\mathbf{k}$$

$$\text{Now } \hat{\mathbf{n}} = \frac{\nabla S}{|\nabla S|} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{2}$$

$$\text{Then } \int_S \text{curl } \mathbf{F} \cdot \hat{\mathbf{n}} dS = \int_S (-z\mathbf{j} - 3\mathbf{k}) \cdot \left( \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{2} \right) dS = \frac{1}{2} \int_S (-yz - 3z) dS$$

$$x = 2 \sin \phi \cos \theta; y = 2 \sin \phi \sin \theta; z = 2 \cos \phi, dS = 4 \sin \phi d\phi d\theta$$

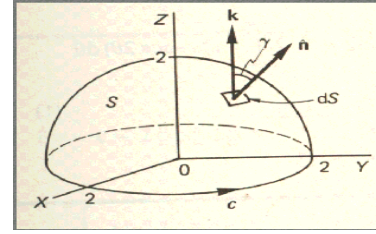
$$\therefore \int_S \text{curl } \mathbf{F} \cdot \hat{\mathbf{n}} dS = \frac{1}{2} \iint_S (-2 \sin \phi \sin \theta 2 \cos \phi - 6 \cos \phi) 4 \sin \phi d\phi d\theta$$

$$= - \int_0^{2\pi} \int_0^{\pi/2} (2 \sin^2 \phi \cos \phi \sin \theta + 3 \sin \phi \cos \phi) d\phi d\theta = -4 \int_0^{2\pi} \left[ \frac{2 \sin^3 \phi \sin \theta}{3} + \frac{3 \sin^2 \phi}{2} \right]_0^{\pi/2} d\theta$$

$$= -4 \int_0^{2\pi} \left( \frac{2}{3} \sin \theta + \frac{3}{2} \right) d\theta = -12\pi \quad (2)$$

So we have from our two results (1) and (2)

$$\int_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}$$



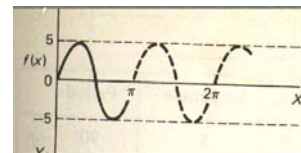
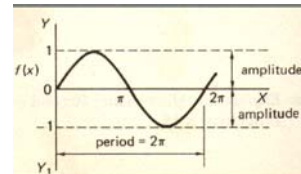
**LECTURE No. 39****PERIODIC FUNCTIONS****Periodic functions**

A function  $f(x)$  is said to be periodic if its function values repeat at regular intervals of the independent variable. The regular interval between repetitions is the period of the oscillations.  $f(x + p) = f(x)$

**Graphs of  $y = A \sin nx$** 

(a)  $y = \sin x$  The obvious example of a periodic function is  $y = \sin x$ , which goes through its complete range of values while  $x$  increases from  $0^\circ$  to  $360^\circ$ .

The period is therefore  $360^\circ$  or  $2\pi$  radians and the amplitude, the maximum displacement from the position of rest.

**Example:**

$$y = 5 \sin 2x$$

The amplitude is 5. The period is  $180^\circ$  and there are thus 2 complete cycles in  $360^\circ$ .

Thinking along the same lines, the function  $y = A \sin nx$

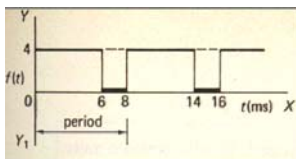
has amplitude =  $A$ ; period =  $\frac{360^\circ}{n} = \frac{2\pi}{n}$ ;  $n$  cycles in  $360^\circ$ .

Graphs of  $y = A \cos nx$  have the same characteristics

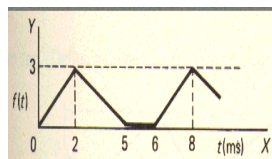
**Example**

Functions	Amplitude	Period
$y = 3 \sin 5x$	3	$72^\circ$
$y = 2 \cos 3x$	2	$120^\circ$
$y = \sin \frac{x}{2}$	1	$720^\circ$
$y = 4 \sin 2x$	4	$180^\circ$

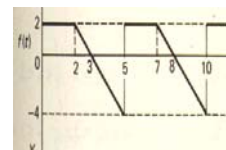
$$y = A \sin nx$$

**Example**

period = 8 ms



period = 6 ms



period = 5 cm

**Analytical description of a periodic function**

A periodic function can be defined analytically in many cases.

**Example**

(a) Between  $x = 0$  and  $x = 4$ ,  $y = 3$ ,

i.e.  $f(x) = 3 \quad 0 < x < 4$

(b) Between  $x = 4$  and  $x = 6$ ,  $y = 0$ ,

i.e.  $f(x) = 0 \quad 4 < x < 6$ .

So we could define the function by

$$f(x) = 3 \quad 0 < x < 4$$

$$f(x) = 0 \quad 4 < x < 6$$

$$f(x) = f(x + 6)$$

the last line indicating that the function is periodic with period 6 units

$$f(x) = 2 - x \quad 0 < x < 3$$

$$f(x) = -1 \quad 3 < x < 5$$

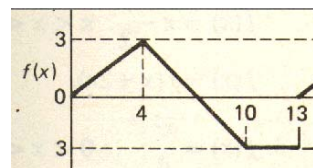
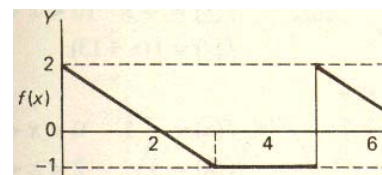
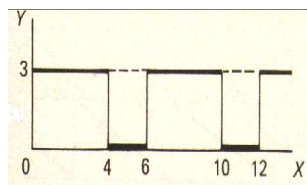
$$f(x) = f(x + 5)$$

$$f(x) = \frac{3x}{4} \quad 0 < x < 4$$

$$f(x) = 7 - x \quad 4 < x < 10$$

$$f(x) = -3 \quad 10 < x < 13$$

$$f(x) = f(x + 13)$$

**Example**

Sketch the graphs of the following inserting relevant values.

1.  $f(x) = 4 \quad 0 < x < 5$

$f(x) = 0 \quad 5 < x < 8$

$f(x) = f(x + 8)$

$f(x) = 3x - x^2 \quad 0 < x < 3$

$f(x) = f(x + 3)$

$f(x) = 2 \sin x \quad 0 < x < \pi$

$f(x) = 0 \quad \pi < x < 2\pi$

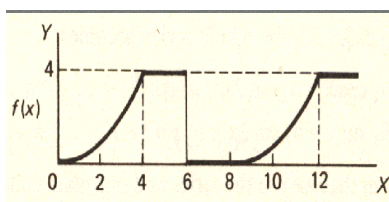
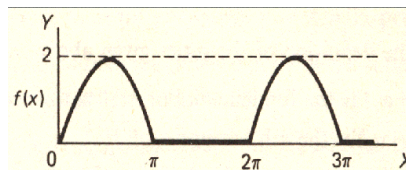
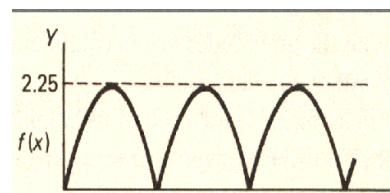
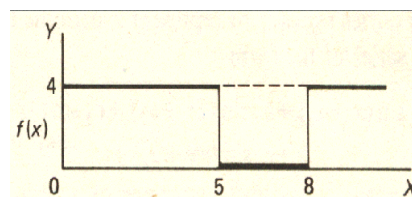
$f(x) = f(x + 2\pi)$

$f(x) = \frac{x^2}{4} \quad 0 < x < 4$

$f(x) = 4 \quad 4 < x < 6$

$f(x) = 0 \quad 6 < x < 8$

$f(x) = f(x + 8)$



### Useful integrals

The following integrals appear frequently in our work on Fourier series, so it will help if we obtain the result in readiness. In each case,  $m$  and  $n$  are integers other than zero.

$$(a) \int_{-\pi}^{\pi} \sin nx \, dx = \left[ \frac{-\cos nx}{n} \right]_{-\pi}^{\pi} = \frac{1}{n} \{-\cos n\pi + \cos n\pi\} = 0$$

$$(b) \int_{-\pi}^{\pi} \cos nx \, dx = \left[ \frac{\sin nx}{n} \right]_{-\pi}^{\pi} = \frac{1}{n} \{\sin n\pi + \sin n\pi\} = 0$$

$$(c) \int_{-\pi}^{\pi} \sin^2 nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} (1 - \cos 2nx) \, dx = \frac{1}{2} \left[ x - \frac{\sin 2nx}{2n} \right]_{-\pi}^{\pi} = \pi \quad (n \neq 0)$$

$$(d) \int_{-\pi}^{\pi} \cos^2 nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} (1 + \cos 2nx) \, dx = \frac{1}{2} \left[ x + \frac{\sin 2nx}{2n} \right]_{-\pi}^{\pi} = \pi \quad (n \neq 0)$$

$$(e) \int_{-\pi}^{\pi} \sin nx \cos mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \{\sin(n+m)x + \sin(n-m)x\} \, dx = \frac{1}{2} \{0 + 0\} = 0$$

from result (a) with  $n \neq m$

$$(f) \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \{\cos(n+m)x + \cos(n-m)x\} \, dx = \frac{1}{2} \{0 + 0\} = 0$$

from result (b) with  $n \neq m$

If  $n = m$ , then

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx \text{ becomes } \int_{-\pi}^{\pi} \cos^2 nx \, dx = \pi \quad \text{from (d) above.}$$

$$(g) \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = -\frac{1}{2} \int_{-\pi}^{\pi} (-2) \sin nx \sin mx \, dx = -\frac{1}{2} \int_{-\pi}^{\pi} \{\cos(n+m)x - \cos(n-m)x\} \, dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \{0 - 0\} = 0 \quad \text{from result (b) with } n \neq m$$

If  $n = m$ , then

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx \text{ becomes } \int_{-\pi}^{\pi} \sin^2 nx \, dx = \pi \quad \text{from (c) above}$$

### Summary of integrals

$$(a) \int_{-\pi}^{\pi} \sin nx \, dx = 0, \quad (b) \int_{-\pi}^{\pi} \cos nx \, dx = 0, \quad (c) \int_{-\pi}^{\pi} \sin^2 nx \, dx = \pi \quad (n \neq 0)$$

$$\begin{aligned}
 & \text{(d) } \int_{-\pi}^{\pi} \cos^2 nx \, dx = \pi \quad (n \neq 0) \quad \text{(e) } \int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0 \\
 & \text{(f) } \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0 \quad (n \neq m) \quad \text{(g) } \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0 \quad (n \neq m) \\
 & \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \pi \quad (n = m) \quad \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \pi \quad (n = m)
 \end{aligned}$$

**Note**

We have evaluated the integrals between  $-\pi$  and  $\pi$ , but, provided integration is carried out over a complete periodic interval of  $2\pi$ , the results are the same. Thus, the limits could just as well be  $-\pi$  to  $\pi$ ,  $0$  to  $2\pi$ ,  $-\pi/2$  to  $3\pi/2$ , etc. We can therefore choose the limits to suit the particular problems.

**Fourier series****Periodic functions of period  $2\pi$** 

The basis of a Fourier series is to represent a periodic function by a trigonometrical series of the form.

$$f(x) = A_0 + c_1 \sin(x + \alpha_1) + c_2 \sin(2x + \alpha_2) + c_3 \sin(3x + \alpha_3) + \dots + c_n \sin(nx + \alpha_n) + \dots$$

where  $A_0$  is a constant term

$c_1, c_2, c_3, \dots, c_n$  denote the amplitudes of the compound sine terms  $\alpha_1, \alpha_2, \alpha_3, \dots$  are constant auxiliary angles.

