Composition of Functions

Definition: For functions f and g, define $f \circ g$, the *composition* of f and g by,

$$(f \circ g)(x) = f(g(x))$$

$\text{Dom}(f \circ g) = \{x \in \text{dom}(g) | g(x) \in \text{dom}(f)\}$

Example: Suppose $f(x) = x - 2$ and $g(x) = x^2$.

(a) Find $(f \circ g)$ and $(g \circ f)$.

$(f \circ g)(x) = f(g(x)) = f(x^2) = x^2 - 2$

$(g \circ f)(x) = g(f(x)) = g(x - 2) = (x - 2)^2 = x^2 - 4x + 4$

***Note:** $(f \circ g) \neq (g \circ f)$

(b) Find $(f \circ g)(2)$ and $(g \circ f)(2)$.

$(f \circ g)(2) = f(g(2)) = f(2^2) = f(4) = 4 - 2 = 2$

$(g \circ f)(2) = g(f(2)) = g(2 - 2) = g(0) = 0^2 = 0$

Example: For $f(x) = 3x + 4$ and $g(x) = 5$, find $(f \circ g)$ and $(g \circ f)$.

$(f \circ g)(x) = f(g(x)) = f(5) = 3(5) + 4 = 19$

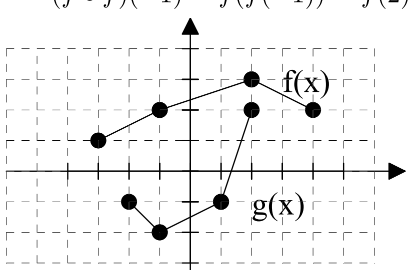
$(g \circ f)(x) = g(3x + 4) = 5$

Example: For f and g below, note that
$f(-3) = 1$, $f(-1) = 2$, $f(2) = 3$, $f(4) = 2$, and
$g(-2) = -1$, $g(-1) = -2$, $g(1) = -1$, $g(2) = 2$.
Find:

$(f \circ g)(2) = f(g(2)) = f(2) = 3$,

$(g \circ f)(2) = g(f(2)) = g(3) =$ undefined,

$(f \circ f)(-1) = f(f(-1)) = f(2) = 3$



Example: $f(x) = x + \frac{1}{x}$. Find $(f \circ f)$.

$$(f \circ f)(x) = f(f(x)) = (x + \tfrac{1}{x}) + (\tfrac{1}{x+\frac{1}{x}}) = x + \tfrac{1}{x} + \tfrac{x}{x^2+1}$$

Next we want to write a function as a composition of 2 simpler functions.

Example: Write $(x^2+2)^6$ as a composition $f(g(x))$.

$(x^2+2)^6$ has an inner function $g(x) = x^2 + 2$.

Then the outer function $f(x)$ does what remains to be done: $f(x) = x^6$.
Check: $f(g(x)) = f(x^2+2) = (x^2+2)^6$.

Example: Write $4\frac{1}{x} + 3$ as a composition $f(g(x))$.

$4(\frac{1}{x}) + 3$ as inner function $g(x) = \frac{1}{x}$.

Then the outer function $f(x)$ does what remains to be done: $f(x) = 4x + 3$.
Check: $f(g(x)) = f(\frac{1}{x}) = 4(\frac{1}{x}) + 3$.

Example: Write $\sqrt{x+1}$ as a composition $f(g(x))$.

$\sqrt{x+1}$ has inner function $g(x) = x + 1$

So $f(x) = \sqrt{x}$.

Check: $f(g(x)) = f(x+1) = \sqrt{x+1}$.

Example: Write $x^4 + x^2 + 1$ as a composition.

$x^4 + x^2 + 1 = (x^2)^2 + x^2 + 1. \quad \Rightarrow \quad g(x) = x^2 \quad \Rightarrow \quad f(x) = x^2 + x + 1.$

Check: $f(g(x)) = f(x^2) = (x^2)^2 + x^2 + 2 = x^4 + x^2 + 1.$

Example: Write $\frac{1}{1+|x|}$ as the composition of 3 functions $h(f(g(x)))$.

$\frac{1}{1+|x|}$ has inner function $g(x) = |x|$

$\frac{1}{(1+x)}$ has inner function $f(x) = 1 + x$

Thus $h(x) = \frac{1}{x}$.