Each sine term,  $c_n \sin(nx + \alpha_n)$  can be expanded thus:

$$\begin{aligned}
 c_n \sin(nx + \alpha_n) &= c_n \{\sin nx \cos \alpha_n + \cos nx \sin \alpha_n\} = (c_n \sin \alpha_n) \cos nx + (c_n \cos \alpha_n) \sin nx \\
 &= a_n \cos nx + b_n \sin nx
 \end{aligned}$$

The whole series becomes.

$$f(x) = A_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$$

## LECTURE No.40

## FOURIER SERIES

As we know that  $A_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$ ; which can be written as in the expanded form

$$\begin{aligned} & A_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots + (a_n \cos nx + b_n \sin nx) + \cdots \\ &= A_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx + \cdots \end{aligned}$$

$$f(x) = A_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx + \cdots$$

**Fourier coefficients**

We have defined Fourier series in the form

$$f(x) = A_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}; \quad n \text{ a positive integer}$$

(a) To find  $A_0$ , we integrate  $f(x)$  with respect to  $x$  from  $-\pi$  to  $\pi$ .

$$\int_{-\pi}^{\pi} f(x) dx = \int_{-\pi}^{\pi} A_0 dx + \sum_{n=1}^{\infty} \text{Error!} = [A_0 x]_{-\pi}^{\pi} + \text{Error!} \{0 + 0\} = 2A_0 \pi$$

$$2A_0 \pi = \int_{-\pi}^{\pi} f(x) dx$$

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2} a_0; \quad \text{Where } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

(b) To find  $a_n$  we multiply  $f(x)$  by  $\cos mx$  and integrate from  $-\pi$  to  $\pi$ .

$$\int_{-\pi}^{\pi} f(x) \cos mx dx = \int_{-\pi}^{\pi} A_0 \cos mx dx + \sum_{n=1}^{\infty} \text{Error!}$$

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos mx dx &= A_0 \{0\} + \sum_{n=1}^{\infty} \{a_n (0) + b_n (0)\} = 0 & \text{for } n \neq m \\ &= 0 + a_n \pi + 0 = a_n \pi & \text{for } n = m \end{aligned}$$

$$\therefore a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

(c) To find  $b_n$  we multiply  $f(x)$  by  $\sin mx$  and integrate from  $-\pi$  to  $\pi$ .

$$\int_{-\pi}^{\pi} f(x) \sin mx dx = \int_{-\pi}^{\pi} A_0 \sin mx dx + \sum_{n=1}^{\infty} \text{Error!}$$

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \sin mx dx &= A_0 \{0\} + \sum_{n=1}^{\infty} \{a_n (0) + b_n (0)\} = 0 & \text{for } n \neq m \\ &= 0 + 0 + b_n \pi = b_n \pi & \text{for } n = m \end{aligned}$$

$$\therefore b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

### Result For Fourier Series

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\};$$

$$(a) \quad a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = 2 \times \text{mean value of } f(x) \text{ over a period}$$

$$(b) \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = 2 \times \text{mean value of } f(x) \cos nx \text{ over a period.}$$

$$(c) \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 2 \times \text{mean value of } f(x) \sin nx \text{ over a period.}$$

In each case,  $n = 1, 2, 3, \dots$

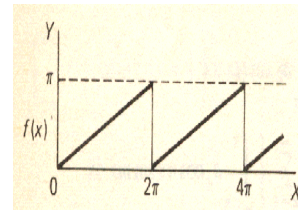
### **Example**

Determine the Fourier series to represent the periodic function shown.

It is more convenient here to take the limits as 0 to  $2\pi$ .  
The function can be defined as

$$f(x) = \frac{x}{2} \quad 0 < x < 2\pi$$

$$f(x) = f(x + 2\pi) \quad \text{period} = 2\pi.$$



Now to find the coefficients

$$(a) \quad a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) \, dx = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{x}{2}\right) \, dx = \frac{1}{4\pi} [x^2]_0^{2\pi} = \pi$$

$$a_0 = \pi$$

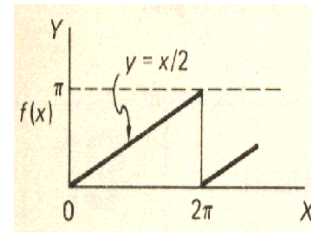
$$(b) \quad a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \left(\frac{x}{2}\right) \cos nx \, dx$$

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} x \cos nx \, dx = \frac{1}{2\pi} \text{ Error! = Error!Error!}$$

$$\therefore a_n = 0$$

$$(a) \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx \quad \text{So we now have}$$



$$b_n = \frac{1}{\pi} \int_0^{2\pi} \frac{x}{2} \sin nx \, dx = \frac{1}{2\pi} \text{ Error!} = - \text{Error!} [2\pi - 0] = - \text{Error!}$$

$$a_0 = \pi; \quad a_n = 0; \quad b_n = -\frac{1}{n}$$

Now the general expression for a Fourier series is

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\} \quad \text{Therefore in this case}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \{b_n \sin nx\} = \frac{\pi}{2} + \left\{-\frac{1}{1} \sin x - \frac{1}{2} \sin 2x - \frac{1}{3} \sin 3x - \dots\right\} \quad \text{since } a_n = 0$$

$$f(x) = \frac{\pi}{2} - \left\{\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots\right\}$$

### Dirichlet Conditions

If the Fourier series is to represent a function  $f(x)$ , then putting  $x = x_1$  will give an infinite series in  $x_1$  and the value of this should converge to the value of  $f(x_1)$  as more and more terms of the series are evaluated. For this to happen, the following conditions must be fulfilled.

(a) The function  $f(x)$  must be defined and single-valued.

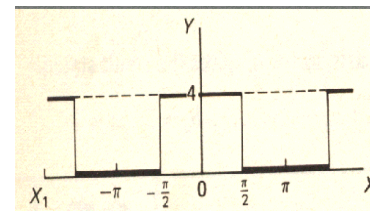
(b)  $f(x)$  must be continuous or have a finite number of finite discontinuities within a periodic interval.

(c)  $f(x)$  and  $f'(x)$  must be piecewise continuous in the periodic interval.

If these Dirichlet conditions are satisfied, the Fourier series converges to  $f(x_1)$ , if  $x = x_1$  is a point of continuity

### Example

Find the Fourier series for the function shown.



Consider one cycle between  $x=0$  and  $x=\pi$ .

The function can be defined by

$$f(x) = 0 \quad -\pi < x < -\frac{\pi}{2}$$

$$f(x) = 4 \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$f(x) = 0 \quad \frac{\pi}{2} < x < \pi$$

$$f(x) = f(x + 2\pi)$$

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$$

The expression for  $a_0$  is  $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx$  This gives

$$a_0 = \frac{1}{\pi} \text{ Error!} = \text{Error!} [4x] \text{Error!} = 4$$

$$a_0 = 4$$

$$(b) \ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} 4 \cos nx \, dx$$

$$= \frac{1}{\pi} \left[ \frac{4 \sin nx}{n} \right]_{-\pi}^{\pi} = \frac{4}{n\pi} [\sin n\pi - \sin(-n\pi)] = \frac{4}{n\pi} [\sin n\pi + \sin n\pi] = \frac{8 \sin n\pi}{n\pi}$$

$$a_n = \frac{8}{n\pi} \sin \frac{n\pi}{2}$$

Then considering different integer values of  $n$ , we have

$$\begin{aligned} \text{If } n \text{ is even} & \quad a_n = 0 \\ \text{If } n = 1, 5, 9, \dots & \quad a_n = \frac{8}{n\pi} \\ \text{If } n = 3, 7, 11, \dots & \quad a_n = -\frac{8}{n\pi} \end{aligned}$$

(c) To find  $b_n$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} 4 \sin nx \, dx$$

$$b_n = \frac{4}{\pi} \int_{-\pi/2}^{\pi/2} \sin nx \, dx = \frac{4}{\pi} \left[ -\frac{\cos nx}{n} \right]_{-\pi/2}^{\pi/2} = -\frac{4}{n\pi} \left\{ \cos \frac{n\pi}{2} - \cos \left( -\frac{n\pi}{2} \right) \right\} = 0$$

$$b_n = 0$$

So with  $a_0 = 4$ :  $a_n$  as stated above;  $b_n = 0$ ;

The Fourier series is

$$f(x) = 2 + \frac{8}{\pi} \left\{ \cos x - \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x - \frac{1}{7} \cos 7x + \dots \right\}$$

In this particular example, there are, in fact, no sine terms.

### Effect Of Harmonics

It is interesting to see just how accurately the Fourier series represents the function with which it is associated. The complete representation requires an infinite number of terms, but we can, at least, see the effect of including the first few terms of the series.

Let us consider the waveform shown. We established earlier that the function

$$f(x) = 0 \quad -\pi < x < -\frac{\pi}{2}$$

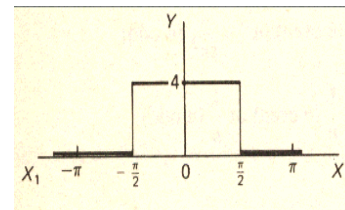
$$f(x) = 4 \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$f(x) = 0 \quad \frac{\pi}{2} < x < \pi$$

$$f(x) = f(x + 2\pi)$$

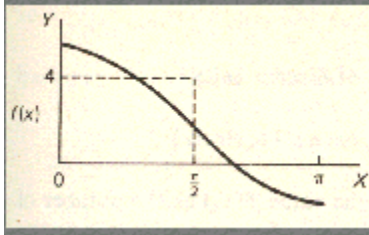
gives the Fourier series

$$f(x) = 2 + \frac{8}{\pi} \left\{ \cos x - \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x - \frac{1}{7} \cos 7x + \dots \right\}$$

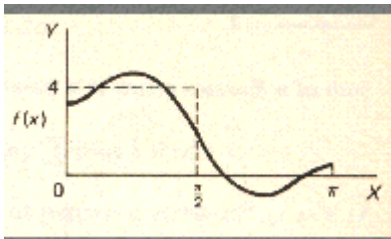


If we start with just one cosine term, we can then see the effect of including subsequent harmonics. Let us restrict our attention to just the right-hand half of the symmetrical waveform. Detailed plotting of points gives the following development.

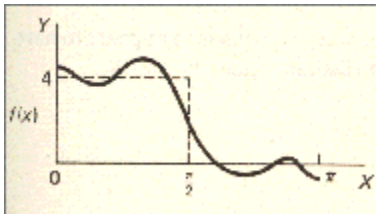
$$(1) f(x) = 2 + \frac{8}{\pi} \cos x$$



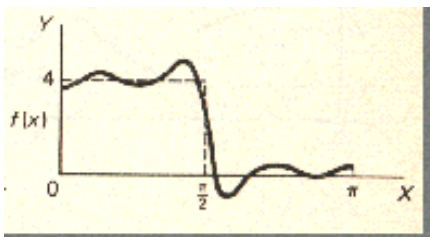
$$(2) f(x) = 2 + \frac{8}{\pi} \left\{ \cos x - \frac{1}{3} \cos 3x \right\}$$



$$(3) f(x) = 2 + \frac{8}{\pi} \left\{ \cos x - \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x \right\}$$



$$(4) f(x) = 2 + \frac{8}{\pi} \left\{ \cos x - \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x - \frac{1}{7} \cos 7x \right\}$$



As the number of terms is increased, the graph gradually approaches the shape of the original square waveform. The ripples increase in number and decrease in amplitude, but a perfectly square waveform is unattainable in practice. For practical purpose, the first few terms normally suffice to give an accuracy of acceptable level.

## LECTURE No.41

## EXAMPLES

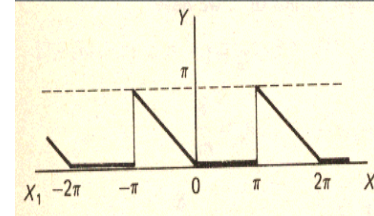
**Example 1**

Find the Fourier series for the function defined by

$$f(x) = -x \quad -\pi < x < 0$$

$$f(x) = 0 \quad 0 < x < \pi$$

$$f(x) = f(x + 2\pi)$$

**Solution:**

The general expressions for  $a_0$ ,  $a_n$ ,  $b_n$  are

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^0 (-x) dx + \frac{1}{\pi} \int_0^{\pi} 0 dx = \frac{1}{\pi} \int_{-\pi}^0 (-x) dx = \frac{1}{\pi} \left[ -\frac{x^2}{2} \right]_{-\pi}^0 = \frac{\pi}{2}$$

(b) To find  $a_n$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^0 (-x) \cos nx dx + \frac{1}{\pi} \int_0^{\pi} 0 dx = -\frac{1}{\pi} \int_{-\pi}^0 x \cos nx dx$$

$$= -\frac{1}{\pi} \text{Error!} = -\text{Error! Error!}$$

$$= -\frac{1}{\pi} \left\{ \frac{1}{n} \left[ \frac{\cos nx}{n} \right]_{-\pi}^0 \right\} = -\frac{1}{\pi n^2} \left\{ \left[ \frac{\cos nx}{n} \right]_{-\pi}^0 \right\} = -\frac{1}{\pi n^2} [\cos 0 - \cos n\pi]$$

$$= -\frac{1}{\pi n^2} \{1 - \cos n\pi\}$$

But  $\cos n\pi = 1$  (n even) and  $\cos n\pi = -1$  (n odd)

$$a_n = -\frac{2}{\pi n^2} \text{ (n odd) and } a_n = 0 \text{ (n even)}$$

(c) Now to find  $b_n$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_{-\pi}^0 (-x) \sin nx dx = -\frac{1}{\pi} \int_{-\pi}^0 x \sin nx dx$$

$$= -\frac{1}{\pi} \text{Error!} = -\text{Error! Error!} = -\text{Error!}$$

$$b_n = -\frac{1}{n} \text{ (n even) and } b_n = \frac{1}{n} \text{ (n odd)}$$

$$\text{So we have } a_0 = \frac{\pi}{2}; \quad a_n = 0 \text{ (n even) and } a_n = -\frac{2}{\pi n^2} \text{ (n odd)}$$

$$b_n = -\frac{1}{n} \text{ (n even) and } b_n = \frac{1}{n} \text{ (n odd)}$$

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi} \left( \cos x + \frac{1}{9} \cos 3x + \frac{1}{25} \cos 5x + \dots \right) + \left( \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \dots \right)$$

It is just a case of substituting  $n = 1, 2, 3$ , etc.

In this particular example, we have a constant term and both sine and cosine terms.

### **Odd And Even Functions**

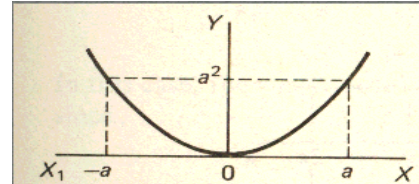
#### **(a) Even functions**

A function  $f(x)$  is said to be even if  $f(-x) = f(x)$  i.e. the function value for a particular negative value of  $x$  is the same as that for the corresponding positive value of  $x$ . The graph of an even function is therefore symmetrical about the  $y$ -axis.

$y = f(x) = x^2$  is an even function  
since

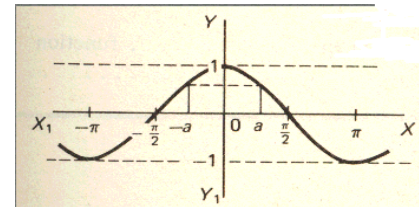
$$f(-2) = 4 = f(2)$$

$$f(-3) = 9 = f(3) \text{ etc.}$$



$y = f(x) = \cos x$  is an even function  
since  $\cos(-x) = \cos x$

$$f(-a) = \cos a = f(a)$$



#### **(b) Odd functions**

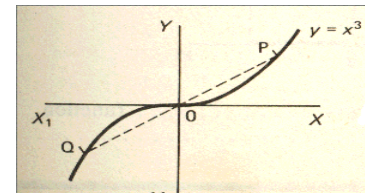
A function  $f(x)$  is said to be odd if  $f(-x) = -f(x)$

i.e. the function value for a particular negative value of  $x$  is numerically equal to that for the corresponding positive value of  $x$  but opposite in sign. The graph of an odd function is thus symmetrical about the origin.

$y = f(x) = x^3$  is an odd function since

$$f(-2) = -8 = -f(2)$$

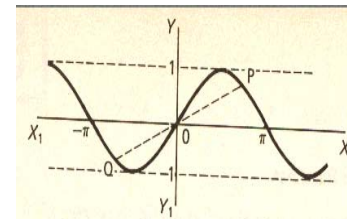
$$f(-5) = -125 = -f(5) \text{ etc.}$$



$y = f(x) = \sin x$  is an odd function

Since  $\sin(-x) = -\sin x$

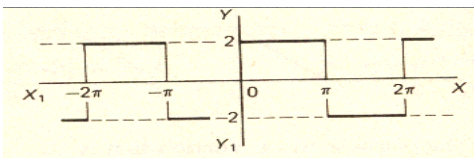
$$f(-a) = -f(a).$$



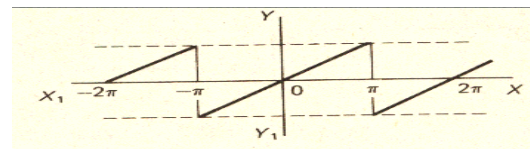
#### **Remarks:**

(1) An even function  $f(-x) = f(x)$  is symmetrical about the  $y$ -axis

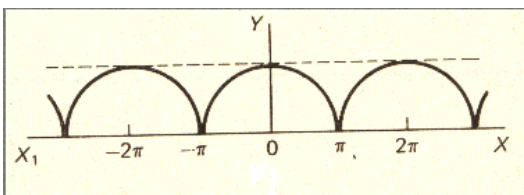
(2) An odd function  $f(-x) = -f(x)$  symmetrical about the origin.

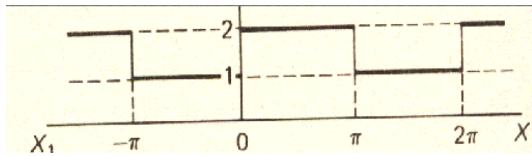


odd.



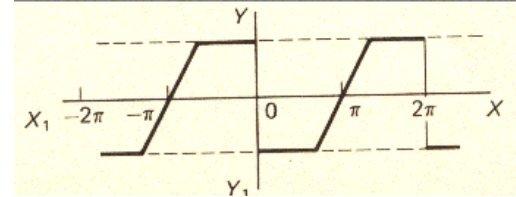
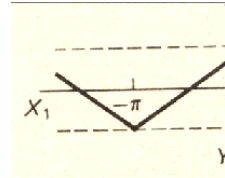
Odd





Even  
Odd

Even  
neither



### Products Of Odd And Even Functions

(a) Two even functions

Let  $F(x) = f(x) g(x)$  where  $f(x)$  and  $g(x)$  are even functions.

Then  $F(-x) = f(-x) g(-x) = f(x) g(x)$  since  $f(x)$  and  $g(x)$  are even.

$$\therefore F(-x) = F(x)$$

$F(x)$  is even

(b) Two odd functions

Let  $F(x) = f(x) g(x)$ , where  $f(x)$  and  $g(x)$  are odd functions.

Then  $F(-x) = f(-x) g(-x) = \{-f(x)\} \{-g(x)\}$

since  $f(x)$  and  $g(x)$  are odd.

$$= f(x) g(x) = F(x)$$

$$\therefore F(-x) = F(x)$$

$F(x)$  is even

(c) One odd and one even function

Let  $F(x) = f(x) g(x)$  where  $f(x)$  is odd and  $g(x)$  even.

Then  $F(-x) = f(-x) g(-x) = -f(x) g(x) = -F(x)$

$$\therefore F(-x) = -F(x)$$

$F(x)$  is odd

**So if  $f(x)$  and  $g(x)$  are both even, then  $f(x) g(x)$  is even and if  $f(x)$  and  $g(x)$  are both odd, then  $f(x) g(x)$  is even but if either  $f(x)$  or  $g(x)$  is even and the other odd. Then  $f(x) g(x)$  is odd.**

### **Example 2**

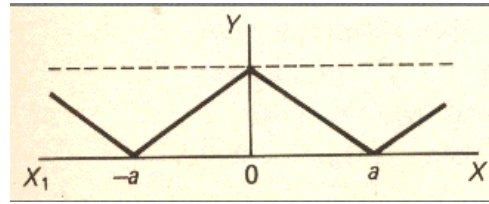
State whether each of the following products is odd, even, or neither.

1.  $x^2 \sin 2x$       odd    (E) (O) = (O)
2.  $x^3 \cos x$       odd    (O) (E) = (O)
3.  $\cos 2x \cos 3x$     even    (E) (E) = (E)
4.  $x \sin nx$       even    (O) (O) = (E)
5.  $3 \sin x \cos 4x$     odd    (O) (E) = (O)
6.  $(2x + 3) \sin 4x$     neither (N) (O) = (N)
7.  $\sin^2 x \cos 3x$     even    (E) (E) = (E)
8.  $x^3 e^x$       neither (O) (N) = (N)
9.  $(x^4 + 4) \sin 2x$     odd    (E) (O) = (O)

**Two useful facts emerge from odd and even functions.**

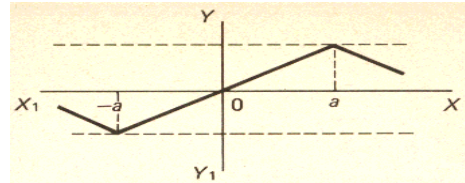
(a) For an even function

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$



(b) For an odd function

$$\int_{-a}^a f(x) dx = 0$$



### **THEOREM 1**

If  $f(x)$  is defined over the interval  $-\pi < x < \pi$  and  $f(x)$  is even, then the Fourier series for  $f(x)$  contains cosine terms only. Included in this is  $a_0$  which may be regarded as  $a_n \cos nx$  with  $n = 0$ .

**Proof:**

$$(a) \quad a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} f(x) dx \quad \therefore \quad a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$(b) \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx.$$

But  $f(x) \cos nx$  is the product of two even functions and therefore itself even.

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx \quad \therefore \quad a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$(c) \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

Since  $f(x) \sin nx$  is the product of an even function and an odd function, it is itself odd.

$$\therefore \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = 0.$$

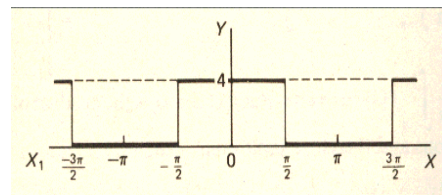
$$\therefore \quad b_n = 0$$

Therefore, there are no sine terms in the Fourier series for  $f(x)$ .

### **Example 3**

The waveform shown is symmetrical about the y-axis. The function is therefore even and there will be no sine terms in the series.

$$\therefore \quad f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$



$$\begin{aligned} \text{(a) } a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi/2} 4 dx \\ &= \frac{2}{\pi} [4x]_0^{\pi/2} = 4 \end{aligned}$$

$$\begin{aligned} \text{(b) } a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi/2} 4 \cos nx dx = \frac{8}{\pi} \left[ \frac{\sin nx}{n} \right]_0^{\pi/2} \\ &= \frac{8}{\pi n} \sin \frac{n\pi}{2} \end{aligned}$$

$$\begin{aligned} \text{But } \sin \frac{n\pi}{2} &= 0 && \text{for } n \text{ even} \\ &= 1 && \text{for } n = 1, 5, 9, \dots \\ &= -1 && \text{for } n = 3, 7, 11, \dots \end{aligned}$$

$$a_n = 0 \quad (n \text{ even});$$

$$a_n = \frac{8}{\pi n} \quad (n = 1, 5, 9, \dots);$$

$$a_n = -\frac{8}{\pi n} \quad (n = 3, 7, 11, \dots)$$

(c)  $b_n = 0$ , since  $f(x)$  is an even function. Therefore, the required series is

$$f(x) = 2 + \frac{\pi}{8} \left\{ \cos x - \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x - \frac{1}{7} \cos 7x + \dots \right\}$$

### **Theorem 2:**

If  $f(x)$  is an odd function defined over the interval  $-\pi < x < \pi$ , then the Fourier series for  $f(x)$  contains sine terms only.

#### **Proof:**

Since  $f(x)$  is an odd function

$$\int_{-\pi}^0 f(x) dx = - \int_0^{\pi} f(x) dx.$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx.$$

But  $f(x)$  is odd

$$\therefore a_0 = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

Remembering that  $f(x)$  is odd and  $\cos nx$  is even, the product  $f(x) \cos nx$  is odd.

$$\therefore a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (\text{odd function}) dx$$

$$\therefore a_n = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

and since  $f(x)$  and  $\sin nx$  are each odd, the product  $f(x) \sin nx$  is even.

$$\text{Then } b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (\text{even function}) dx = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$$

$$\therefore b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$$

So,

If  $f(x)$  is odd function then  $a_0 = 0$ ;  $a_n = 0$ ;

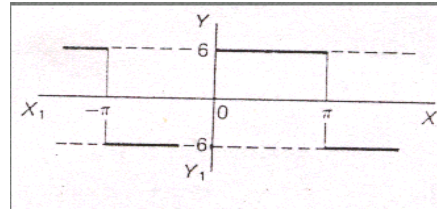
$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx \text{ i.e. the Fourier series contains sine terms only.}$$

#### Example 4

$$f(x) = -6 \quad -\pi < x < 0$$

$$f(x) = 6 \quad 0 < x < \pi$$

$$f(x) = f(x + 2\pi)$$



We can see that this is an odd function;

$$a_0 = 0 \text{ and } a_n = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx.$$

$f(x) \sin nx$  is a product of two odd functions and is therefore even.

$$\therefore b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$$

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} 6 \sin nx dx = \frac{12}{\pi} \left[ \frac{-\cos nx}{n} \right]_0^{\pi} = \frac{12}{\pi} \left[ \frac{\cos nx}{n} \right]_{\pi}^0 = \frac{12}{\pi} [\cos 0 - \cos n\pi] \\ &= \frac{12}{\pi n} (1 - \cos n\pi). \end{aligned}$$

$$b_n = 0 \quad (n \text{ even}) \quad b_n = \frac{24}{\pi n} \quad (n \text{ odd})$$

$$\text{So the series is } f(x) = \frac{24}{\pi} \left\{ \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right\}$$

**Remark:** If  $f(x)$  is neither an odd nor an even function, then we must obtain expressions for  $a_0$ ,  $a_n$  and  $b_n$  in full.