Check: $h(f(g(x))) = h(f(|x|)) = h(1 + |x|) = \frac{1}{1+|x|}$

Inverse Functions

Definition: f^{-1}, the *inverse* of f, is the function, if any, such that

$$(f \circ f^{-1})(x) = x \qquad \text{when } f^{-1}(x) \text{ is defined and}$$
$$(f^{-1} \circ f)(x) = x \qquad \text{when } f(x) \text{ is defined}$$

Example: $f(x) = 2x$, $g(x) = \frac{x}{2}$

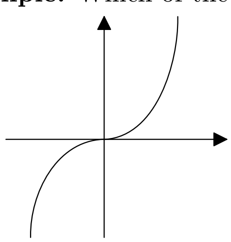
Consider $f(g(x)) = f(\frac{x}{2}) = 2(\frac{x}{2}) = x$ and $g(f(x)) = g(2x) = \frac{2x}{2} = x$. Thus, $g(x)$ is an inverse function of $f(x)$. I can write $f^{-1}(x) = g(x) = \frac{x}{2}$.

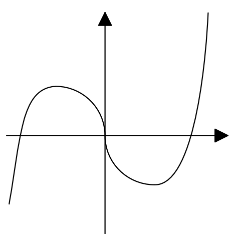
Definition: f is 1-1 ("one-to-one") $\iff$ $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

Example: $f(x) = 3x$ is 1-1 but
$g(x) = x^2$ is not 1-1 since $1 \neq -1$ but $(-1)^2 = 1^2$.

Horizontal Line Test: If every horizontal line intersects the graph of a function f in at most one point, the f is one-to-one.

Example: Which of the following has an inverse?





Answer: The first graph has an inverse and the second graph doesn't.

Theorem: The function f has an inverse if and only if f is 1-1.

Theorem: $y = f^{-1}(x) \iff f(y) = x$.

To find $f^{-1}(x)$ for complicated functions:
(1) Switch x and y in $y = f(x)$, i.e. write $x = f(y)$.
(2) Solve for y. After that you can replace y by $f^{-1}(x)$.

Example: $f(x) = x^3$, find $f^{-1}(x)$.
$f(y) = x \iff y^3 = x \iff y = x^{\frac{1}{3}} \iff f^{-1}(x) = x^{\frac{1}{3}}$.

*****Note:** $f^{-1}(x) \neq (f(x))^{-1}$.

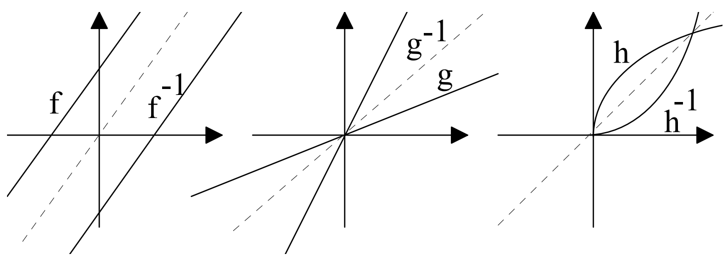
$f^{-1}(x)$ is the inverse of $f(x)$ and $(f(x))^{-1} = \frac{1}{f(x)}$ is the reciprocal of $f(x)$.

Example: $f(x) = \frac{2x+1}{x-1}$, find $f^{-1}(x)$.

$$\begin{gathered} y = \tfrac{2x+1}{x-1} \\ x = \tfrac{2y+1}{y-1} \\ x(y-1) = 2y+1 \\ xy - x = 2y+1 \\ xy - 2y = x+1 \\ y(x-2) = x+1 \\ y = \tfrac{x+1}{x-2} \end{gathered}$$

Now let's look at how the graph of f is related to the graph of f^{-1}.

Since $y = f^{-1}(x) \iff f(y) = x$. Thus the graph of $y = f^{-1}(x)$ is the graph of $f(y) = x$ which is just the graph of $f(x) = y$ with x and y interchanged. Interchanging x and y reflects the plane around the major diagonal $y = x$.



Theorem: The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ across the major diagonal $y = x$.

Theorem: The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.