## LECTURE No.42

## EXAMPLES

**Example**

Determine the Fourier series for the function shown.

**Solution:** This is neither odd nor even.

Therefore we must find  $a_0$ ,  $a_n$  and  $b_n$ .

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$$

$$f(x) = \frac{2}{\pi} x \quad 0 < x < \pi$$

$$= 2, \quad \pi < x < 2\pi$$

$$(a) a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \text{ Error! = Error! Error! = Error! } \{ \pi + 4\pi - 2\pi \}$$

$$\therefore a_0 = 3$$

$$(b) a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \text{ Error!}$$

$$= \frac{1}{\pi} \text{ Error! = Error! Error!}$$

$$= \frac{2}{\pi} \text{ Error! = Error! Error!}$$

$$= \frac{2}{\pi} \left\{ \frac{1}{n\pi} \left[ \frac{\cos nx}{n} \right]_0^{\pi} \right\} = \frac{2}{\pi^2 n^2} \{ \cos n\pi - \cos 0 \} = \frac{2}{\pi^2 n^2} \{ \cos n\pi - 1 \}$$

$$a_n = 0 \quad (n \text{ even}); \quad a_n = \frac{-4}{\pi^2 n^2} \quad (n \text{ odd})$$

(c) To find  $b_n$ , we proceed in the same general manner

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \text{ Error!}$$

$$= \frac{2}{\pi} \text{ Error! = Error! Error!}$$

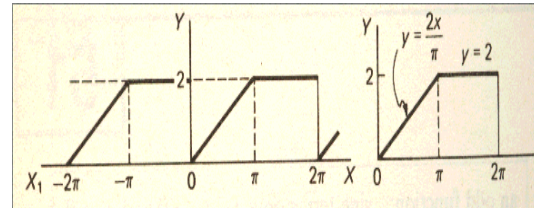
$$= \frac{2}{\pi} \left\{ -\frac{1}{n} \cos n\pi + (0-0) - \frac{1}{n} (\cos 2\pi n - \cos n\pi) \right\} = \frac{2}{\pi} \left\{ -\frac{1}{n} \cos n\pi + (0-0) - \frac{1}{n} \cos 2\pi n + \frac{1}{n} \cos n\pi \right\}$$

$$= \frac{2}{\pi} \left\{ -\frac{1}{n} \cos 2n\pi \right\} = -\frac{2}{\pi n} \cos 2n\pi$$

But  $\cos 2n\pi = 1$ .

$$\therefore b_n = -\frac{2}{\pi n}$$

So the required series is



$$f(x) = \frac{3}{2} - \frac{4}{\pi^2} \left\{ \cos x + \frac{1}{9} \cos 3x + \frac{1}{25} \cos 5x + \dots \right\} - \frac{2}{\pi} \left\{ \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \frac{1}{4} \sin 4x + \dots \right\}$$

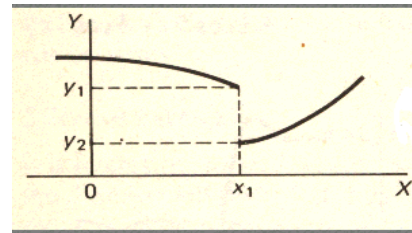
### Sum of a Fourier series at a point of discontinuity

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$$

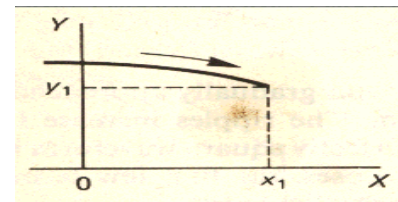
At  $x = x_1$ , the series converges to the value  $f(x_1)$  as the number of terms including increases to infinity.

A particular point of interest occurs at a point of finite discontinuity or 'jump' of the function  $y = f(x)$ .

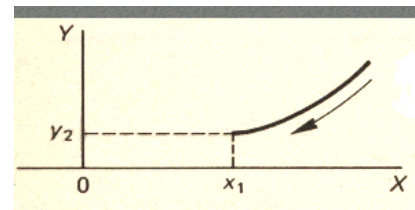
At  $x = x_1$ , the function appears to have two distinct values,  $y_1$  and  $y_2$ .



If we approach  $x = x_1$  from below that value, the limiting value of  $f(x)$  is  $y_1$ .



If we approach  $x = x_1$  from above that value, the limiting value of  $f(x)$  is  $y_2$ .



To distinguish between these two values we write

$y_1 = f(x_1 - 0)$  denoting immediately before  $x = x_1$

$y_2 = f(x_1 + 0)$

denoting immediately after  $x = x_1$ .

In fact, if we substitute  $x = x_1$  in the Fourier series for  $f(x)$ , it can be shown that the series converges to the value

$$\frac{1}{2} \{f(x_1 - 0) + f(x_1 + 0)\} = \frac{1}{2} (y_1 + y_2), \text{ the average of } y_1 \text{ and } y_2.$$

### Example

Consider the function

$$f(x) = 0 \quad -\pi < x < 0$$

$$f(x) = a \quad 0 < x < \pi$$

$$f(x) = f(x + 2\pi)$$

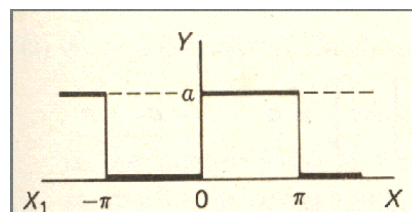
$$(a) \quad a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} a dx$$

$$= \frac{1}{\pi} [ax]_0^{\pi} = a$$

$$\therefore a_0 = a$$

$$(b) \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{\pi} a \cos nx dx = \frac{a}{\pi} \left[ \frac{\sin nx}{n} \right]_0^{\pi} = 0$$

$$\therefore a_n = 0$$



$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_0^{\pi} a \sin nx dx = \frac{a}{\pi} \left[ \frac{-\cos nx}{n} \right]_0^{\pi} = \frac{a}{n\pi} (1 - \cos n\pi)$$

But  $\cos n\pi = 1$  (n even) and  $\cos n\pi = -1$  (n odd)

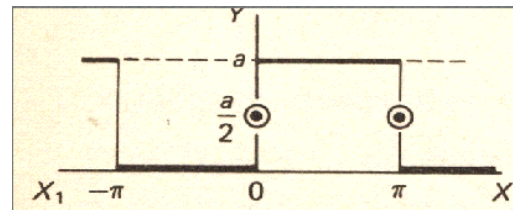
$$b_n = 0 \quad (\text{n even}) \quad \text{and} \quad b_n = \frac{2a}{n\pi} \quad (\text{n odd})$$

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \frac{a}{2} + \frac{2a}{\pi} \left\{ \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right\}$$

A finite discontinuity, or 'jump', occurs at  $x = 0$ . If we substitute  $x = 0$  in the series obtained, all the sine terms vanish and we get  $f(x) = a/2$ , which is, in fact, the average of the two function values at  $x = 0$ .

Note also that at  $x = \pi$ , another finite discontinuity occurs and substituting  $x = \pi$  in the series gives the same result. Because of this ambiguity, the function is said to be 'undefined' at  $x = 0$ ,  $x = \pi$ , etc.



### Half-Range Series

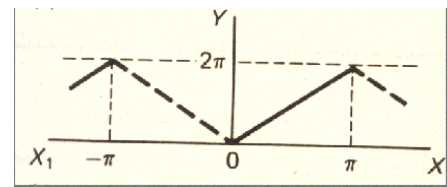
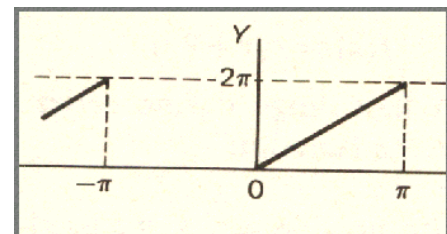
Sometime a function of period  $2\pi$  is defined over the range 0 to  $\pi$ , instead of the normal  $-\pi$  to  $\pi$ , or 0 to  $2\pi$ . We then have a choice of how to proceed.

For example, if we are told between

$x = 0$  and  $x = \pi$ ,  $f(x) = 2x$ ,

then, since the period is  $2\pi$ , we have no evidence of how the function behave between  $x = -\pi$  and  $x = 0$ .

If the waveform were as shown in (a), the function would be an even function, symmetrical about the y-axis and the series would have only cosine terms (including possibly  $a_0$ ).



On the other hand, if the waveform were as shown in (b), the function would be odd, being symmetrical about the origin and the series would have only sine terms.

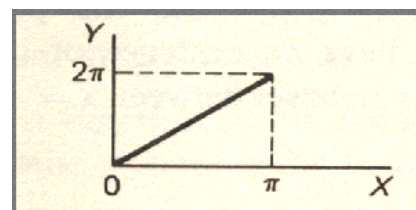
### Example

A function  $f(x)$  is defined by

$$f(x) = 2x \quad 0 < x < \pi$$

$$f(x) = f(x + 2\pi)$$

Obtain a half-range cosine series to represents the function.



To obtain a cosine series, i.e. a series with no sine terms, we need an even function.

Therefore, we assume the waveform between

$x = -\pi$  and  $x = 0$  to be as shown, making the total graph symmetrical about the y-axis.

Now we can find expressions for the Fourier coefficients as usual.

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} 2x dx = \frac{2}{\pi} \left[ x^2 \right]_0^{\pi} = 2\pi$$

$$\therefore a_0 = 2\pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} 2x \cos nx dx = \frac{4}{\pi} \int_0^{\pi} x \cos nx dx = \frac{4}{\pi} \text{ Error!}$$

$$= \frac{4}{\pi} \left\{ (0-0) - \frac{1}{n} \left[ \frac{-\cos nx}{n} \right]_0^{\pi} \right\} = \left( \frac{4}{\pi n^2} \right) (\cos n\pi - 1)$$

$$\cos n\pi = 1 \quad (n \text{ even}) \text{ and } \cos n\pi = -1 \quad (n \text{ odd})$$

$$\therefore a_n = 0 \quad (n \text{ even}) \text{ and } a_n = -\frac{8}{\pi n^2} \quad (n \text{ odd})$$

All that now remains is  $b_n$  which is zero, since  $f(x)$  is an even function, i.e.  $b_n = 0$

$$\text{So } a_0 = 2\pi, \quad a_n = 0 \quad (n \text{ even}) \text{ and } a_n = -\frac{8}{\pi n^2} \quad (n \text{ odd}),$$

$$b_n = 0. \quad \text{Therefore}$$

$$f(x) = \pi - \frac{8}{\pi} \left\{ \cos x + \frac{1}{9} \cos 3x + \frac{1}{25} \cos 5x + \dots \right\}$$

### Example

Determine a half-range sine series to represent the function  $f(x)$  defined by

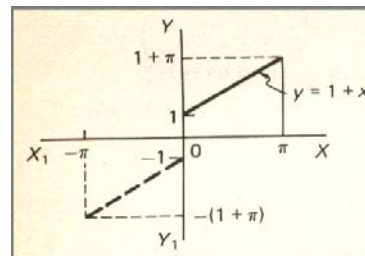
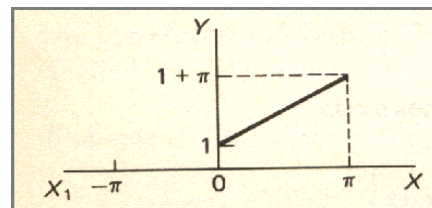
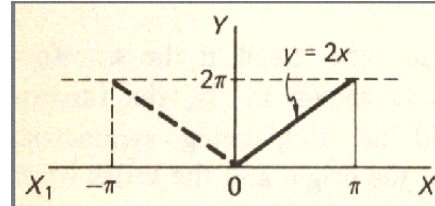
$$f(x) = 1 + x \quad 0 < x < \pi$$

$$f(x) = f(x + 2\pi)$$

We choose the waveform between  $x = -\pi$  and  $x = 0$  so that the graph is symmetrical about the origin. The function is then an odd function and the series will contain only sine terms.

$$\therefore a_0 = 0 \quad \text{and} \quad a_n = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$



$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} (1+x) \sin nx \, dx = \frac{2}{\pi} \text{Error!}$$

$$= \frac{2}{\pi} \left\{ -\frac{1+\pi}{n} \cos n\pi + \frac{1}{n} + \frac{1}{n} \left[ \frac{\sin nx}{n} \right]_0^{\pi} \right\} = \frac{2}{\pi} \left\{ \frac{1}{n} - \frac{1+\pi}{n} \cos n\pi \right\} = \frac{2}{\pi n} \{1 - (1+\pi) \cos n\pi\}$$

$$\cos n\pi = 1 \quad (n \text{ even}) \quad \text{and} \quad \cos n\pi = -1 \quad (n \text{ odd})$$

$$\therefore b_n = -\frac{2}{n} \quad (n \text{ even})$$

$$\therefore b_n = \frac{4+2\pi}{\pi n} \quad (n \text{ odd})$$

Substituting in the general expression  $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$  we have

$$f(x) = \frac{4+2\pi}{\pi} \left\{ \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right\} - 2 \left\{ \frac{1}{2} \sin 2x + \frac{1}{4} \sin 4x + \frac{1}{6} \sin 6x + \dots \right\}$$

and the required series obtained

$$f(x) = \left( \frac{4}{\pi} + 2 \right) \left\{ \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right\} - 2 \left\{ \frac{1}{2} \sin 2x + \frac{1}{4} \sin 4x + \frac{1}{6} \sin 6x + \dots \right\}$$

So knowledge of odd and even functions and of half-range series saves a deal of unnecessary work on occasions.

**LECTURE No. 43****FUNCTIONS WITH PERIODS OTHER THAN  $2\pi$** 

So far, we have considered functions  $f(x)$  with period  $2\pi$ . In practice, we often encounter functions defined over periodic intervals other than  $2\pi$ , e.g. from 0 to  $T$ ,  $-\frac{T}{2}$  to  $\frac{T}{2}$  etc.

**Functions with Period  $T$** 

If  $y = f(x)$  is defined in the range  $-\frac{T}{2}$  to  $\frac{T}{2}$ , i.e. has a period  $T$ , we can convert this to an interval of  $2\pi$  by changing the units of the independent variable.

In many practical cases involving physical oscillations, the independent variable is time ( $t$ ) and the periodic interval is normally denoted by  $T$ , i.e.

$$f(t) = f(t + T)$$

Each cycle is therefore completed in  $T$  seconds and the frequency  $f$  hertz (oscillations per second) of the

periodic function is therefore given by  $f = \frac{1}{T}$ . If the

angular velocity,  $\omega$  radians per seconds, is defined by  $\omega = 2\pi f$ , then

$$\omega = \frac{2\pi}{T} \text{ and } T = \frac{2\pi}{\omega}$$

The angle,  $x$  radians, at any time  $t$  is therefore  $x = \omega t$  and the Fourier series to represent the function can be expressed as

$$f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \{a_n \cos n\omega t + b_n \sin n\omega t\}$$

which can also be written in the form

$$f(t) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} B_n \sin (n\omega t + \phi_n) \quad n = 1, 2, 3, \dots$$

**Fourier Coefficients**

With the new variable, the Fourier coefficients become:

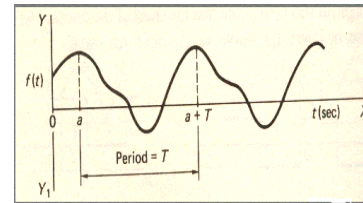
$$f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \{a_n \cos n\omega t + b_n \sin n\omega t\}$$

$$a_0 = \frac{2}{T} \int_0^T f(t) dt = \frac{\omega}{\pi} \int_0^{2\pi/\omega} f(t) dt$$

$$a_n = \frac{2}{T} \int_0^T f(t) \cos n\omega t dt = \frac{\omega}{\pi} \int_0^{2\pi/\omega} f(t) \cos n\omega t dt$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin n\omega t dt = \frac{\omega}{\pi} \int_0^{2\pi/\omega} f(t) \sin n\omega t dt$$

We can see that there is very little difference between these expressions and those that have gone before. The limits can, of course, be 0 to  $T$ ,  $-\frac{T}{2}$  to  $\frac{T}{2}$ ,  $-\frac{\pi}{\omega}$  to  $\frac{\pi}{\omega}$ , 0 to  $\frac{2\pi}{\omega}$  etc. as is convenient, so long as they cover a complete period.



**Example 1**

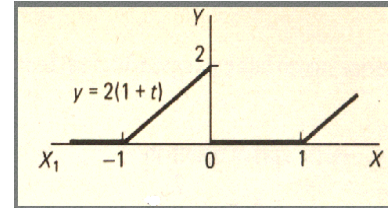
Determine the Fourier series for a periodic function defined by

$$f(t) = 2(1+t) \quad -1 < t < 0$$

$$f(t) = 0 \quad 0 < t < 1$$

$$f(t) = f(t+2)$$

The first step is to sketch the wave which is.



$$f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \{a_n \cos n\omega t + b_n \sin n\omega t\}$$

$$T = 2$$

$$a_0 = \frac{2}{T} \int_{-T/2}^{T/2} f(t) dt = \frac{2}{2} \int_{-1}^1 f(t) dt = \int_{-1}^0 2(1+t) dt + \int_0^1 (0) dt = \left[ 2t + t^2 \right]_{-1}^0 = 1$$

$$\therefore a_0 = 1$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos n\omega t dt = \frac{2}{2} \int_{-1}^1 f(t) \cos n\omega t dt = \int_{-1}^0 2(1+t) \cos n\omega t dt$$

$$a_n = 2 \text{ Error!} = 2 \text{ Error!}$$

$$= \frac{2}{n^2 \omega^2} (1 - \cos n\omega)$$

$$\text{Now } T = \frac{2\pi}{\omega}$$

$$\therefore \omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi \quad \therefore a_n = \frac{2}{n^2 \omega^2} (1 - \cos n\pi)$$

$$\therefore a_n = 0 \text{ (n even)}$$

$$= \frac{4}{n^2 \omega^2} \text{ (n odd)}$$

$$\text{Now for } b_n, \quad b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin n\omega t dt$$

$$b_n = \frac{2}{2} \int_{-1}^0 2(1+t) \sin n\omega t dt = 2 \text{ Error!}$$

$$= 2 \left\{ -\frac{1}{n\omega} + \frac{1}{n\omega} \left[ \frac{\sin n\omega t}{n\omega} \right]_{-1}^0 \right\} = 2 \left\{ -\frac{1}{n\omega} + \frac{1}{n^2 \omega^2} (\sin n\omega) \right\}$$

$$\text{As before } \omega = \pi \quad \therefore b_n = -\frac{2}{n\omega}$$

So the first few terms of the series give

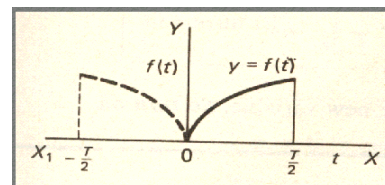
$$f(t) = \frac{1}{2} + \frac{4}{\omega^2} \left\{ \cos \omega t + \frac{1}{9} \cos 3\omega t + \frac{1}{25} \cos 5\omega t + \dots \right\} - \frac{2}{\omega} \left\{ \sin \omega t + \frac{1}{2} \sin 2\omega t + \frac{1}{3} \sin 3\omega t + \frac{1}{4} \sin 4\omega t + \dots \right\}$$

**Half-Range Series**

The theory behind the half-range sine and cosine series still applies with the new variable.

(a) Even function

Half-range cosine series



$$y = f(t) \quad 0 < t < \frac{T}{2}$$

$$f(t) = f(t + T)$$

symmetrical about the y-axis.

With an even function, we know that  $b_n = 0$

$$\therefore f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t$$

$$\text{where } a_0 = \frac{4}{T} \int_0^{T/2} f(t) dt \quad \text{and} \quad a_n = \frac{4}{T} \int_0^{T/2} f(t) \cos n\omega t dt$$

(b) Odd function

Half-range sine series

$$y = f(t) \quad 0 < t < \frac{T}{2}$$

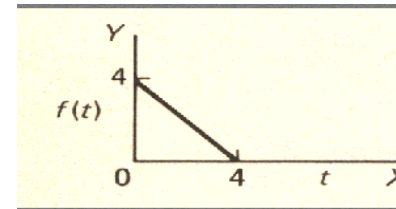
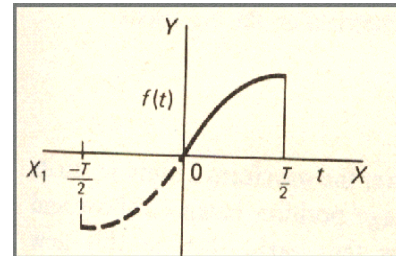
$$f(t) = f(t + T)$$

symmetrical about the origin.

$$\therefore a_0 = 0 \text{ and } a_n = 0$$

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t;$$

$$b_n = \frac{4}{T} \int_0^{T/2} f(t) \sin n\omega t dt$$



### Example 2

A function  $f(t)$  is defined by

$$f(t) = 4 - t, \quad 0 < t < 4.$$

We have to form a half-range cosine series to represent the function in this interval.

First we form an even function, i.e. symmetrical about the y-axis.

$$\begin{aligned} a_0 &= \frac{4}{T} \int_0^{T/2} f_1(t) dt = \frac{4}{8} \int_0^4 (4 - t) dt = \frac{1}{2} \int_0^4 (4 - t) dt = \frac{1}{2} \left[ 4t - \frac{t^2}{2} \right]_0^4 = \frac{1}{2} \left[ 4(4) - \frac{(4)^2}{2} \right] \\ &= \frac{1}{2} [16 - 8] = \frac{1}{2} (8) = 4 \end{aligned}$$

$$a_n = \frac{4}{T} \int_0^{T/2} f_1(t) \cos n\omega t dt = \frac{4}{8} \int_0^4 (4 - t) \cos n\omega t dt$$

Simple integration by parts gives

$$a_n = \frac{1}{2} \left\{ -\frac{2 \sin 4n\omega}{n\omega} - \frac{1}{n^2\omega^2} (\cos 4n\omega - 1) \right\}$$

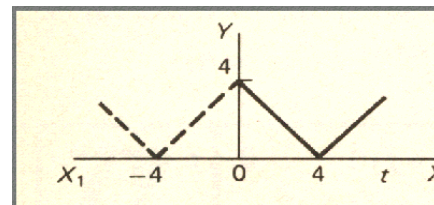
$$\text{But } \omega = \frac{2\pi}{T} = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$\therefore a_n = \frac{1}{2} \left\{ -\frac{2 \sin n\pi}{n\omega} - \frac{1}{n^2\omega^2} (\cos n\pi - 1) \right\}$$

$$n = 1, 2, 3, \dots$$

$$\sin n\pi = 0;$$

$$\cos n\pi = 1 \quad (n \text{ even}); \quad \cos n\pi = -1 \quad (n \text{ odd})$$



$$\therefore a_n = 0 \quad (n \text{ even}) \quad a_n = \frac{1}{n^2 \omega^2} \quad (n \text{ odd})$$

$$\therefore f(t) = 2 + \frac{1}{\omega^2} \left\{ \cos \omega t + \frac{1}{9} \cos 3\omega t + \frac{1}{25} \cos 5\omega t + \dots \right\} \quad \text{where } \omega = \frac{\pi}{4}.$$

**Example 3**

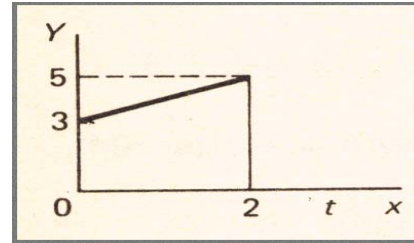
A function  $f(t)$  is defined by

$$f(t) = 3 + t \quad 0 < t < 2$$

$$f(t) = f(t + 4)$$

Obtain the half-range sine series for the function in this range.

Sine series required. Therefore, we form an function, symmetrical about the origin.

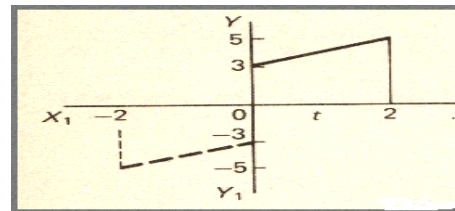


$$a_0 = 0; \quad a_n = 0; \quad T = 4$$

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t$$

$$\therefore b_n = \frac{4}{T} \int_0^{T/2} f(t) \sin n\omega t \, dt = \int_0^2 (3 + t) \sin n\omega t \, dt$$

$$\int_0^2 (3 + t) \sin n\omega t \, dt = \left| \frac{(3+t)\cos n\omega t}{-n\omega} \right|_0^2 - \int_0^2 \frac{\cos n\omega t}{-n\omega} \, dt$$



$$= \frac{(3+2)\cos n\omega 2}{-n\omega} - \frac{3}{-n\omega} + \frac{1}{n\omega} \left| \frac{\sin n\omega t}{n\omega} \right|_0^2 = \frac{3}{n\omega} - \frac{5}{n\omega} \cos 2n\omega + \frac{1}{n^2 \omega^2} \left[ \frac{\sin 2n\omega}{n\omega} - 0 \right]$$

$$\text{But } T = \frac{2\pi}{\omega} \quad \therefore \omega = \frac{2\pi}{T} = \frac{\pi}{2}$$

$$= \frac{3}{n\omega} - \frac{5}{n\omega} \cos n\pi + \frac{1}{n^2 \omega^2} \left[ \frac{\sin n\pi}{n\omega} \right]$$

$$b_n = \frac{1}{n\omega} (3 - 5 \cos 2n\omega) + \frac{1}{n^2 \omega^2} (\sin 2n\omega)$$

$$\therefore b_n = \frac{1}{n\omega} (3 - 5 \cos n\pi) + \frac{1}{n^2 \omega^2} (\sin n\pi) = -\frac{2}{n\omega} \quad (n \text{ even})$$

$$= \frac{8}{n\omega} \quad (n \text{ odd})$$

$$\therefore f(t) = \frac{2}{\omega} \left\{ 4 \sin \omega t - \frac{1}{2} \sin 2\omega t + \frac{4}{3} \sin 3\omega t - \frac{1}{4} \sin 4\omega t \dots \right\}$$

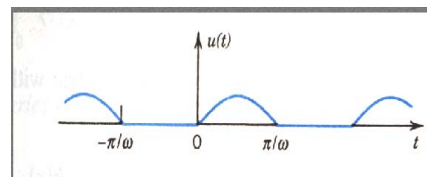
**Half-Wave Rectifier**

A sinusoidal voltage  $E \sin \omega t$ , where  $t$  is time, is passed through a half-wave rectifier that clips the negative portion of the wave

Find the Fourier series of the resulting periodic functions.

$$u(t) = 0 \quad \text{if} \quad -T/2 < t < 0$$

$$= E \sin \omega t \quad 0 < t < T/2 \quad \text{here}$$



$$T = \frac{2\pi}{\omega}$$

$$a_0 = \frac{2}{T} \int_{-T/2}^{T/2} u(t) dt = \frac{2}{T} \int_{-T/2}^0 0 dt + \frac{2}{T} \int_0^{T/2} E \sin \omega t dt = \frac{2}{T} \int_0^{T/2} E \sin \omega t dt = \frac{\omega}{\pi} \int_0^{\pi/\omega} E \sin \omega t dt$$

$$= \frac{\omega}{\pi} E \left| \frac{-\cos \omega t}{\omega} \right|_0^{\pi/\omega} = \frac{2E}{\pi}$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} u(t) \cos n \omega t dt = \frac{2}{T} \int_0^{T/2} E \sin \omega t \cos n \omega t dt = \frac{\omega E}{2\pi} \int_0^{\pi/\omega} 2 \sin \omega t \cos n \omega t dt$$

$$= \frac{\omega E}{2\pi} \int_0^{\pi/\omega} [\sin (1+n) \omega t + \sin (1-n) \omega t] dt$$

If  $n = 1$  then integral on the right is zero and if  $n = 2, 3, \dots$  then we obtain.

$$a_n = \frac{\omega E}{2\pi} \left[ -\frac{\cos (1+n) \omega t}{(1+n)\omega} - \frac{\cos (1-n) \omega t}{(1-n)\omega} \right]_0^{\pi/\omega} = \frac{\omega E}{2\pi} \left[ \frac{-\cos (1+n) \pi + 1}{(1+n)\omega} + \frac{-\cos (1-n) \pi + 1}{(1-n)\omega} \right]$$

$$= \frac{\omega E}{2\pi\omega} \left[ \frac{-\cos (1+n) \pi + 1}{(1+n)} + \frac{-\cos (1-n) \pi + 1}{(1-n)} \right]$$

if  $n$  is odd then  $a_n = 0$

$$\text{if } n \text{ is even then } a_n = \frac{E}{2\pi} \left( \frac{2}{1+n} + \frac{2}{1-n} \right) = \frac{E}{2\pi} \left[ \frac{2 - 2n + 2 + 2n}{(1+n)(1-n)} \right] = \frac{2E}{(1-n)(1+n)\pi}$$

$$= \frac{2E}{(1-n^2)\pi}$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} u(t) \sin n \omega t dt = \frac{2}{T} \int_0^{T/2} E \sin \omega t \sin n \omega t dt = -\frac{\omega E}{2\pi} \int_0^{\pi/\omega} 2 \sin \omega t \sin n \omega t dt$$

$$= -\frac{\omega E}{2\pi} \int_0^{\pi/\omega} [\cos(1+n)\omega t - \cos(1-n)\omega t] dt$$

If  $n = 1$

$$b_n = -\frac{\omega E}{2\pi} \int_0^{\pi/\omega} [\cos 2\omega t - 1] dt = -\frac{\omega E}{2\pi} \left[ \frac{\sin 2\omega t}{2\omega} - t \right]_0^{\pi/\omega} = -\frac{\omega E}{2\pi} (-\pi/\omega) = E/2$$

if  $n \neq 1$

$$b_n = -\frac{\omega E}{2\pi} \int_0^{\pi/\omega} [\cos(1+n)\omega t - \cos(1-n)\omega t] dt = -\frac{\omega E}{2\pi} \left[ \frac{\sin(1+n)\omega t}{(1+n)\omega} - \frac{\sin(1-n)\omega t}{(1-n)\omega} \right]_0^{\pi/\omega}$$

$$= -\frac{\omega E}{2\pi} \left[ \frac{\sin(1+n)\pi}{(1+n)\omega} - \frac{\sin(1-n)\pi}{(1-n)\omega} \right] = 0 \quad \text{for } n = 2, 3, 4, \dots$$

$$u(t) = \frac{1}{2} a_0 + \sum_{n=2}^{\infty} a_n \cos n \omega t$$

$$u(t) = \frac{E}{\pi} + \frac{E}{2} \sin \omega t - \frac{2E}{\pi} \left( \frac{1}{1.3} \cos 2\omega t + \frac{1}{3.5} \cos 4\omega t + \dots \right)$$

## LECTURE No. 44

## LAPLACE TRANSFORMS

**Laplace Transform**

The Laplace Transform of a function  $F(t)$  is denoted by  $L\{F(t)\}$  and is defined as the integral of  $F(t)e^{-st}$  between the limits  $t = 0$  and  $t = \infty$ .

$$L\{F(t)\} = \int_0^{\infty} e^{-st} F(t) dt.$$

In all cases, the constant parameter  $s$  is assumed to be positive and large enough to ensure that the product  $F(t)e^{-st}$  converges to 0 as  $t \rightarrow \infty$ , whatever the function  $F(t)$ .

In determining the Laplace Transform of any function, you will appreciate that the limits are substituted for  $t$  so that the result will be a function of  $s$ .

**Example 1:**

Laplace Transform of  $F(t) = a$ , where  $a$  is constant.

$$\begin{aligned} \text{Solution: } L\{a\} &= \int_0^{\infty} a e^{-st} dt = a \int_0^{\infty} e^{-st} dt = a \left[ \frac{e^{-st}}{-s} \right]_0^{\infty} = \frac{-a}{s} \left[ \frac{1}{e^{st}} \right]_0^{\infty} \\ &= \frac{-a}{s} \left( \frac{1}{e^{\infty}} - \frac{1}{e^0} \right) = -\frac{a}{s} \left( \frac{1}{\infty} - \frac{1}{1} \right) = -\frac{a}{s} (0 - 1) = \frac{a}{s} \\ L\{a\} &= \frac{a}{s} \end{aligned}$$

**Example 2:** Show that  $L\{t^3\} = \frac{3!}{s^4}$ 

$$\text{Solution: } L\{t^3\} = \int_0^{\infty} t^3 e^{-st} dt = \left[ t^3 \frac{e^{-st}}{-s} \right]_0^{\infty} - \int_0^{\infty} 3t^2 \frac{e^{-st}}{-s} dt = \frac{-1}{s} \left[ \frac{t^3}{e^{st}} \right]_0^{\infty} + \frac{3}{s} \int_0^{\infty} t^2 e^{-st} dt$$

As  $e^{st}$  approaches to  $\infty$  more rapidly as compared to  $t^3$ , when  $t$  approaches  $\infty$ .

Therefore,  $\frac{t^3}{e^{st}}$  becomes 0 at  $t = \infty$ .

$$\begin{aligned} &= \frac{-1}{s} (0 - 0) + \frac{3}{s} \left\{ \left[ t^2 \frac{e^{-st}}{-s} \right]_0^{\infty} - \int_0^{\infty} (2t) \frac{e^{-st}}{-s} dt \right\} = 0 + \frac{3}{s} \left\{ \frac{-1}{s} (0 - 0) + \frac{2}{s} \int_0^{\infty} t e^{-st} dt \right\} \\ &= \frac{3}{s} \frac{2}{s} \int_0^{\infty} t e^{-st} dt = \frac{3 \cdot 2}{s^2} \left\{ \left[ t \frac{e^{-st}}{-s} \right]_0^{\infty} - \int_0^{\infty} (1) \frac{e^{-st}}{-s} dt \right\} = \frac{3 \cdot 2}{s^2} \left\{ \frac{-1}{s} (0 - 0) + \frac{1}{s} \int_0^{\infty} e^{-st} dt \right\} \end{aligned}$$

$$= \frac{3.2}{s^3} \frac{(-1)}{s} \int_0^{\infty} e^{-st} (-s) dt = \frac{-3.2.1}{s^4} e^{-st} \Big|_0^{\infty} = \frac{-3.2.1}{s^4} (0-1) = \frac{3!}{s^4}$$

So  $L\{t^3\} = \frac{3!}{s^4}$

### **Example 3**

Find the Laplace transform of the form  $e^{at}$ , that is  $F(t) = e^{at}$  where  $a$  is a constant.

**Solution :**  $L\{e^{at}\} = \int_0^{\infty} e^{at} e^{-st} dt = \int_0^{\infty} e^{-(s-a)t} dt = \left[ \frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty} = -\frac{1}{s-a} \left[ \frac{1}{e^{(s-a)t}} \right]_0^{\infty}, s > a$

$$= -\frac{1}{s-a} [0 - 1] = \frac{1}{s-a}$$

$$L\{e^{at}\} = \frac{1}{s-a}$$

**Remark:** We already have two standard transforms:

$$(1) \quad L\{a\} = \frac{a}{s} \quad L\{4\} = \frac{4}{s} \quad L\{-5\} = \frac{-5}{s}$$

$$(2) \quad L\{e^{at}\} = \frac{1}{s-a} \quad L\{e^{4t}\} = \frac{1}{s-4} \quad L\{e^{-2t}\} = \frac{1}{s+2}$$

**Remark:** Laplace transform is always a function of  $s$ .

$$L\{F(t)\} = f(s)$$

### **Complex Numbers Power of $i$**

Every time a factor  $i^4$  occurs, it can be replaced by the factor 1, so that the power of  $i$  is reduced to one of the four results above.

$$i^9 = i^{4 \times 2 + 1} = i^{4 \times 2} i = (i^4)^2 i = (1)^2 i = i$$

$$i^{20} = i^{4 \times 5} = (i^4)^5 = (1)^5 = 1$$

$$i^{30} = i^{4 \times 7 + 2} = (i^4)^7 i^2 = (1)^7 (-1) = -1$$

$$i^{15} = i^{4 \times 3 + 3} = (i^4)^3 i^3 = (1)^3 (-i) = -i$$

### **Complex Numbers**

$z = 3 + 5i$  is called a complex number where 3 is real part and 5 is imaginary part of the complex number  $z$ .

In general,  $z = a + bi$  is called a complex number, where  $a$  is the real part and  $b$  is an imaginary part of the complex number  $z$ .

$$\text{Complex Number} = (\text{Real Part}) + i (\text{Imaginary Part})$$

### **Conjugate complex numbers**

For a complex number  $a + ib$ , the complex number  $a - ib$  is called the conjugate of  $a + ib$ . Conjugate complex numbers are identical except the signs in the middle for the brackets.

- $4 + 5i$  and  $4 - 5i$  are conjugate complex numbers

- $6 + 2i$  and  $2 + 6i$  are not conjugate complex numbers
- $5 - 3i$  and  $-5 + 3i$  are not conjugate complex numbers

**Remember**

The product of complex number by its conjugate is always entirely real.

$$(3 + 4i)(3 - 4i) = (3)^2 - (4i)^2 = 9 - (-16) = 9 + 16 = 25$$

$$(a + bi)(a - bi) = a^2 + b^2$$

**Euler Formula**

As we know that the series expansion of  $e^x$ ,  $\cos x$  and  $\sin x$  are given as

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Replace  $x$  by  $(it)$ , we get

$$e^{(it)} = 1 + (it) + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \dots$$

$$e^{(it)} = 1 + (it) - \frac{(t)^2}{2!} - \frac{i(t)^3}{3!} + \frac{(t)^4}{4!} + \frac{i(t)^5}{5!} - \frac{(t)^6}{6!} - \frac{i(t)^7}{7!} + \dots$$

$$e^{(it)} = \left[ 1 - \frac{(t)^2}{2!} + \frac{(t)^4}{4!} - \frac{(t)^6}{6!} + \dots \right] + i \left[ t - \frac{(t)^3}{3!} + \frac{(t)^5}{5!} - \frac{(t)^7}{7!} + \dots \right]$$

$$e^{it} = \cos t + i \sin t$$

$$\text{where } \cos t = 1 - \frac{(t)^2}{2!} + \frac{(t)^4}{4!} - \frac{(t)^6}{6!} + \dots \quad \text{and} \quad \sin t = t - \frac{(t)^3}{3!} + \frac{(t)^5}{5!} - \frac{(t)^7}{7!} + \dots$$

$$\operatorname{Re}(e^{it}) = \cos t \quad \text{and} \quad \operatorname{Im}(e^{it}) = \sin t$$

**Example 4:** Find the Laplace Transform of  $F(t) = \sin at$ .

$$\text{Solution: } L\{\sin at\} = L\{I(e^{iat})\} = I\{L(e^{iat})\} = I\left\{\int_0^\infty e^{iat} e^{-st} dt\right\} = I\left\{\int_0^\infty e^{-(s-ia)t} dt\right\}$$

$$= I\left\{\left[\frac{e^{-(s-ia)t}}{-(s-ia)}\right]_0^\infty\right\} = I\left\{\frac{1}{-(s-ia)}(0-1)\right\} = I\left\{\frac{1}{(s-ia)}\right\}$$

$$= I\left\{\frac{1}{(s-ia)} \times \frac{s+ia}{(s+ia)}\right\} = I\left\{\frac{s+ia}{s^2+a^2}\right\} = \frac{a}{s^2+a^2}$$

$$L\{\sin at\} = \frac{a}{s^2+a^2}$$

Similarly, we can use the same method to determine  $L\{\cos at\}$ .

**Example 5:** Find the Laplace Transform of  $F(t) = \cos at$ .

**Solution:** Since  $\cos at$  is the real part of  $e^{iat}$ , written as  $R(e^{iat})$

$$L\{\cos at\} = R\left\{\frac{s+ia}{s^2+a^2}\right\} = R\left\{\frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}\right\} = \frac{s}{s^2+a^2}$$

**Example 6: The Laplace Transform of  $F(t) = t^n$ , where  $n$  is a positive integer.**

**Solution:** By the definition,  $L\{t^n\} = \int_0^\infty t^n e^{-st} dt$

$$\begin{aligned} L\{t^n\} &= \int_0^\infty t^n e^{-st} dt = \left[ t^n \frac{e^{-st}}{-s} \right]_0^\infty - \int_0^\infty n t^{n-1} \frac{e^{-st}}{-s} dt \quad \text{Integrating by parts,} \\ &= -\frac{1}{s} \left[ t^n e^{-st} \right]_0^\infty + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} dt = -\frac{1}{s} [0-0] + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} dt \end{aligned}$$

$$\therefore L\{t^n\} = \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} dt \quad \text{----- (1)}$$

**Notice** that  $\int_0^\infty t^{n-1} e^{-st} dt$  is identical to  $\int_0^\infty t^n e^{-st} dt$  except that  $n$  is replaced by  $n-1$ .

$$\text{If } I_n = \int_0^\infty t^n e^{-st} dt, \text{ then } I_{n-1} = \int_0^\infty t^{n-1} e^{-st} dt$$

$$\text{So the result (1) becomes } I_n = \frac{n}{s} I_{n-1} \quad \text{----- (2)}$$

This is reduction formula and, if we now replace  $n$  by  $n-1$ , we get  $I_{n-1} = \frac{n-1}{s} I_{n-2}$

$$\text{Use it in equation (2), we get } I_n = \frac{n}{s} \frac{n-1}{s} I_{n-2} \quad \text{----- (3)}$$

If we replace  $n$  by  $n-2$  again in the result (2), we have  $I_{n-2} = \frac{n-2}{s} I_{n-3}$

$$\begin{aligned} \text{Use it in equation (3), we get } I_n &= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} I_{n-3} \\ &= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdot \frac{n-3}{s} I_{n-4} \end{aligned}$$

$$\text{So finally, we have } I_n = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdot \frac{n-3}{s} \cdot \frac{n-4}{s} \cdots \frac{2}{s} \cdot \frac{1}{s} I_0$$

$$\text{But } I_0 = L\{t^0\} = L\{1\} = \frac{1}{s}$$

$$\text{So } I_n = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdot \frac{n-3}{s} \cdot \frac{n-4}{s} \cdots \frac{2}{s} \cdot \frac{1}{s} = \frac{n!}{s^{n+1}}$$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

**Remark :**  $L\{t\} = \frac{1}{s^2}$  for  $n = 1$        $L\{t^2\} = \frac{2!}{s^3}$  for  $n = 2$   
 $L\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4}$  for  $n = 3$

**Example 7:** Find Laplace Transform of  $F(t) = \sinh(at)$ .

**Solution:** By the exponential definitions of  $\sinh at$ ,  $\sinh(at) = \frac{1}{2}(e^{at} - e^{-at})$

$$\begin{aligned} L\{\sin(at)\} &= \int_0^{\infty} (\sin at) e^{-st} dt = \int_0^{\infty} \left( \frac{1}{2}(e^{at} - e^{-at}) \right) e^{-st} dt = \frac{1}{2} \int_0^{\infty} (e^{-(s-a)t} - e^{-(s+a)t}) dt \\ &= \frac{1}{2} \left\{ \left[ \frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty} - \left[ \frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty} \right\} = \frac{1}{2} \left\{ \frac{1}{-(s-a)}(0-1) - \frac{1}{-(s+a)}(0-1) \right\} \\ &= \frac{1}{2} \left\{ \frac{1}{(s-a)} - \frac{1}{(s+a)} \right\} = \frac{1}{2} \left\{ \frac{s+a-s+a}{(s-a)(s+a)} \right\} = \left\{ \frac{a}{s^2-a^2} \right\} \\ L\{\sin(at)\} &= \left\{ \frac{a}{s^2-a^2} \right\} \end{aligned}$$

**Example 8:** Find Laplace Transform of  $F(t) = \cosh(at)$ .

**Solution:** By the exponential definitions of  $\cosh at$ ,  $\cosh(at) = \frac{1}{2}(e^{at} + e^{-at})$

$$\begin{aligned} L\{\cosh(at)\} &= L\left\{ \frac{1}{2}(e^{at} + e^{-at}) \right\} = \frac{1}{2} \int_0^{\infty} (e^{at} + e^{-at}) e^{-st} dt = \frac{1}{2} \left[ \int_0^{\infty} e^{-(s-a)t} dt + \int_0^{\infty} e^{-(s+a)t} dt \right] \\ &= \frac{1}{2} \left\{ \left[ \frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty} + \left[ \frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty} \right\} = \frac{1}{2} \left\{ \frac{1}{-(s-a)}(0-1) + \frac{1}{-(s+a)}(0-1) \right\} \\ &= \frac{1}{2} \left\{ \frac{1}{(s-a)} + \frac{1}{(s+a)} \right\} = \frac{1}{2} \left\{ \frac{s+a+s-a}{(s-a)(s+a)} \right\} = \frac{1}{2} \left\{ \frac{2s}{s^2-a^2} \right\} = \left\{ \frac{s}{s^2-a^2} \right\} \\ L\{\cosh(at)\} &= \frac{s}{s^2-a^2} \end{aligned}$$

**Remark:** We can combine these transforms by adding or subtracting as necessary, but they must not be multiplied together to form the transform of a product.

**Example 9**

(a)  $L\{2 \sin 3t + \cos 3t\} = 2L\{\sin 3t\} + L\{\cos 3t\} = 2 \cdot \frac{3}{s^2+9} + \frac{s}{s^2+9} = \frac{s+6}{s^2+9}$

(b)  $L\{4e^{2t} + 3 \cosh 4t\} = 4L\{e^{2t}\} + L\{3 \cosh 4t\}$   
 $= 4 \cdot \frac{1}{s-2} + 3 \cdot \frac{s}{s^2-16} = \frac{4(s^2-16) + 3s(s-2)}{(s-2)(s^2-16)} = \frac{7s^2-6s-64}{(s-2)(s^2-16)}$

## LECTURE No. 45

## THEOREMS

**Theorem 1: (The First Shift Theorem)**

The first Shift theorem states that if  $L\{F(t)\} = f(s)$ , then  $L\{e^{(-at)}F(t)\} = f(s+a)$ .

The transform  $L\{e^{(-at)}F(t)\}$  is thus the same as  $L\{F(t)\}$  with  $s$  everywhere in the result replaced by  $(s+a)$ .

**Example 1:**

$$\text{If } L\{\sin 2t\} = \frac{2}{s^2+4}, \text{ then } L\{e^{-3t} \sin 2t\} = \frac{2}{(s+3)^2+4} = \frac{2}{s^2+6s+13}$$

**Example 2:**

$$L\{t^2\} = \frac{2}{s^3}, \text{ then } L\{t^2 e^{4t}\} \text{ is the same with } s \text{ replaced by } (s-4).$$

$$\text{So } L\{t^2 e^{4t}\} = \frac{2}{(s-4)^3}$$

**Theorem 2: (Multiplying F(t) by t)**

$$\text{If } L\{F(t)\} = f(s), \text{ then } L\{t F(t)\} = -\frac{d}{ds}\{f(s)\}$$

$$\text{Example 3: Since } L\{\sin 2t\} = \frac{2}{s^2+4}$$

$$\text{So } L\{t \sin 2t\} = -\frac{d}{ds}\left[\frac{2}{(s^2+4)}\right] = \frac{4s}{(s^2+4)^2}$$

$$\text{Example 4: } L\{t \cos 3t\} = -\frac{d}{ds}\left(\frac{s}{s^2-9}\right) = -\frac{(s^2-9)-s(2s)}{(s^2-9)^2} = -\frac{s^2-9-2s^2}{(s^2-9)^2} = \frac{s^2+9}{(s^2-9)^2}$$

We could, if necessary, take this a stage further and find

$$L\{t \cos 3t\} = -\frac{d}{ds}\left(\frac{s^2+9}{(s^2-9)^2}\right) = \frac{2s(s^2+27)}{(s^2-9)^3}$$

Theorem Obviously extends the range of function that we can deal with. So, in general

$$\text{If } L\{F(t)\} = f(s) \text{ then } L\{t^n(F(t))\} = (-1)^n \frac{d^n}{ds^n}\{f(s)\}$$

**Theorem (Dividing by t)**

$$\text{If } L\{F(t)\} = f(s) \text{ then } L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(s)ds$$

$$\text{Example 5 : Determine } L\left\{\frac{\sin at}{t}\right\}$$

$$\text{As } L\{\sin at\} = \frac{a}{s^2+a^2}$$

$$\therefore L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \frac{a}{s^2+a^2} ds = \left[\tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \tan^{-1}\left(\frac{a}{s}\right)$$

**Example 6 :** Determine  $L\left\{\frac{1-\cos 2t}{t}\right\}$

$$\text{As } L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$$

Then by Theorem 3,

$$\begin{aligned} L\left\{\frac{1-\cos 2t}{t}\right\} &= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds = \left[ \ln s - \frac{1}{2} \ln(s^2+4) \right]_s^\infty = \left[ \frac{1}{2} \cdot 2 \ln s - \frac{1}{2} \ln(s^2+4) \right]_s^\infty \\ &= \left[ \frac{1}{2} \ln s^2 - \frac{1}{2} \ln(s^2+4) \right]_s^\infty = \frac{1}{2} \left[ \ln s - \ln(s^2+4) \right]_s^\infty = \frac{1}{2} \left[ \ln \frac{s^2}{(s^2+4)} \right]_s^\infty \end{aligned}$$

$$\text{When } s \rightarrow \infty \text{ then } \ln \frac{s^2}{(s^2+4)} \rightarrow \ln 1 = 0$$

$$L\left\{\frac{1-\cos 2t}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds = -\frac{1}{2} \left[ \ln \frac{s^2}{(s^2+4)} \right] = \ln \left[ \frac{s^2}{(s^2+4)} \right]^{-\frac{1}{2}} = \ln \sqrt{\frac{s^2+4}{s^2}}$$

### Standard Forms

<b>F(t)</b>	<b><math>\mathcal{L}\{F(t)\} = f(s)</math></b>
a	$\frac{a}{s}$
$e^{at}$	$\frac{1}{s-a}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$t^n$	$\frac{n!}{s^{n+1}}$

(n a positive integer)

### Theorem 1    The First Shift Theorem

If  $L\{F(t)\} = f(s)$  then  $L\{e^{(-at)} F(t)\} = f(s+a)$

### Theorem 2    Multiplying by t

If  $L\{F(t)\} = f(s)$  then  $L\{t(F(t))\} = -\frac{d}{ds}\{f(s)\}$

### Theorem 3    Dividing by t

If  $L\{F(t)\} = f(s)$  then  $L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(s) ds$ , provided  $\lim_{t \rightarrow 0} \left\{\frac{F(t)}{t}\right\}$  exists.

### Inverse Transforms

Here we have the reverse process i.e. given a Laplace transform, we have to find the function of  $t$  to which it belongs. For example, we know that  $\frac{a}{s^2 + a^2}$  is the Laplace

Transform of **sin at**, so we can now write  $L^{-1}\left[\frac{a}{s^2 + a^2}\right] = \sin at$ , the symbol  $L^{-1}$  indicating the inverse transform and not a reciprocal.

$$(a) \quad L^{-1}\left\{\frac{1}{s-2}\right\} = e^{-2t}$$

$$(b) \quad L^{-1}\left\{\frac{s}{s^2 + 25}\right\} = \cos 5t$$

$$(c) \quad L^{-1}\left\{\frac{4}{s}\right\} = 4$$

$$(d) \quad L^{-1}\left\{\frac{12}{s^2 - 9}\right\} = 4 \sinh 3t$$

But what about  $L^{-1}\left\{\frac{3s}{s^2 - s - 6}\right\}$ , it happens that we can write  $\frac{3s}{s^2 - s - 6}$  as the sum of two simpler functions  $\frac{1}{s+2} + \frac{1}{s-3}$  which, of course, makes all the difference, since we can now proceed.  $L^{-1}\left\{\frac{3s}{s^2 - s - 6}\right\} = L^{-1}\left\{\frac{1}{s+2}\right\} + L^{-1}\left\{\frac{1}{s-3}\right\} = e^{-2t} + 2e^{3t}$

### Rules of Partial Fractions

1. The numerator must be of lower degree than denominator. If it is not, then we first divide out.
2. Factorise the denominator into its prime factors. These determine the shapes of the partial fraction.
3. A linear factor  $(s+a)$  gives a partial fraction  $\frac{A}{s+a}$  is a constant to be determined.
4. A repeated factor  $(s+a)^2$  gives  $\frac{A}{s+a} + \frac{B}{(s+a)^2}$
5. Similarly  $(s+a)^3$  gives  $\frac{A}{s+a} + \frac{B}{(s+a)^2} + \frac{C}{(s+a)^3}$
6. A quadratic Factor  $(s^2 + ps + q)$  gives  $\frac{Ps+Q}{s^2 + ps + q}$
7. Repeated quadratic Factor  $(s^2 + ps + q)^2$  gives  $\frac{Ps+Q}{s^2 + ps + q} + \frac{Rs+T}{(s^2 + ps + q)^2}$

So  $\frac{s-19}{(s-2)(s+5)}$  has partial fraction of the form  $\frac{A}{(s-2)} + \frac{B}{(s+5)}$  and  $\frac{3s^2 - 4s + 11}{(s+3)(s-2)^2}$

has partial fraction  $\frac{A}{(s+3)} + \frac{B}{(s-2)} + \frac{C}{(s-2)^2}$

**Example 7**

To determine  $L^{-1} \left\{ \frac{5s+1}{s^2-s-12} \right\}$

a) First we check that the numerator is of lower degree than the denominator. In fact this is so.

b) Factorize the denominator

$$\frac{5s+1}{s^2-s-12} = \frac{5s+1}{(s-4)(s+3)} = \frac{A}{(s-4)} + \frac{B}{(s+3)}$$

We therefore have an identity

$$\frac{5s+1}{s^2-s-12} = \frac{A}{(s-4)} + \frac{B}{(s+3)}$$

which is true for any value of s we care to substitute

If we multiply through by the denominator  $(s^2-s-12)$  we have

$$5s+1 \equiv A(s+3) + B(s-4)$$

We now substitute convenient values for s

i) Let  $(s-4)=0$  that is  $s=4$  therefore  $21 = A(7) + B(0) \Rightarrow A=3$

ii) Let  $(s+3)=0$  that is  $s=-3$  therefore  $B=2$

So 
$$\frac{5s+1}{s^2-s-12} = \frac{3}{(s-4)} + \frac{2}{(s+3)}$$

$$L^{-1} \left\{ \frac{5s+1}{s^2-s-12} \right\} = 3e^{4t} + 2e^{-3t}$$

**Example 8** Determine  $L^{-1} \left\{ \frac{9s-8}{s^2-2s} \right\}$

$$L\{F(t)\} = \frac{9s-8}{s^2-2s}$$

a) Numerator of first degree ; denominator of second degree.

b) 
$$\frac{9s-8}{s^2-2s} = \frac{A}{s} + \frac{B}{s-2}$$

c) Multiply by  $s(s-2)$

$$\therefore 9s-8 \equiv A(s-2) + B(s)$$

d) Put  $s=0$

$$\therefore -8 \equiv A(-2) + B(0) \therefore A=4$$

e) Put  $s=2$ , i.e.  $s=2$

$$\therefore 10 = A(0) + B(0) \therefore B=5$$

$$\therefore F(t) = L^{-1} \left\{ \frac{4}{s} + \frac{5}{s-2} \right\} = 4 + 5e^{2t}$$

## Table of inverse transforms

Standard transforms

$f(s)$	$F(t)$
$\frac{a}{s}$	$a$
$\frac{1}{s+a}$	$e^{-at}$
$\frac{n!}{s^{n+1}}$	$t^n$ (n a positive integer)
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$ (n a positive integer)
$\frac{a}{s^2+a^2}$	$\sin at$
$\frac{s}{s^2+a^2}$	$\cos at$
$\frac{a}{s^2-a^2}$	$\sinh at$
$\frac{s}{s^2-a^2}$	$\cosh at$

### Transforms Of Derivatives

Let  $F'(t)$  denote the first derivative of  $F(t)$  with respect to  $t$ ,  $F''(t)$  denote the second derivative of  $F(t)$  with respect to  $t$ , etc.

Then  $L\{F'(t)\} = \int_0^{\infty} e^{-st} F'(t) dt$  by definition

Integrating by Parts

$$L\{F'(t)\} = \left[ e^{-st} F(t) \right]_0^{\infty} - \int_0^{\infty} F(t) (-se^{-st}) dt$$

when  $t \rightarrow 0, e^{-st} F(t) \rightarrow 0$

$$L\{F'(t)\} = -F(0) + s \int_0^{\infty} e^{-st} F(t) dt$$

$$L\{F'(t)\} = -F(0) + sL\{F(t)\}$$

$$L\{F''(t)\} = -F'(0) + sL\{F'(t)\} = -F'(0) + s[-F(0) + sL\{F(t)\}]$$

$$L\{F''(t)\} = s^2 L\{F(t)\} - sF(0) + F'(0)$$

$$L\{F'''(t)\} = s^3 L\{F(t)\} - s^2 F(0) - sF'(0) + F''(0)$$

$$L\{F^{iv}(t)\} = s^4 L\{F(t)\} - s^3 F(0) - s^2 F'(0) - sF''(0) + F'''(0)$$

**Differential Equation And Its Solution**

$$\frac{dx}{dt} - 2x = 4 \text{ -----(1)}$$

Its Solution is  $x = -2 + 3e^{2t}$ , To verify it we find  $\frac{dx}{dt}$

$$\frac{dx}{dt} = 6e^{2t} \text{ then}$$

$$\begin{aligned} \frac{dx}{dt} - 2x &= 6e^{2t} - 2(-2 + 3e^{2t}) \\ &= 6e^{2t} + 4 - 6e^{2t} = 4 \end{aligned}$$

So equation (1) is satisfied. Hence  $x = -2 + 3e^{2t}$  is solution of  $\frac{dx}{dt} - 2x = 4$

**Example 9:** Solve the differential equation  $\frac{dx}{dt} - 2x = 4$  given that at  $t = 0$ ,  $x = 1$

Taking Laplace transform as

$$\begin{aligned} L\left[\frac{d}{dt}(x(t)) - 2L(x(t))\right] &= L(4) \Rightarrow L\left[\frac{d}{dt}(x(t))\right] - 2L(x(t)) = L(4) \Rightarrow sL(x(t)) - x(0) - 2L(x(t)) = \frac{4}{s} \\ (s-2)L(x(t)) - x(0) &= \frac{4}{s} \Rightarrow (s-2)L(x(t)) - 1 = \frac{4}{s} \Rightarrow (s-2)L(x(t)) = \frac{4}{s} + 1 \Rightarrow (s-2)L(x(t)) = \frac{4+s}{s} \\ \Rightarrow L(x(t)) &= \frac{4+s}{s(s-2)} \Rightarrow x(t) = L^{-1}\left[\frac{4+s}{s(s-2)}\right] \text{-----(1)} \end{aligned}$$

First we do the partial fraction of  $\frac{4+s}{s(s-2)}$

$$\frac{4+s}{s(s-2)} = \frac{A}{s} + \frac{B}{(s-2)}$$

$$\Rightarrow 4+s = A(s-2) + B(s) \text{-----(2)}$$

$$\text{Put } s = 0 \text{ in equation (2)} \quad ; \quad 4 = -2A \quad ; \quad A = -2$$

$$\text{Put } s = 2 \text{ in equation (2)} \quad ; \quad 6 = B(2) \quad ; \quad B = 3$$

$$\text{So } \frac{4+s}{s(s-2)} = \frac{-2}{s} + \frac{3}{s-2}$$

Equation # (1) becomes

$$x(t) = L^{-1}\left[\frac{4+s}{s(s-2)}\right] = L^{-1}\left[\frac{-2}{s}\right] + L^{-1}\left[\frac{3}{s-2}\right] = -2 + 3e^{2t}$$

**Solution of differential equation by laplace transforms**

To solve a differential equation by Laplace transforms, we go through Laplace transforms, we go through four distinct stages.

- Re- write the equation in term of Laplace transforms.
- Insert the given initial conditions.
- Rearrange the equation algebraically to give the transform of the solution.

(d) Determine the inverse transform to obtain the particular solution.

**Solve the equation**

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2e^{3t} \text{ given that at } t = 0, x = 5 \text{ and } \frac{dx}{dt} = 7$$

$$x''(t) - 3x'(t) + 2x(t) = 2e^{3t}$$

$$\text{Given } x(0) = 5, x'(0) = 7$$

$$L(x''(t)) - 3L(x'(t)) + 2L(x(t)) = 2L(e^{3t})$$

$$s^2L\{x(t)\} - sx(0) - x'(0) - 3\{sL(x(t)) - x(0)\} + 2Lx(t) = \frac{2}{s-3}$$

We rewrite the equation in term of its transforms.

$$L\left[\frac{d^2x}{dt^2}\right] - 3L\left[\frac{dx}{dt}\right] + 2L[x] = 2L[e^{3t}]$$

$$[s^2L(x(t)) - sx(0) - x'(0)] - 3[sL(x(t)) - x(0)] + 2L(e^{3t})$$

$$\text{At } t = 0, x = 5, \frac{dx}{dt} = 7$$

$$\text{So } x(0) = 5, x'(0) = 7$$

$$s^2L(x(t)) - s(5) - 7 - 3\{sL(x(t)) + 3(5)\} + 2L(x(t)) = \frac{2}{s-3}$$

$$s^2L(x(t)) - 3sL(x(t)) + 2L(x(t)) = \frac{2}{s-3} - 8 + 5s$$

$$(s^2 - 3s + 2)L(x(t)) = \frac{2 - 8s + 24 + 5s^2 - 15s}{s-3}$$

$$L(x(t)) = \frac{2 - 8s + 24 + 5s^2 - 15s}{(s-1)(s-2)(s-3)} = \frac{5s^2 - 23s + 24}{(s-1)(s-2)(s-3)}$$

Making Partial fraction of R.H.S, We have

$$L(x(t)) = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)}$$

After solving these we get A = 3, B = 2 and C = 0

$$\text{So } L(x(t)) = \frac{3}{(s-1)} + \frac{2}{(s-2)} + \frac{0}{(s-3)}$$

$$L(x(t)) = \frac{3}{(s-1)} + \frac{2}{(s-2)}$$

$$x(t) = L^{-1}\left\{\frac{3}{(s-1)}\right\} + L^{-1}\left\{\frac{2}{(s-2)}\right\} = 3e^t + 2e^{2t}$$